Adiabatic Statistics and Hall Viscosity of Quantum Hall Systems

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What is a (topological) phase of matter?

--a system with a (bulk) gap above the ground state in the thermo limit. Point "defects" allowed also.

How to distinguish between them?

--overlap with a trial state not so useful

(can be small and state is still in the phase, e.g. in thermo limit)

--properties that remain nonzero in thermo limit, and are unchanged throughout the phase, are preferable. E.g. :

---quantum numbers of states, excitations; fusion rules
---adiabatic statistics of excitations
---ground state degeneracy on torus

--"topological" even when a consequence of a symmetry
--connected with statistics

Why are such properties invariant?

--no local degrees of freedom, only global ones (ignore e.g. spin)
In thermo limit, local few-particle operators (such as a change in the Hamiltonian density) cannot split the degeneracies involved in non-Abelian stats or ground states on torus Wen and Niu

--Statistics and ground state degeneracy, even in Abelian phases (e.g. Laughlin) are described by a Chern-Simons gauge thy action, which cannot be renormalized by a small change in underlying Hamiltonian – *rigidity*

Most basic question is always:

What phase are we in?

Topological (gapped) phases in 2 + 1 are related to modular tensor categories

Conformal blocks as trial wavefunctions give *same* MTC as their RCFT, or else gapless

---non-unitary cases

Hall viscosity: *new fundamental physics*

adiabatic transport approach

connection with orbital spin

effective field theory (w. W. Goldberger)

numerical calculations (w. E. Rezayi)

N. Read, Phys. Rev. B **79**, 045308 (2009)

Gapped (topological) phase in 2+1

$\longrightarrow$ 2+1 Topological Quantum Field Theory (TQFT)

or modular tensor category

Moore + Seiberg, Witten, Reshitikhin + Turaev (1988 – 90)

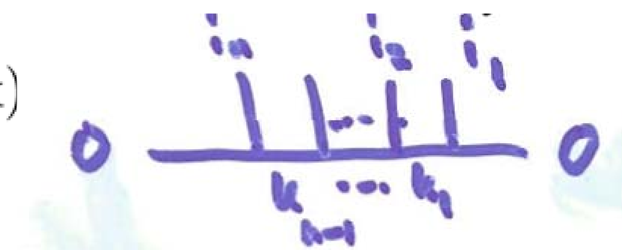
--Finite set of quasiparticle types α ; $\alpha = 0$ is the vacuum/identity op

--Fusion: $$\phi_\alpha \times \phi_\beta = \sum_\gamma N^\gamma_{\alpha\beta}\phi_\gamma, \qquad N^\gamma_{\alpha\beta} \geq 0$$ are integers

$\longrightarrow$degeneracy of *n* well-separated qptcles all of type α

$$\dim \sim (\text{largest eval of } N_\alpha \text{ matrix})$$

as $n \to \infty$.



Operations on these spaces ↑ "time"

Braiding $c_{V_i,V_j} =$ 

→ braid gp rep^n
↓
YB eq

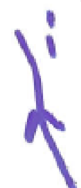
Creation/Destruction

$i_{V_i} =$ 

, $e_{V_i} =$ 

(note order)

Axiom: 

= 

= id

Twist: 2π rotation. Thicken to ribbon:

$\theta_i =$  =  = 

$$\theta_\alpha = e^{2\pi i s_\alpha}$$

(There are consistency conditions also.)

This structure is called a ribbon tensor category

Quantum trace and quantum dimension: (just numbers)

Want 

but 

not yet defined

Use 

so 

= 

$= e'_{V_i}$

Also 

= 
$= i'_{V_i}$

Define $f: V \to V$

$q\mathrm{tr}\, f =$ 

(not only for simple V_i)

Simples = qptcle type i, define

$q\dim V_i =$ 

$= d_i$ Quantum dimension

Also

$\tilde{S}_{ij} =$ 

. <u>Modular</u> TC if $\tilde{S}$ is invertible.

Hermitian structure   Turaev

For any map f, have antilinear map $f \to \bar{f}$ s.t.

$$\bar{\bar{f}} = f$$

$$\overline{id_V} = id_V$$

$$\overline{c_{V,W}} = c_{V,W}^{-1}$$

$$\overline{\theta_V} = \theta_V^{-1}$$

$$\bar{i}_V = e_V c_{V,V^*}(\theta_V \otimes id_{V^*}) = e_V'$$

$$\bar{e}_V = i_V'$$

(Then $S = \tilde{S}/D$ is unitary as matrix.)
Also d_i real.

