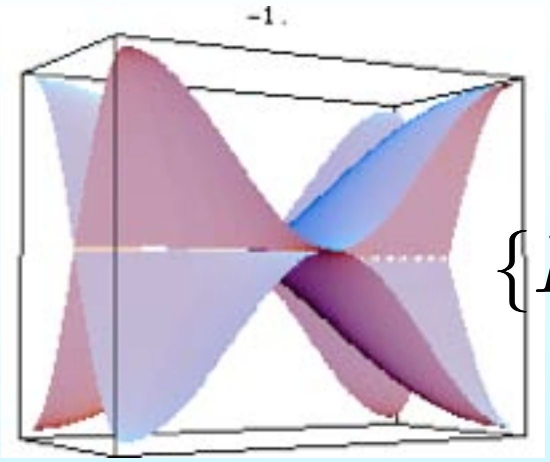
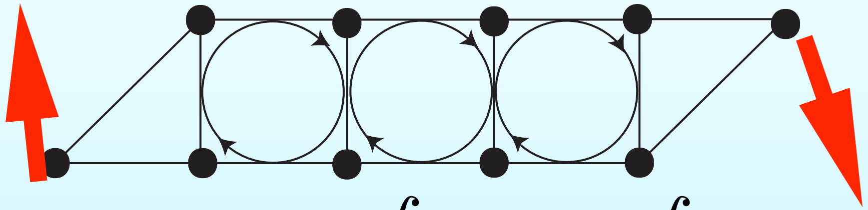
Y. Hatsugai, Cond-mat meets Hep: IPMU Focus Week, Feb 9, 2010

http://rhodia.ph.tsukuba.ac.jp/~hatsugai/modules/pico/?ml_lang=en



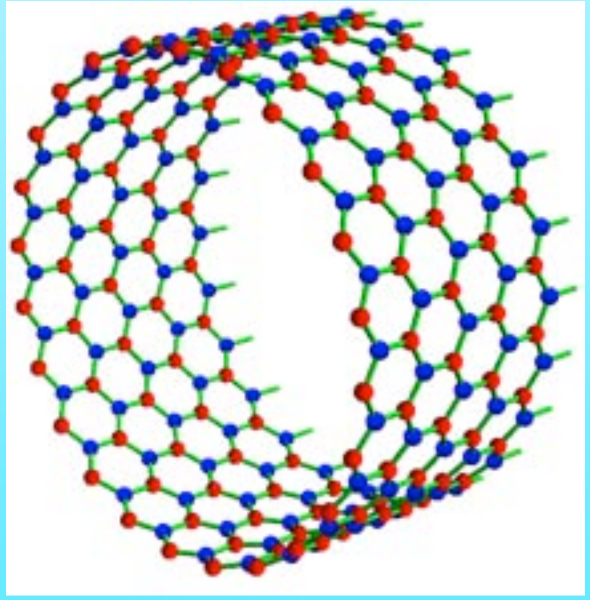
$$\{H,\Gamma\}=0$$



$$i\gamma=\int dA=\int\langle\psi|d\psi\rangle$$

Correspondence between the bulk quantum states and boundary states in topological phases in condensed matter

Zoo of insulators
&
Zoo of boundary states



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University of Tsukuba
筑波大学

Outline

- ☆ Are insulators boring ?
  - ☆ Zoo of insulators : variety to universality
  - ☆ Symmetry breaking & Topological order
  - ☆ Classification of the zoo
- ☆ Observables: Classical to Quantum
  - ☆ Gapless or Gapped
  - ☆ Berry connection & quantization (Chern numbers, etc)
- ☆ Zoo of boundary states
  - ☆ Here & There to symmetry
  - ☆ Bulk-Edge correspondence
- ☆ A lucky example (Integer Quantum Hall states : graphene)
  - ☆ One body to many body
  - ☆ Riemann surface, edge states to Chern

Are insulators boring ??

- Metal is useful. copper, silver, gold: good conductors
  - Lots of applications
- Metal is simple (if free)
  - unstable against for perturbation (without some protection or fine tuning)
  - "high energy" effective theory ?

with interaction: complicated

Anomalous metals, etc Critical : RG
Spin analogue (Gapless spin liquid) is tricky.

- Insulators : Gapped
  - Band insulators
  - Superconductors
  - Integer & Fractional Quantum Hall States
  - Integer spin chains (Haldane)
  - Dimer Models (Shastry-Sutherland)
  - Valence bond solid (VBS) states
  - Half filled Kondo Lattice
  - Spin Hall insulators
  - Kitaev model & string net

Energy gap above the ground state

Are insulators boring ??

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Absence of low energy excitations
Energy gap above the ground state

Lots of variety

Absence of fundamental symmetry breaking (mostly)

Quantum/spin liquids (gapped)

Are insulators boring ??

☆ Insulators : Non metal, gapped

☆ Band insulators

Gapped: Nothing in the gap : cf. Nambu-Goldstone boson

No low lying excitations

No Response against small perturbation

??

? ? ?

~~gapless modes:~~

~~acoustic phonons~~

~~zero sounds~~

~~spin waves~~

Absence of low energy excitations
Energy gap above the ground state

Lots of variety

Absence of fundamental symmetry breaking (mostly)

No responses against for small perturbation

Are insulators boring ??

☆ Quantum liquids (gapped)

- ☆ Band insulators
- ☆ Superconductors
- ☆ Integer & Fractional Quantum Hall States
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Topological Order

X.G.Wen '89

Zoo

Something for classification

☑ Topological order

☑ Berry connections

☑ Edge states

Classical to Quantum (for characterization)

- "Classical" Observables

Unitary invariant

  - Charge density, Spin density,...

$$\mathcal{O} = n_\uparrow \pm n_\downarrow, \cdots$$

$$\langle \mathcal{O} \rangle_G = \langle G | \mathcal{O} | G \rangle = \langle G' | \mathcal{O} | G' \rangle = \langle \mathcal{O} \rangle_{G'}$$

charge, spin, ...

$$|G'\rangle = |G\rangle e^{i\phi}$$

- "Quantum" Observables !

depend on the phase of the state

  - Quantum Interferences:

$$\langle G_1 | G_2 \rangle = \langle G_1' | G_2' \rangle e^{i(\phi_1 - \phi_2)}$$

    - Aharonov-Bohm Effects

$$|G_i\rangle = |G_i'\rangle e^{i\phi_i}$$

    - Berry phases

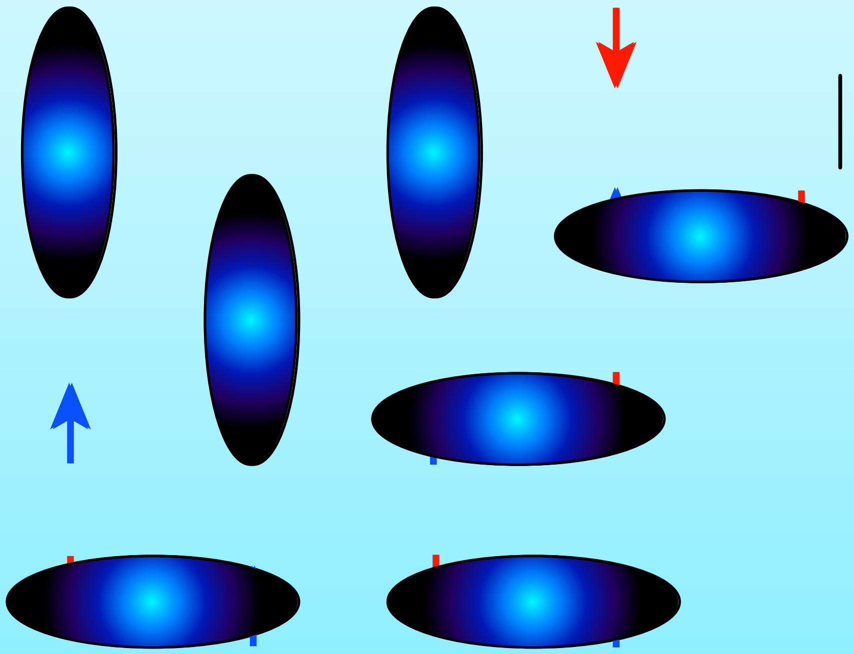
$$\langle G | G + dG \rangle = 1 + \langle G | dG \rangle$$