"Unitary" if <u>positive definite</u>

$$q\mathrm{tr}_V \bar{f} f \geq 0 \quad \text{all } f$$

“Unitary” (*positive definite)* MTC captures $\boxed{\text{positivity of QM .}}$

In particular, $d_\alpha > 0$ and largest eval of N_α is d_α .

Rational conformal field theories (RCFTs) also produce an MTC.

Moore and Seiberg (1988)

Non-unitary RCFTs (in 2D sense) contain some conf weight h<0 and hence some d_α is negative, in every known case.

$\boxed{\text{What is relation between “unitary” in 2D RCFT and in 3D TQFT?}}$

Conformal blocks as trial wavefunctions

Conformal blocks come from RCFTs: Moore and NR (1991)

$$\langle \psi(z_1, \bar{z}_1) \cdots \psi(z_N, \bar{z}_N) \tau(w_1, \bar{w}_1) \cdots \tau(w_1, \bar{w}_1) \rangle_{\mathrm{CFT}} = \sum_a |\Psi_a(w_1, \ldots; z_1, \ldots)|^2$$

Blocks Ψ_a are analytic but multivalued functions in w_l
---"monodromy" under e.g. braiding $\Psi_a \to \sum_b \Psi_b M_{ba}$; M unitary

Many QH trial wavefunctions are conformal blocks, e.g.

$$\Psi_{\mathrm{MR}}(z_1, \ldots, z_N) = \mathcal{A}\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots\right) \prod_{i<j} (z_i - z_j)^Q \cdot e^{-\frac{1}{4}\sum_i |z_i|^2}$$

and with quasiholes at w_l also.

Usual inner product

For braiding and twist in a top phase, must calculate them by **adiabatic transport** (Berry phase/matrix): for Ψ_a **orthonormal**

$$|\Psi_b(w_{(0)})\rangle \to \sum_a |\Psi_a(w_{(0)})\rangle B_{ab}$$

where **holonomy** is

$$B = M\mathcal{P}\exp i\oint_C (A_w \cdot dw + A_{\overline{w}} \cdot d\overline{w}),$$

$$A_{w,l,ab}(w) = i\left\langle \Psi_a(w) \middle| \frac{\partial \Psi_b(w)}{\partial w_l} \right\rangle$$

If also Ψ_a are holomorphic in *w*, then

$$\boxed{B = M}$$

as desired in MR (1991).

As Ψ_a is holomorphic in w_l, issue is orthonormality

$$\langle \Psi_a(w_1,\ldots,w_n) | \Psi_b(w_1,\ldots,w_n)\rangle = \mathcal{Z}_{ab}(w_1,\ldots,w_n)$$

Integrals over z and definition of conformal blocks $\rightarrow$
(go grand canonical)

$$\sum_a \mathcal{Z}_{aa} = \langle e^{\lambda \int d^2 z\ \psi(z,\bar{z})} \prod_k \tau(w_k,\bar{w}_k)\rangle_{\mathrm{CFT}}$$

---CFT perturbed by ψ. What is long-distance behavior of perturbed thy?

---- 1) massive
---- 2) massless
---- 3) other?

1) Massive 2D phase

Correlators $\sum_a \mathcal{Z}_{aa}$ generically go to constants as $|w_k - w_l| \to \infty$

But $\mathcal{Z}_{ab}$ cannot all be non-zero because of monodromy:

$$\mathcal{Z}_{ab} \to \sum_{c,d} (M^\dagger)_{ac} \mathcal{Z}_{cd} M_{db}$$

These imply that

$$\mathcal{Z}_{ab} = \sum_{c,d} (M^\dagger)_{ac} \mathcal{Z}_{cd} M_{db}$$

Schur's lemma implies $\mathcal{Z}_{ab} \propto \delta_{ab}$

Like order/disorder operators
in stat.mech/field thy

$B = M$ follows!