$A = \langle G | dG \rangle$:Berry Connection

$i\gamma = \int A$:Berry Phase

Use Quantum observables for the characterization

Local quantum object to characterize gapped spin liquid



$$|\mathrm{Singlet\ Pair}_{12}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle)$$

$$|G\rangle = \sum_{J=\mathrm{Dimer\ Covering}} c_J \otimes_{ij} |\mathrm{Singlet\ Pair}_{ij}\rangle$$

Anderson

Local Singlet Pairs :

(Basic Objects)

Singlet : quantum order parameter

DO NOT NEED ANY symmetry breaking

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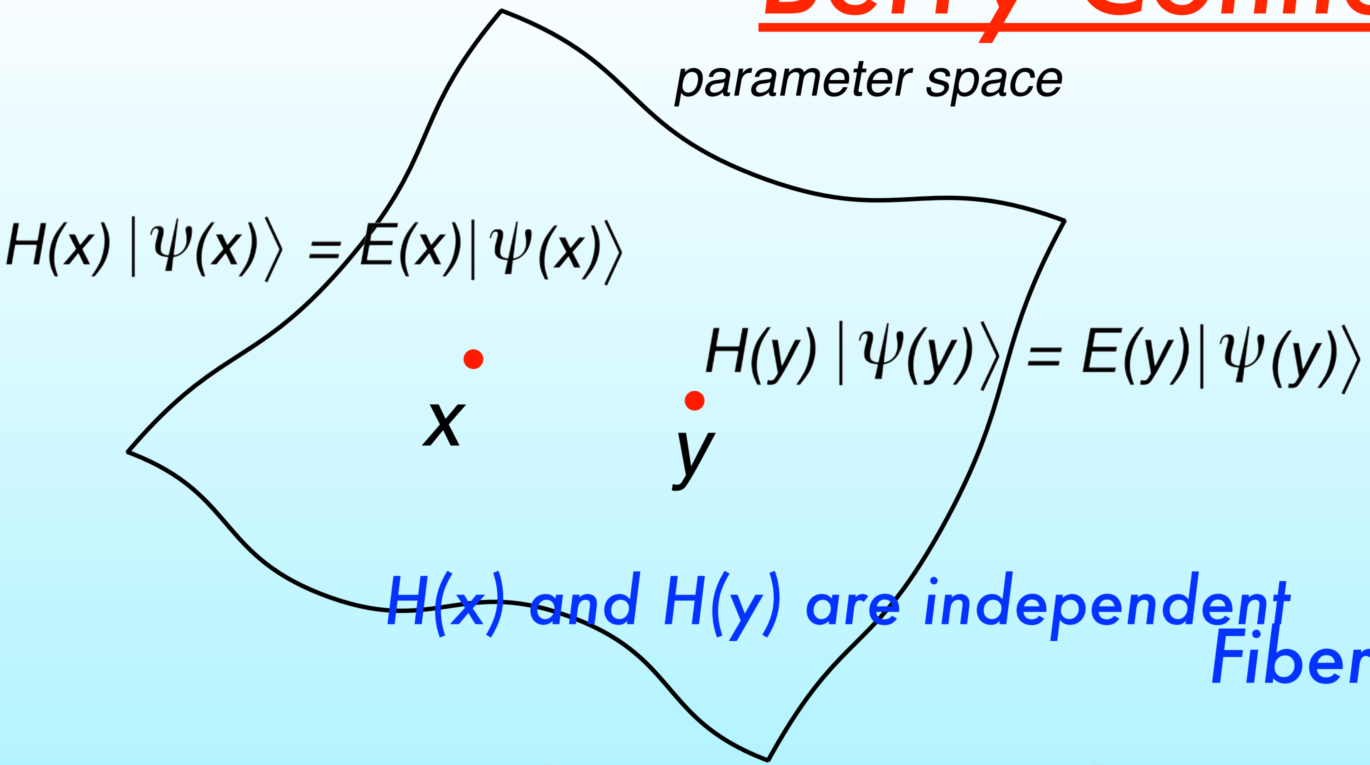
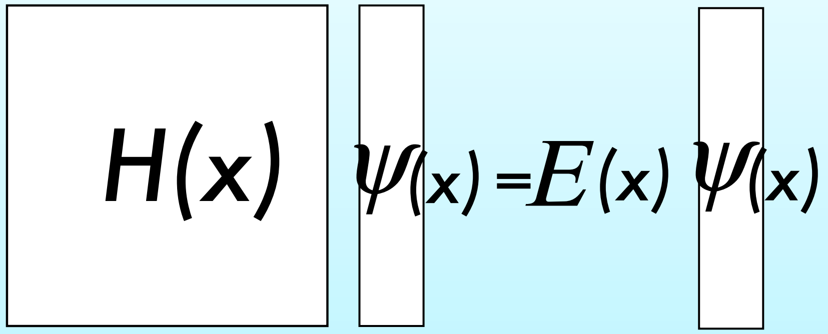
Berry Connection



H(x) and H(y) are independent

Fiber Bundle

Eigenvectors (space) with Parameters



(Abelian)

Information between nearby states

Berry connection : $A_\psi = \langle \psi | d\psi \rangle = \langle \psi | \frac{d}{dx} \psi \rangle dx.$

Gauge Transformation

$$|\psi(x)\rangle = |\psi'(x)\rangle e^{i\Omega(x)}$$

$$A_\psi = A'_\psi + i d\Omega = A'_\psi + i \frac{d\Omega}{dx} dx$$

gauge potential

Geometrical quantities

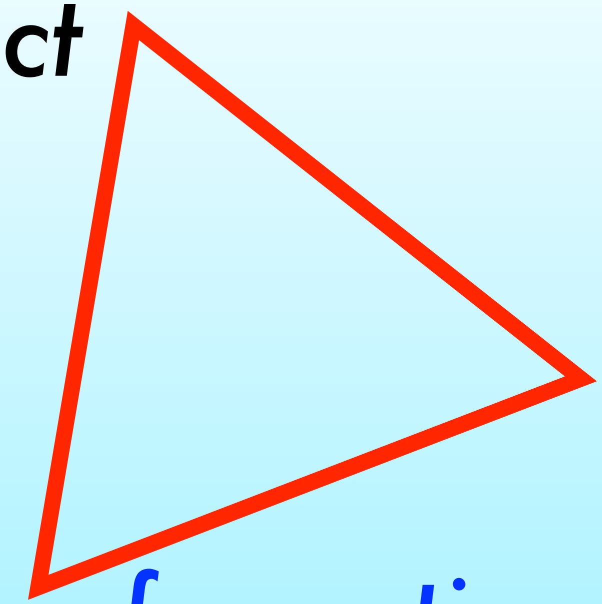
$$i\gamma_C(A_\psi) = \int_C A_\psi$$ **: Berry phase**