2) Massless 2D phase

$$\mathcal{Z}_{ab}(w_1.w_2) \sim \frac{1}{|w_1 - w_2|^{\#}}\left[1 + \mathcal{O}\left(\frac{1}{|w_1 - w_2|^2}\right)\right]$$

$\rightarrow$get power-law corrections to holonomy

---no good in a gapped phase (and other problems)

---probably gapless

3) Other ?

Worse!

Can do quasihole spin (twist) similarly

NR (2008)

Then either MTC obtained is that of the RCFT, or system is gapless.

But use of non-unitary RCFT will produce some negative d_α , not acceptable in QM top. phase.

E.g. Bernevig and Haldane (2008)

Hence those states must be gapless in 2+1 sense!

Second argument using edge states: N.R., PRB (2009)

Hall Viscosity

Avron, Seiler, and Zograf (1995)

Hall viscosity is the viscosity analog of Hall conductivity:

$\sigma_{ij} =$ stress $\qquad u_{ij} =$ strain

$$\sigma_{ij} = \sum_{k,l} \left[\lambda_{ijkl} u_{kl} - \eta_{ijkl} \frac{d}{dt} u_{kl} + \mathcal{O}\left(\frac{d^2}{dt^2} \right) \right], \quad (i,j,k,l = 1,\ldots,d)$$

$\lambda_{ijkl} =$ pressure/elasticity $\qquad \eta_{ijkl} =$ viscosity

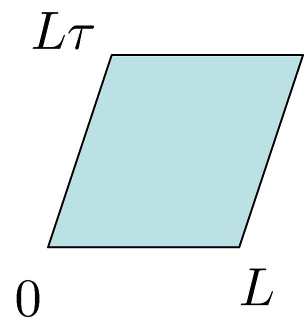
$u_{ij} = u_{ji}$ so $\eta_{ijkl} = \eta_{jikl} = \eta_{ijlk}$

Symmetric part $\eta^{(S)}_{ijkl} = \eta^{(S)}_{klij}$ gives dissipation

Antisymmetric (Hall) part $\eta^{(A)}_{ijkl} = -\eta^{(A)}_{klij}$ non-dissipative

---odd under time-reversal symmetry
---in d=2 isotropic system, only one ind comp: $\eta^{(A)}$

In a top phase, comes from response to varying metric instead of strain, for fixed coordinate system (no independent velocity)
---for torus, equivalent to changing (complex) aspect ratio τ at fixed area:



Hall viscosity is equal to the adiabatic curvature (curl of Berry vector potential) in τ space $(\mathrm{Im}\,\tau > 0)$, divided by area A= $L^2 \mathrm{Im}\,\tau$

Avron, Seiler, and Zograf (1995)

For $\nu \geq 1$ filled Landau levels:

$$\eta^{(A)} = \frac{1}{4}\nu\bar{n}\hbar$$

Avron, Seiler, and Zograf (1995)
(factor of 2: Vignale and Tokatly, 2008)
Levay (1995)

($\bar{n}$ is the particle density) --- ind of τ !

For (i) paired superfluids (e.g. p+ip), and (ii) conformal blocks used as trial QH wavefunctions

$$\boxed{\eta^{(A)} = \frac{1}{2}\bar{s}\bar{n}\hbar}$$

NR (2009)

where $\bar{s}$ is (minus) the mean orbital spin per particle:

(“real” spin neglected here)

$$\bar{s} = \begin{cases} 1/2 & \text{for p-ip} \\ Q/2 & \text{for Laughlin } \nu = 1/Q \text{ state} \\ \nu^{-1}/2 + h_\psi & \text{for general conformal block states} \end{cases}$$

$\bar{s} = \mathcal{S}/2$ where $\mathcal{S}$ is the shift on the sphere: $N_\phi = \nu^{-1}N - \mathcal{S}$

Should be:

---quantized within trans/rot invariant topological phase

---general result for all such phases (Other fluids?)

For classical plasma $\eta^{(A)} = \frac{\bar{n} k_B T}{2\omega_c}$ Lifshitz and Pitaevskii, Physical Kinetics

--- electron in $\mathcal{N}$th LL has orbital spin $\mathcal{N} + 1/2$, cf. Levay (1995)
due to cyclotron motion. This equals kinetic energy / $\hbar\omega_c$.