$$\gamma_C(A_\psi) = \gamma_C(A_{\psi'}) + \int_C d\Omega \leftarrow 2\pi \times (\text{integer}) \text{ if } e^{i\Omega} \text{ is single valued}$$

$$\gamma_C(A_\psi) \equiv \gamma_C(A_{\psi'}) \mod 2\pi$$

only well-defined in mod 2π

How to characterize the local quantum object

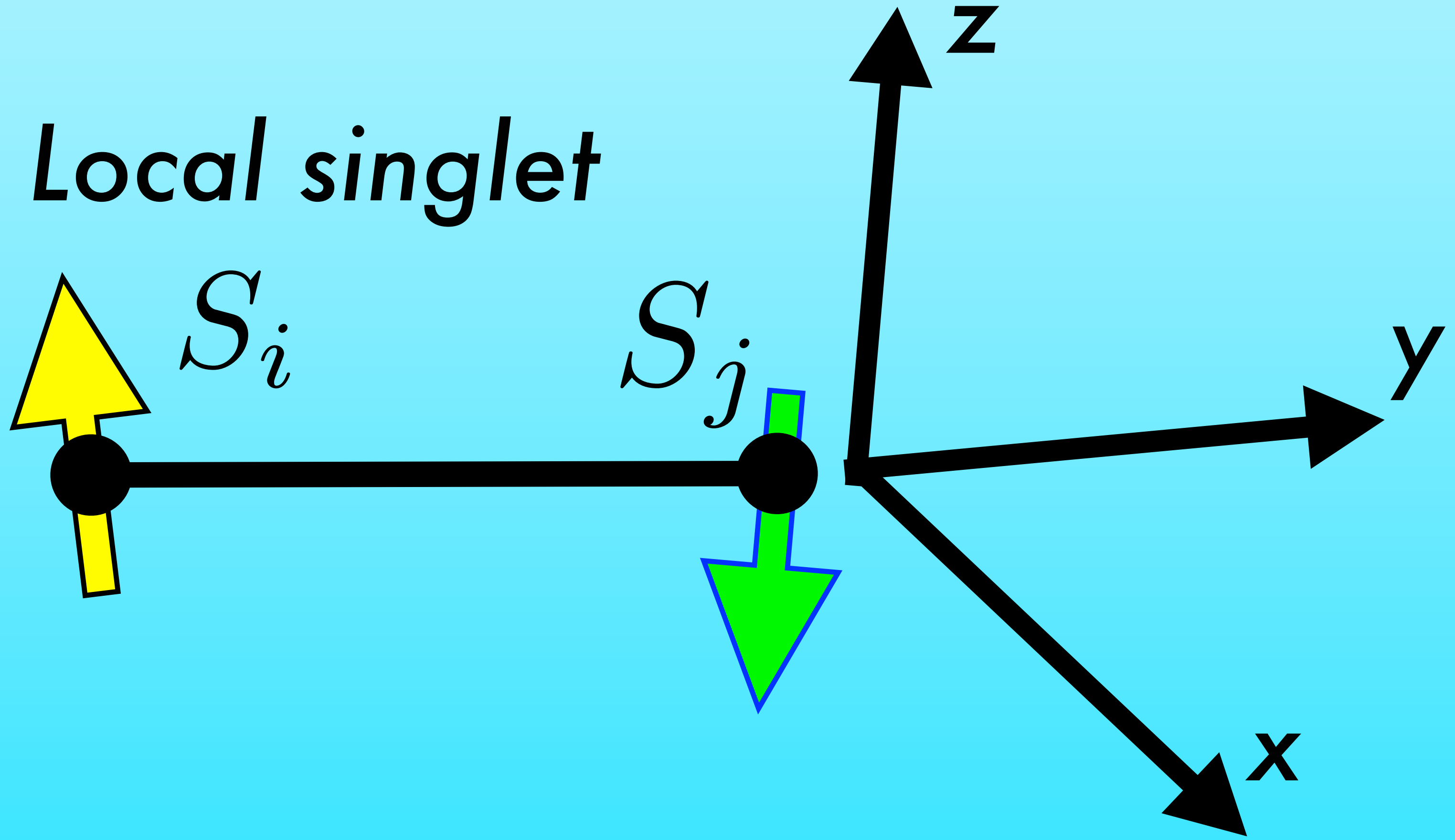


Local quantum object

Berry phases

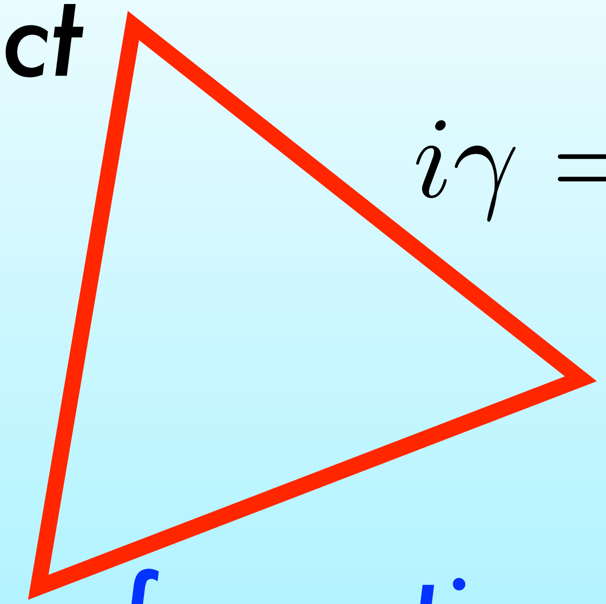
Local gauge transformation

Local singlet



How to characterize the local quantum object

Local quantum object

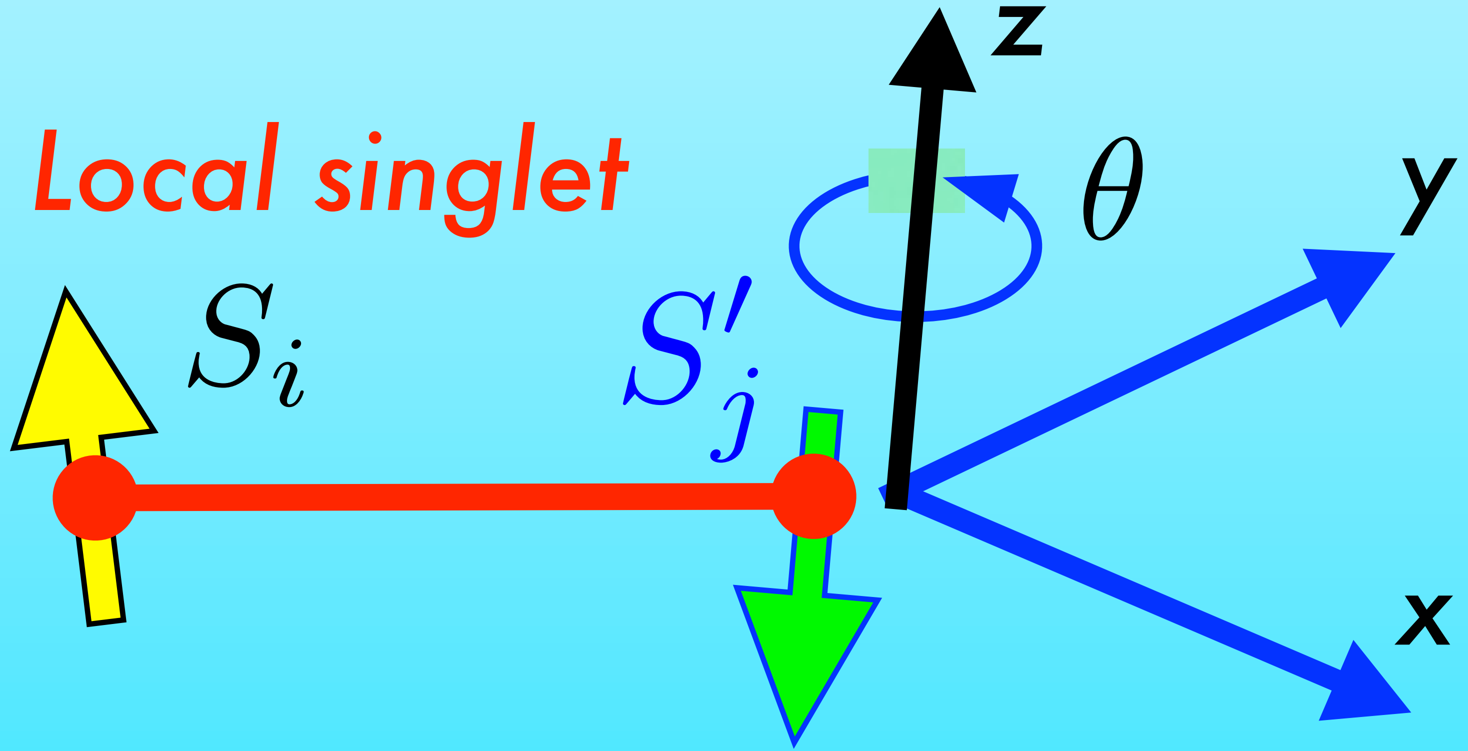


$$i\gamma = \int dA = \int \langle\psi|d\psi\rangle$$

Berry phases

Local gauge transformation

Local singlet



$$|\psi(\theta)\rangle = U(\theta)|\psi(0)\rangle$$

$$U(\theta) = e^{i(S-S_z)\theta}$$

$$A = \langle\psi|d\psi\rangle = Sd\theta$$

$$\gamma = 2\pi S = \pi$$

$$S = 1/2$$

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j \to \boldsymbol{S}_i \cdot \boldsymbol{S}'_j = \frac{1}{2}(e^{-i\theta} S_{i+}S_{j-} + e^{+i\theta} S_{i-}S_{j+}) + S_{iz}S_{jz}$$

Local object in a many spin system

Collection of
weakly coupled quantum local objects
Shastry-Sutherland '81

Topological quantities for
quantum order parameters of
Shastry-Sutherland and the zoo

This is NOT a gauge transformation anymore

for the whole systems

NEED numerical calculation for γ

Topological quantities

collect M states gapped from the else