Hence thermal average at high T (using equipartition) $\Rightarrow$

$$\bar{s} = \frac{k_B T}{\hbar \omega_c}$$

and
$$\eta^{(A)} = \frac{1}{2}\bar{s}\bar{n}\hbar = \frac{\bar{n} k_B T}{2\omega_c}$$
NR (2009)

Relation to spin

Fix coordinates $x^1 = x,\ x^2 = y,\ 0 \le x, y \le 1,$ metric is

$$ds^2 = \sum_{i,j} g_{ij} dx^i dx^j = \frac{A}{\operatorname{Im}\tau}(dx^2 + 2\operatorname{Re}\tau\, dx dy + |\tau|^2 dy^2).$$

Under an "active" coordinate transformation in

$$\mathrm{SL}(2,\mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a,b,c,d \in \mathbf{R},\ ad - bc = 1 \right\},$$

the change in g_{ij} at fixed area *A* is described by $\tau \to \dfrac{a\tau + b}{c\tau + d}.$

E.g. under small transformations $\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \begin{pmatrix} 1+\varepsilon' & 0 \\ 0 & 1-\varepsilon' \end{pmatrix},\ \varepsilon, \varepsilon'$ real,

$\tau = i$ (square) undergoes $i \to i + 2\varepsilon, \quad i \to i + 2\varepsilon' i$ --- two distinct shears.

Commutator is $= I + 2\varepsilon\varepsilon' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ --- an SO(2) rotation! cf. Levay (1995)

Leaves g_{ij} invariant --- rotation of space.

Adiabatic curvature (curl of Berry connection) is given by the commutator of transformations, evaluated on a state, so picks up expectation of generator of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But as this is equivalent to rotation in real space, we identify eigenvalue as “orbital spin”. Note the upper half-plane (τ space) is

$$\cong \mathrm{SL}(2, \mathbf{R})/\mathrm{SO}(2).$$

[Cf. adiabatic rotations of coherent state for SU(2) spin, for spin in z direction, rotations about x and y commute to give S_z, and pick up expectation value. Note the space is $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$.]

In a topological phase, adiabatic curvature is $\mathrm{SL}(2, \mathbf{R})$-invariant on upper-half τ-plane (like that on sphere for spin). Hence Hall viscosity is independent of shape of the system, as it should be for a fluid.

Effective field theory

NR, Goldberger, to appear

Induced action for a 2+1 topological phase in *external* electromagnetic and gravitational fields:

$$S = \frac{\nu}{4\pi}\int d^3x\,\sqrt{g}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{\nu\mathcal{S}}{4\pi}\int d^3x\,\sqrt{g}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu\omega_\lambda + \dots$$

$\uparrow$ Chern-Simons term $\qquad$ $\uparrow$ Wen-Zee term

A_μ = e.m. vector potential, ω_μ = the spin connection for SO(2) *spatial* rotations only.

Vary wrt A_0 and integrate, use Gauss-Bonnet Theorem $\int d^2x\,R = 2\pi(2-2\mathcal{G})$ on genus $\mathcal{G}$ surface ($\mathcal{G}=0$ is the sphere),

$$N = \nu N_\phi + \nu\mathcal{S}(1-\mathcal{G}) \longrightarrow N_\phi = \nu^{-1}N - \mathcal{S}(1-\mathcal{G})$$

so $\mathcal{S}$ is the shift. Coupling of orbital spin to curvature of space.

Wen and Zee (1992)

Induced action has local Galilean, not Lorentz, invariance.

NR, Goldberger, to appear

Apply to variation of τ for torus: use $\omega_\mu \sim g^{-1}\partial_\mu g$ and

$$\nu \nabla \times \mathbf{A}/2\pi = \bar{n} - \nu \bar{s} R/2\pi$$

R vanishes, reproduces the adiabatic Hall viscosity result.
---explains why the shift and Hall viscosity are related by $\bar{s} = \mathcal{S}/2$.

Like the Chern-Simons term, the Wen-Zee term cannot be renormalized, because it is not the integral of a local *gauge-invariant* combination of fields. (Or because of angular momentum conservation in perturbative corrections.)

Hence the result for $\eta^{(A)}$ from trial wavefunctions will hold throughout a trans/rot inv topological phase.

Varying wrt metric gives (complicated) expression for stress tensor of QH state.

Use of Hall viscosity as a diagnostic tool for numerics:
Measure shift on torus (unbiased) instead of extrapolating energies for different shifts on sphere. Don't need the trial state.