$$\Psi = (|\psi_1\rangle, \cdots, |\psi_M\rangle) \quad \langle\psi_j|\psi_k\rangle = \delta_{jk} \quad \Psi^\dagger\Psi = E_M$$

Berry connection & gauge transformation

$$A_g = \Psi_g^\dagger d\Psi_g = g^{-1}Ag + g^{-1}dg$$

$$\Psi_g = \Psi g \qquad g \in U(M)$$

Chern numbers

quantized

$$C_1 = -\frac{1}{2\pi i}\int_{S^2} \mathrm{Tr} F, \quad C_2 = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr} F^2, \cdots$$

$$F_g = dA_g + A_g^2 = g^{-1}Fg$$

Berry phases & generalizations

$$\gamma_1 = -\frac{1}{2\pi i}\int_{S^1} \omega_1, \quad \gamma_3 = -\frac{1}{8\pi^2}\int_{S^3} \omega_3, \cdots$$

$$\omega_1 = \mathrm{Tr}\, A, \omega \quad {}_3 = \mathrm{Tr}\,(AdA + \frac{2}{3}A^3), \cdots$$

Symmetry protected quantization

Anti-Unitary Invariant State and

Z_2 Berry Phase

$$\Theta_N^2 = 1$$

- Anti-Unitary Symmetry $[H(x), \Theta] = 0$
- Invariant State $^\exists \varphi, \quad |\Psi^\Theta\rangle = \Theta|\Psi\rangle = |\Psi\rangle e^{i\varphi}$
  - ex. Unique Eigen State $\simeq |\Psi\rangle$ Gauge Equivalent (Different Gauge)
- To be compatible with the ambiguity,

  the Berry Phases have to be quantized as

$\mathbb{Z}_2$ Berry phase

http://arxiv.org/abs/0909.4831

$$\gamma_C(A^\Psi) = \begin{cases} 0 \\ \pi \end{cases} \mod 2\pi$$

$i^2 = j^2 = k^2 = ijk = -1$

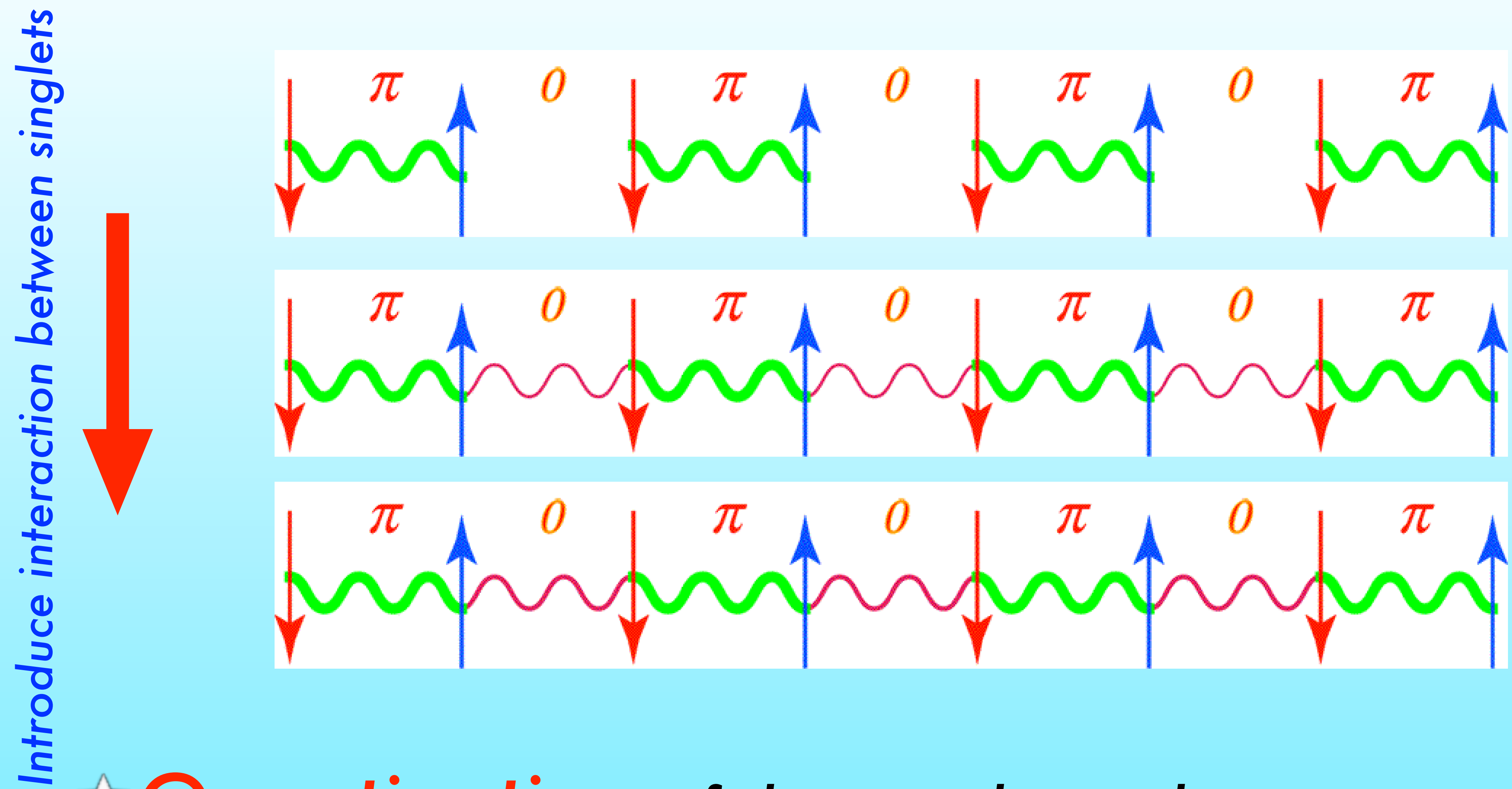
Also quaternionic generalization with Kramers degeneracy $\Theta_N^2 = -1$