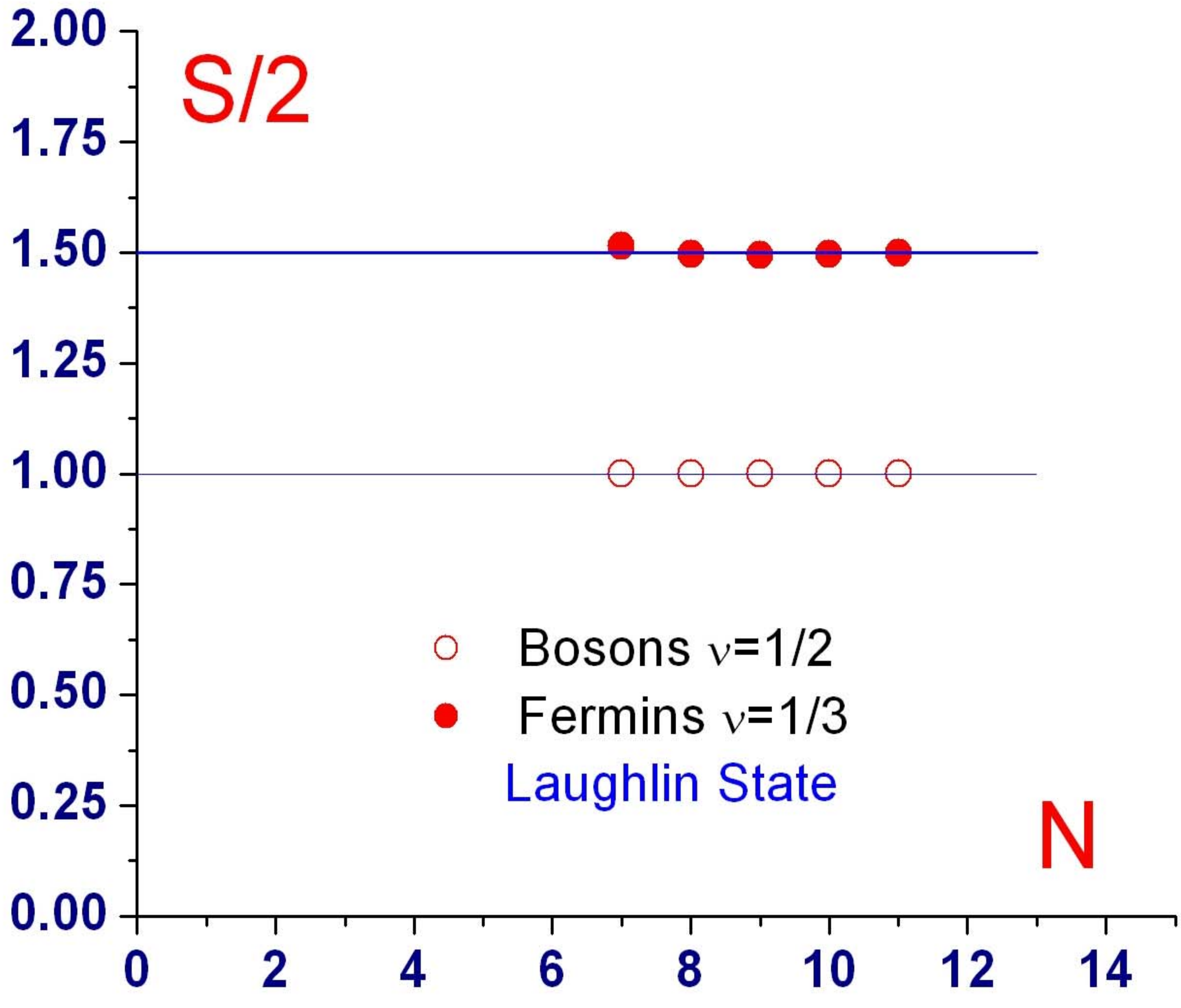
Numerical approach: Rezayi, NR, in progress