$$\gamma_3 = -\frac{1}{8\pi^2}\int_{S^3} \omega_3$$

$$\gamma_C(A^\Psi) = -\gamma_C(A^{\Theta\Psi}) \equiv -\gamma_C(A^\Psi), \ \mathrm{mod} 2\pi$$

YH '09

Adiabatic Continuation & Quantization



☆**Quantization** of the topological quantities **protects**

from **continuous change**

Adiabatic Continuation in a **gapped** system

Topological field theory

↕

Renormalization Group in a **gapless** system

Local field theory

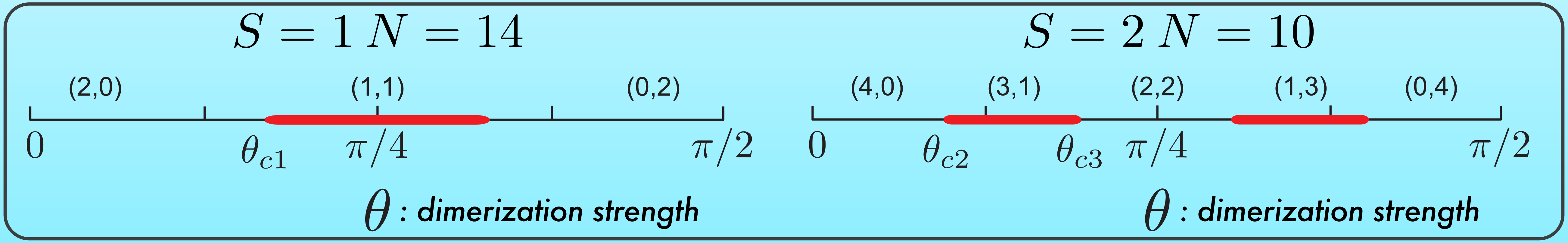
Topological Classification of Gapped Spin Chains

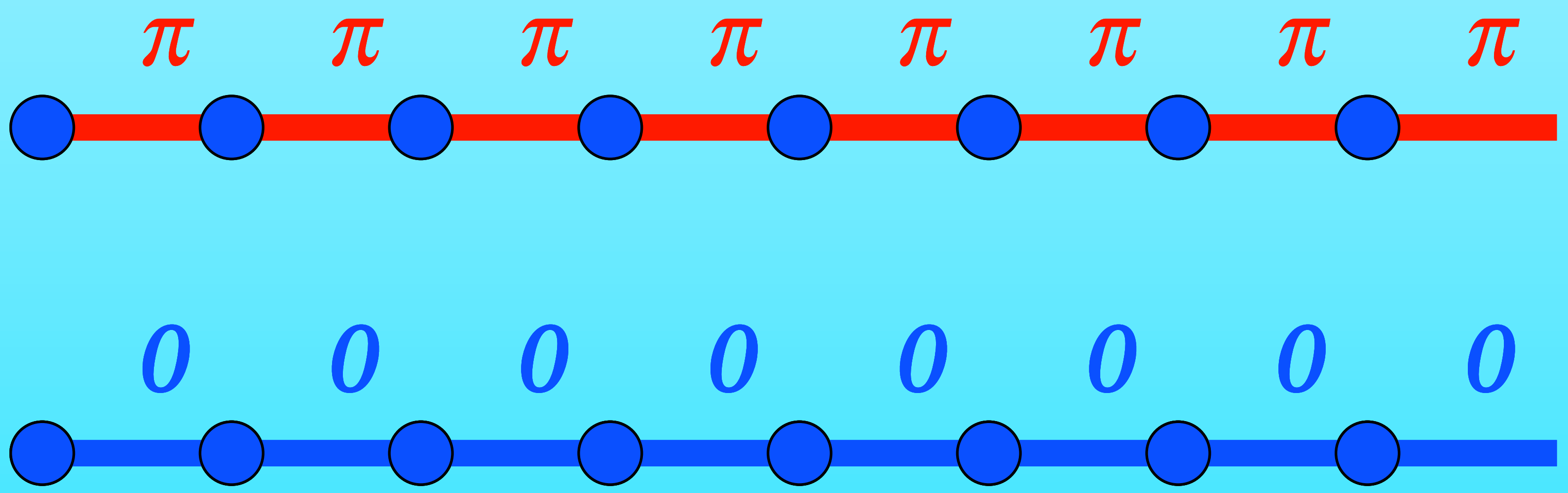
T.Hirano, H.Katsura &YH, Phys.Rev.B77 094431'08

- S=1,2 dimerized Heisenberg model

$$H = \sum_{i=1}^{N/2} (J_1 \boldsymbol{S}_{2i} \cdot \boldsymbol{S}_{2i+1} + J_2 \boldsymbol{S}_{2i+1} \cdot \boldsymbol{S}_{2i+2}) \qquad J_1 = \cos\theta, J_2 = \sin\theta$$

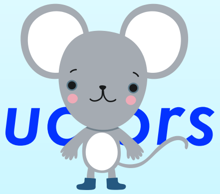
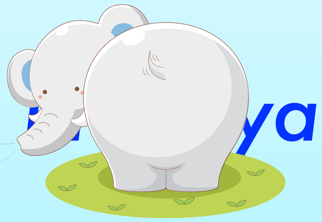
Z_2Berry phase

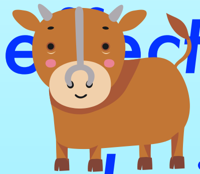
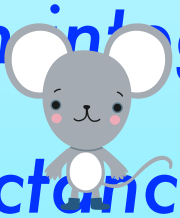


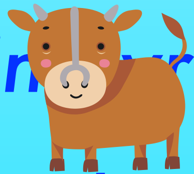


Topological Quantum Phase Transitions with translation invariance

Zoo of Boundary (Edge) States in Cond. Mat.