---evaluate Berry phase γ for a loop C in τ of M discrete steps as

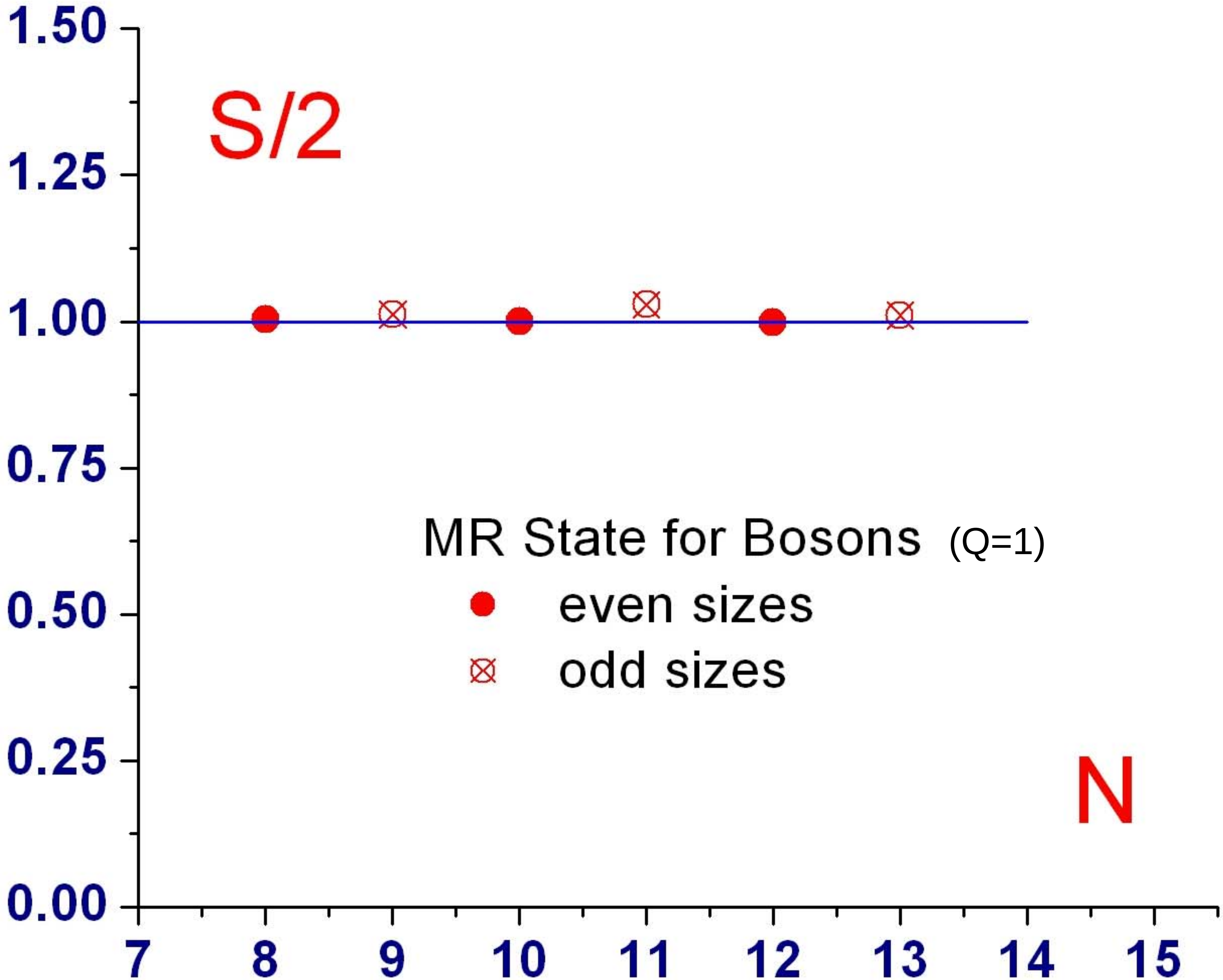
$$e^{i\gamma} = \prod_{l=1}^{M} \langle \Psi(\tau_{l+1}) | \Psi(\tau_l) \rangle$$

Use C small, M large. Divide γ by area of C , and by N to get $\bar{s}$.

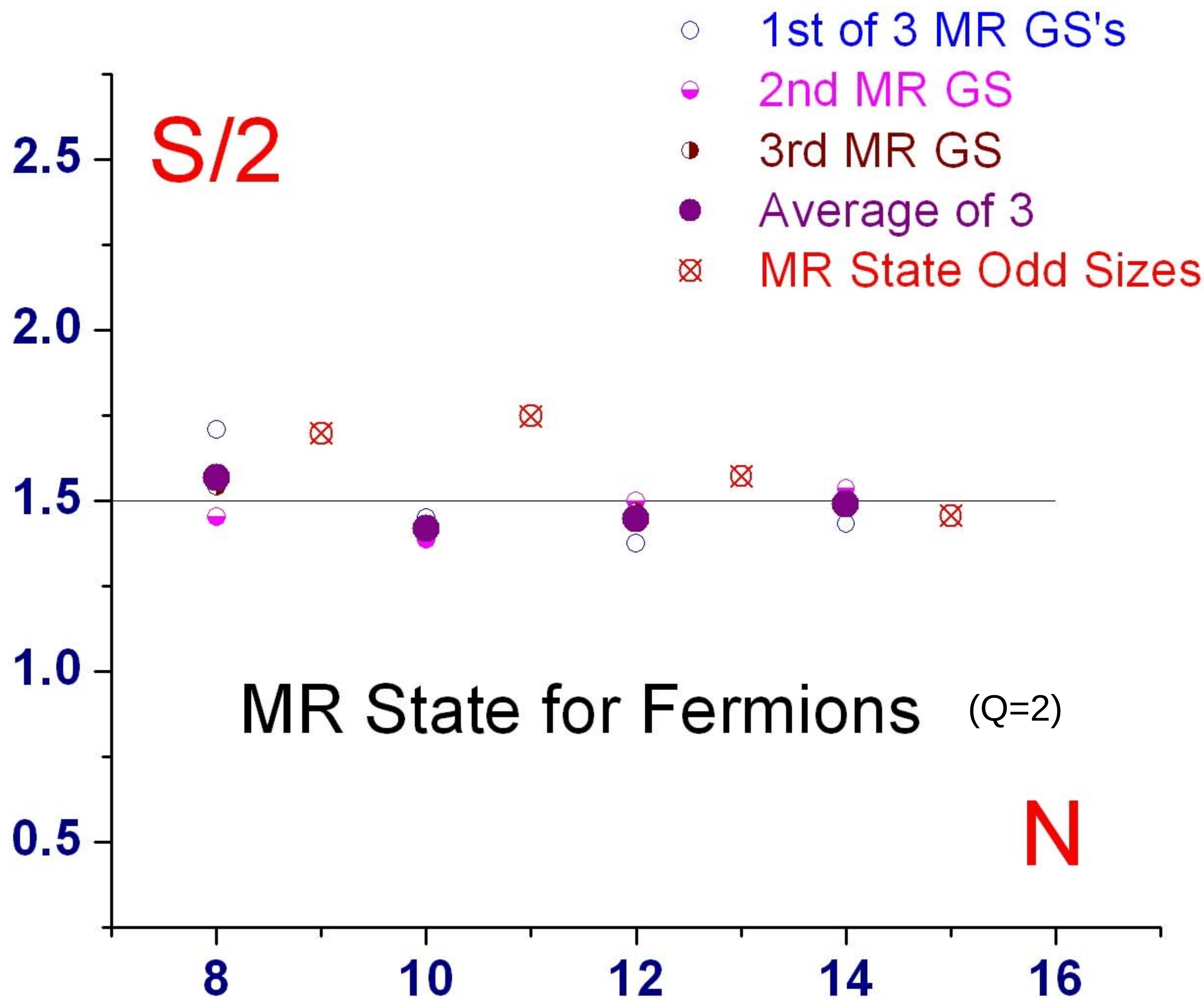
For trial states (Laughlin, Moore-Read), confirm expected values when N sufficiently large.