- ☆ Bound states & phase shift Levinson's theorem, Friedel's sum rule
- ☆ Surface states of Semiconductors (polarized)
- ☆ Solitons in polyacetylene Su-Schriefer-Heeger '79
- ☆ Edge states in quantum Hall effects Halperin '82 YH '93
- ☆ Local moments in integer spin chains near the impurities Kennedy '90
- ☆ Zero bias conductance peaks of the d-wave superconductors Hu, '94
- ☆ Zero energy localized states of graphene Fujita et al.'96 Ryu-YH'02 Arikawa-Aoki-YH'02
- ☆ Quantum Spin Hall Edge states Mele'05 Bernevig-Hughes-Zhang '06
- ☆ Edge states in 2D cold atoms in optical lattice Scarola-Das Sarma., PRL 98, 210403 '07
- ☆ One-way edge modes in gyromagnetic photonic crystals Wang et al., '08, '09
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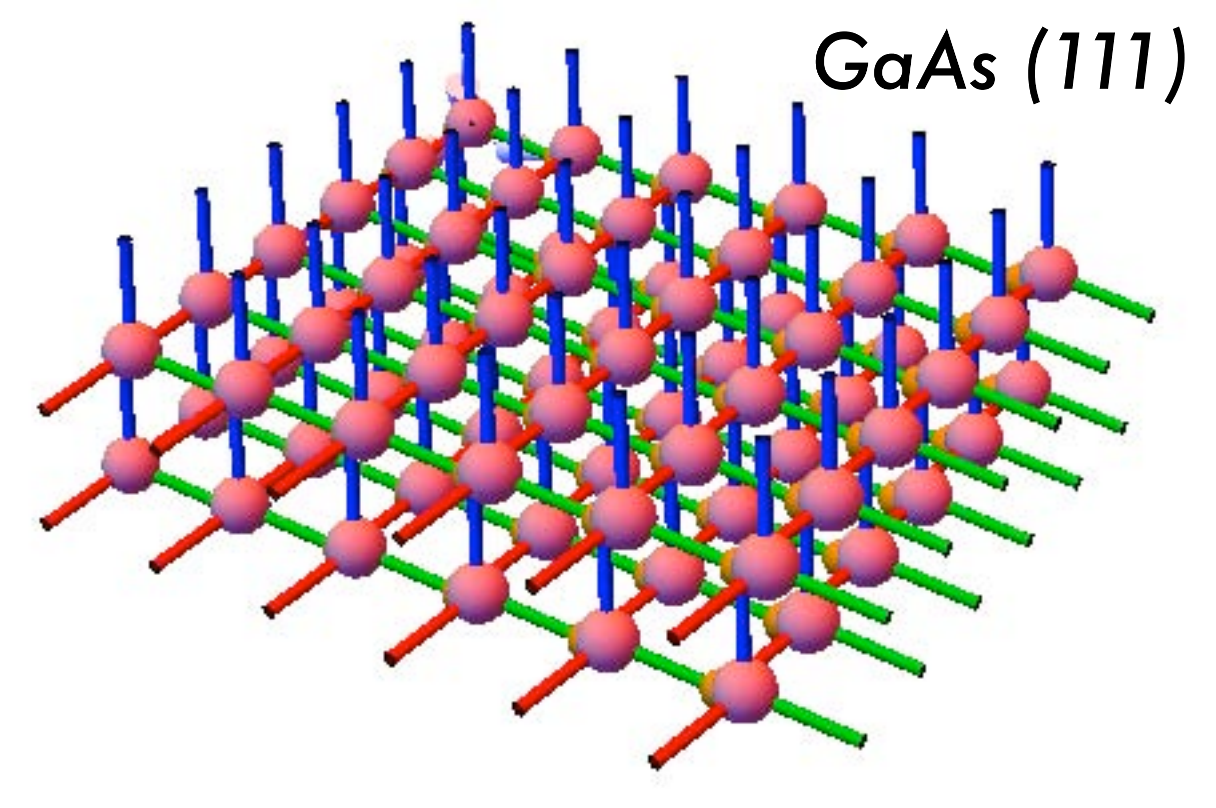

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- Edge 210403 '07
- One-

Wang et al., '08, '09

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Non topological generically implicated by the Bulk (polarization)

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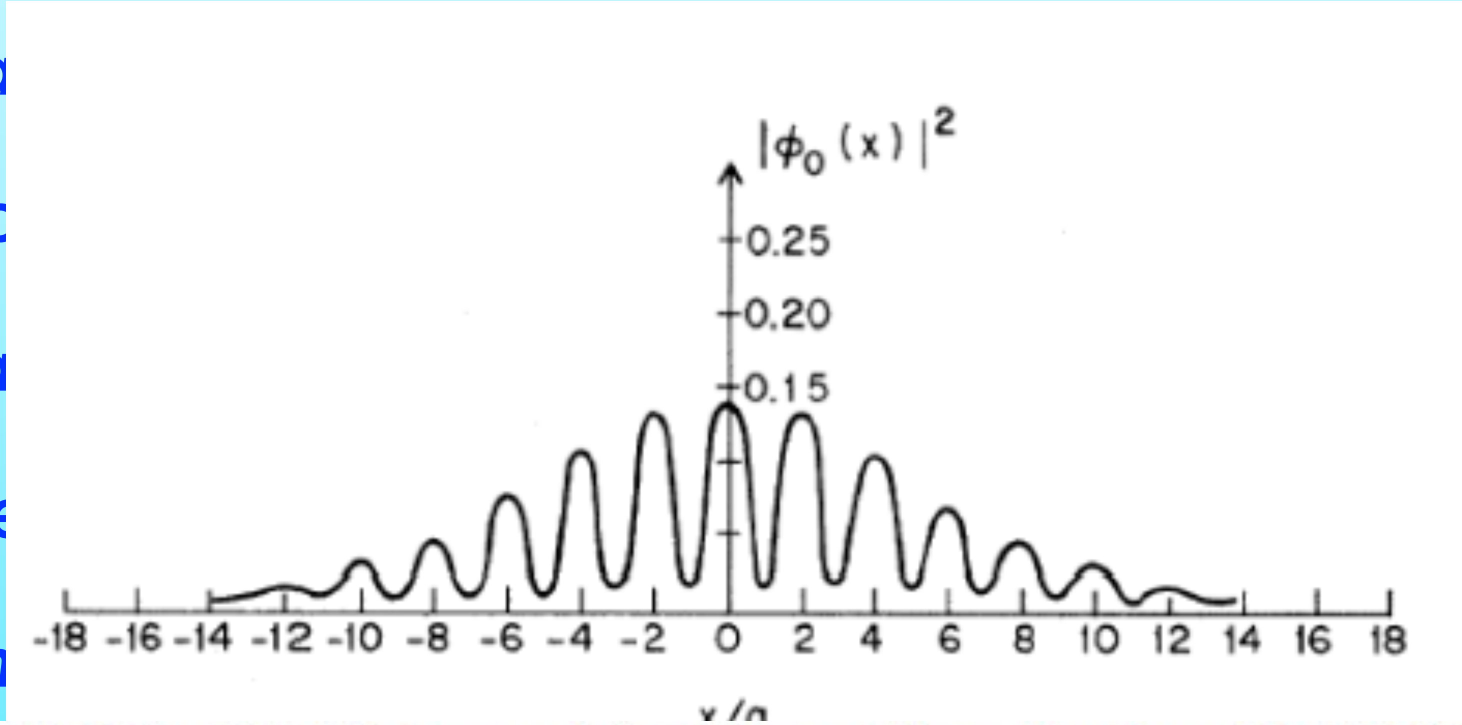

W. P. Su, J. R. Schrie.er and A. J. Heeger, *Phys. Rev. Lett*. **42**, 1698 (1979)

- Edge states in 2D cold atoms in optical lattice   Scarola-Das Sarma., PRL 98, 210403 '07
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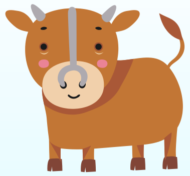
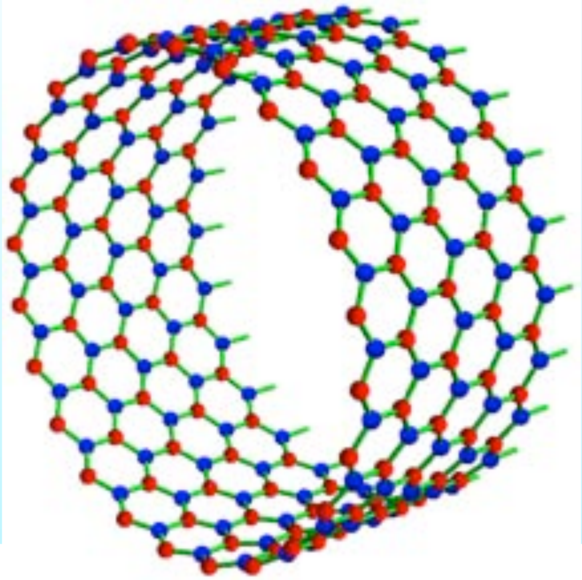


Zoo of Boundary (Edge) States in Cond. Mat.



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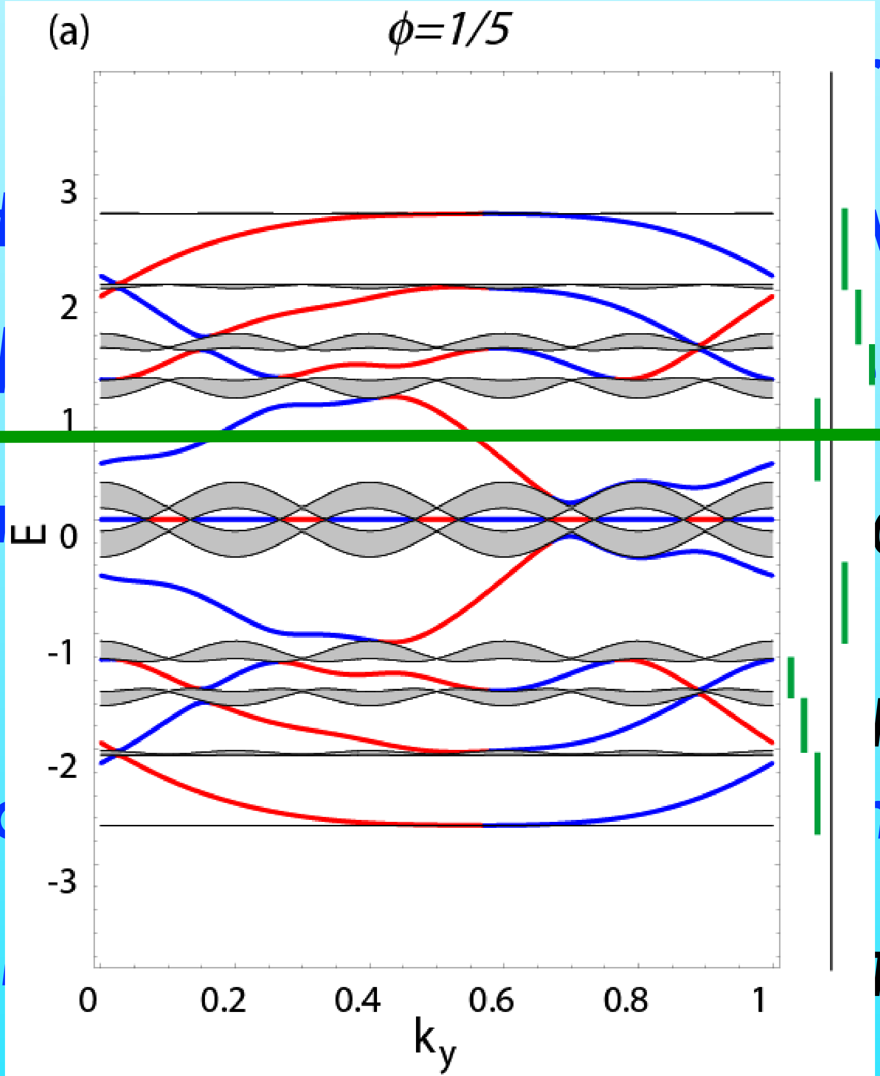
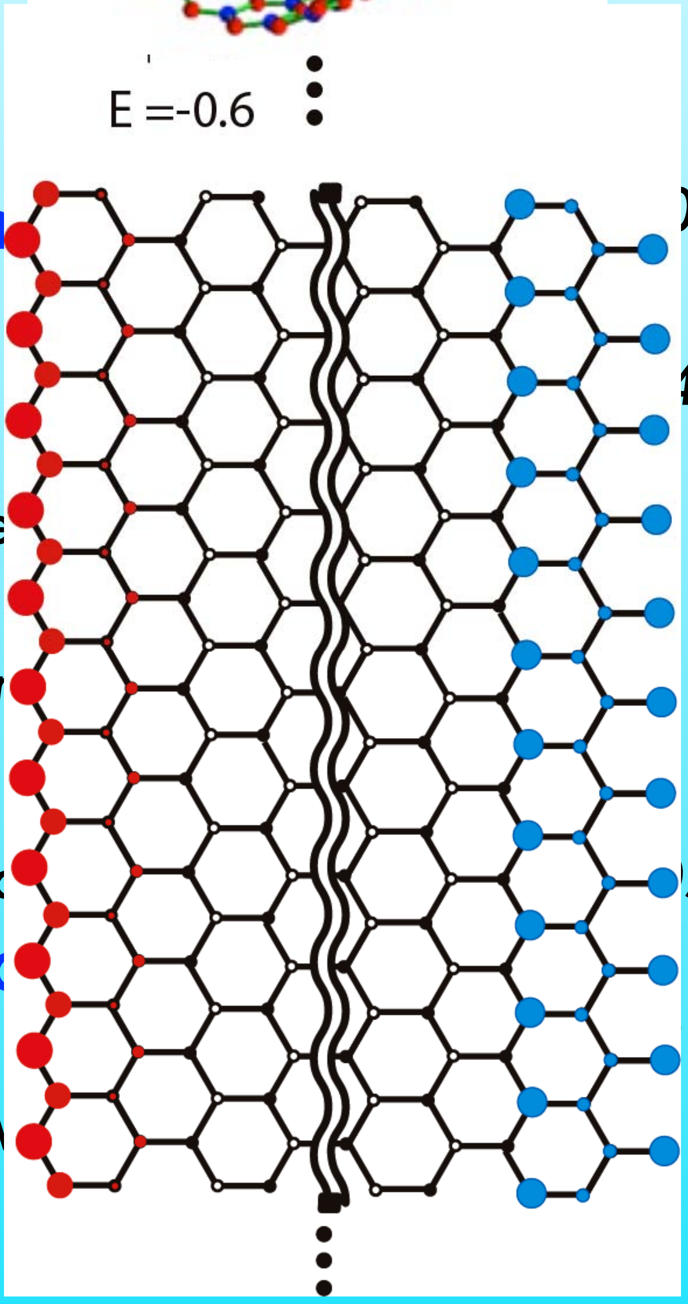




Cond. Mat.