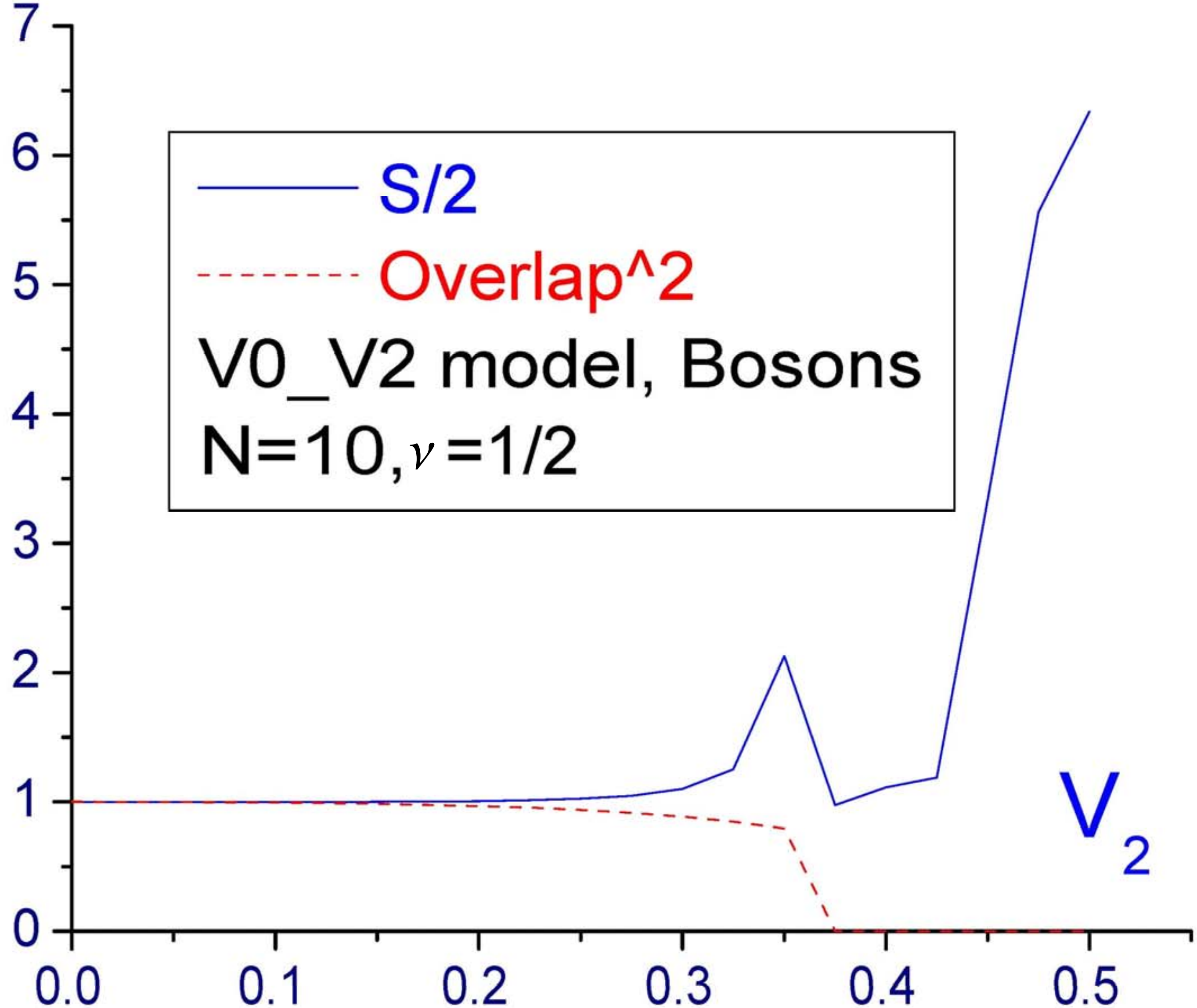


S/2
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10
12
14
Bosons ν=1/2
Fermins ν=1/3
Laughlin State
N



S/2
1.50
1.25
1.00
0.75
0.50
0.25
0.00
MR State for Bosons (Q=1)
even sizes
odd sizes
N
7
8
9
10
11
12
13
14
15


1st of 3 MR GS's
2nd MR GS
3rd MR GS
Average of 3
MR State Odd Sizes
S/2
2.5
2.0
1.5
1.0
0.5
MR State for Fermions (Q=2)
N
8
10
12
14
16



S/2
Overlap^2
V0_V2 model, Bosons
N=10, ν =1/2
V_2
0
1
2
3
4
5
6
7
0.0
0.1
0.2
0.3
0.4
0.5

Conclusion

1) Adiabatic statistics: either given by the monodromy of blocks, or state not gapped

2) Trial functions from a non-unitary RCFT don’t give a top. phase

3) Hall viscosity: new bit of basic physics
   potential use in numerical diagnostics