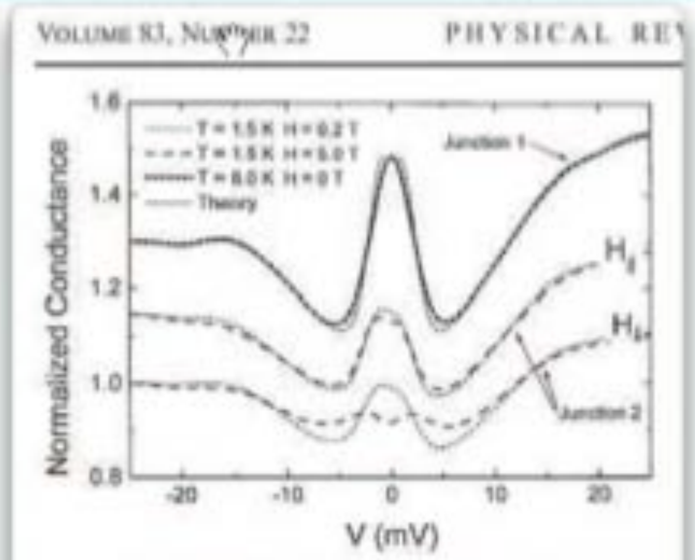


FIG. 1. The temperature dependence of the in-plane tunneling conductance of (110)-YBCO/Pb junctions as function of bias and magnetic field is shown. The field H is always applied parallel to the junction interface, and either parallel or perpendicular to the YBCO ab planes, as labeled. The theoretical curve (solid line) is calculated using the FRS theory [11], as described in the text. For junction 2, low-temperature spectra for low and high applied magnetic field are shown. Note the field-induced splitting in the ZBCP is strongly anisotropic with respect to the field orientation. Data obtained on junction 1 show reproducibility between junctions for data taken at low temperature and field ($T = 1.5$ K, $H = 0.2$ T). Zero-field data taken at a temperature above the T_c of Pb is also shown for junction 1.

Zero Energy Boundary States of Anisotropic Superconductivity

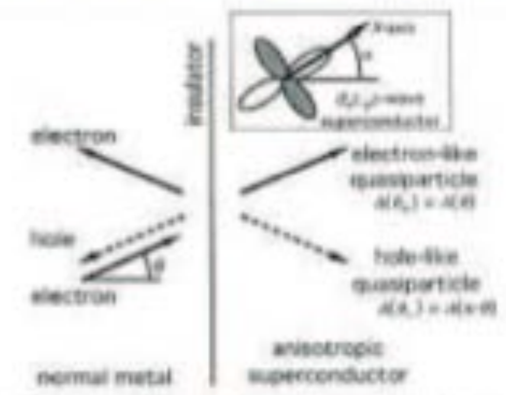
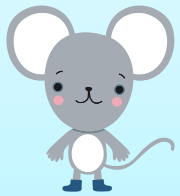


FIG. 1. Schematic illustration of the reflection and transmission processes at the interface. In this figure the quantities θ and α express the injection angle of the electron and the angle between the normal vector of the interface and the x axis of the $d_{x^2-y^2}$-wave superconductor, respectively.

Tanaka-Kashiwaya

M. Aprili, E. Badica, and L. H. Greene, Phys. Rev. Lett. 83, 4630 (1999)

Friedel's sum rule



YH '93

Kennedy '90

- ☆ Zero bias conductance peaks of the d-wave superconductors Hu, '94
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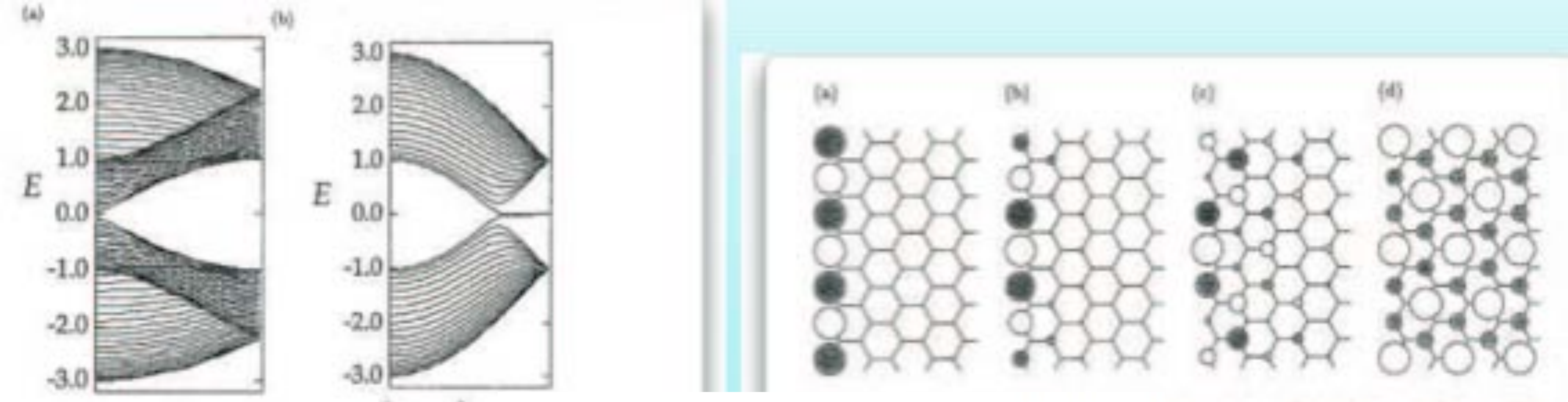
Zoo of Boundary (Edge) States in Cond. Mat.



M. Fujita *et al.*, *J. Phys. Soc. Jpn.* **65**, 1920 (1999)

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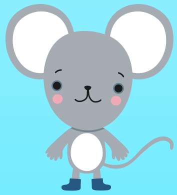
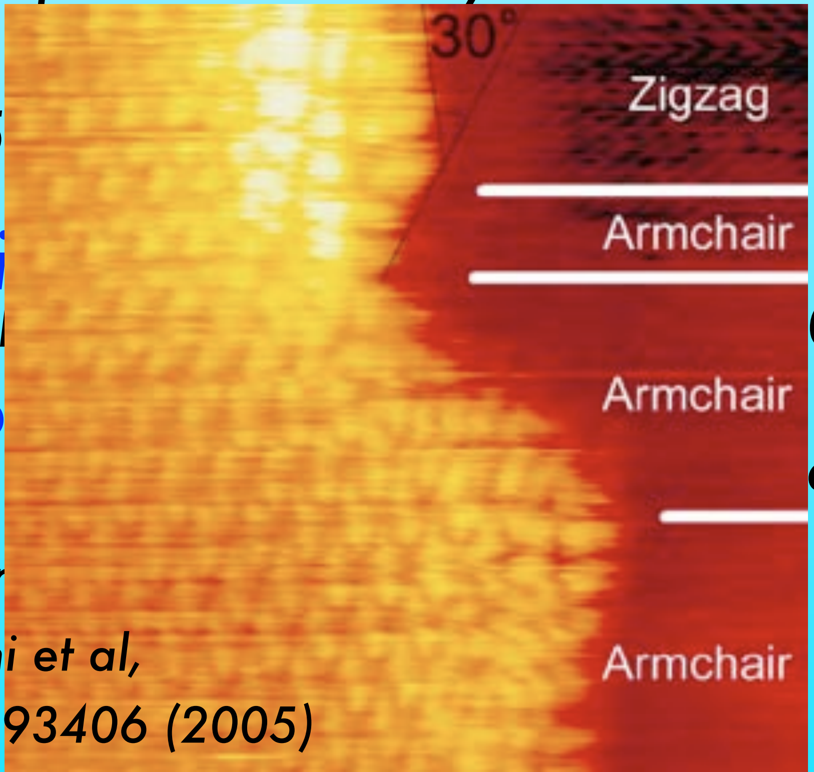


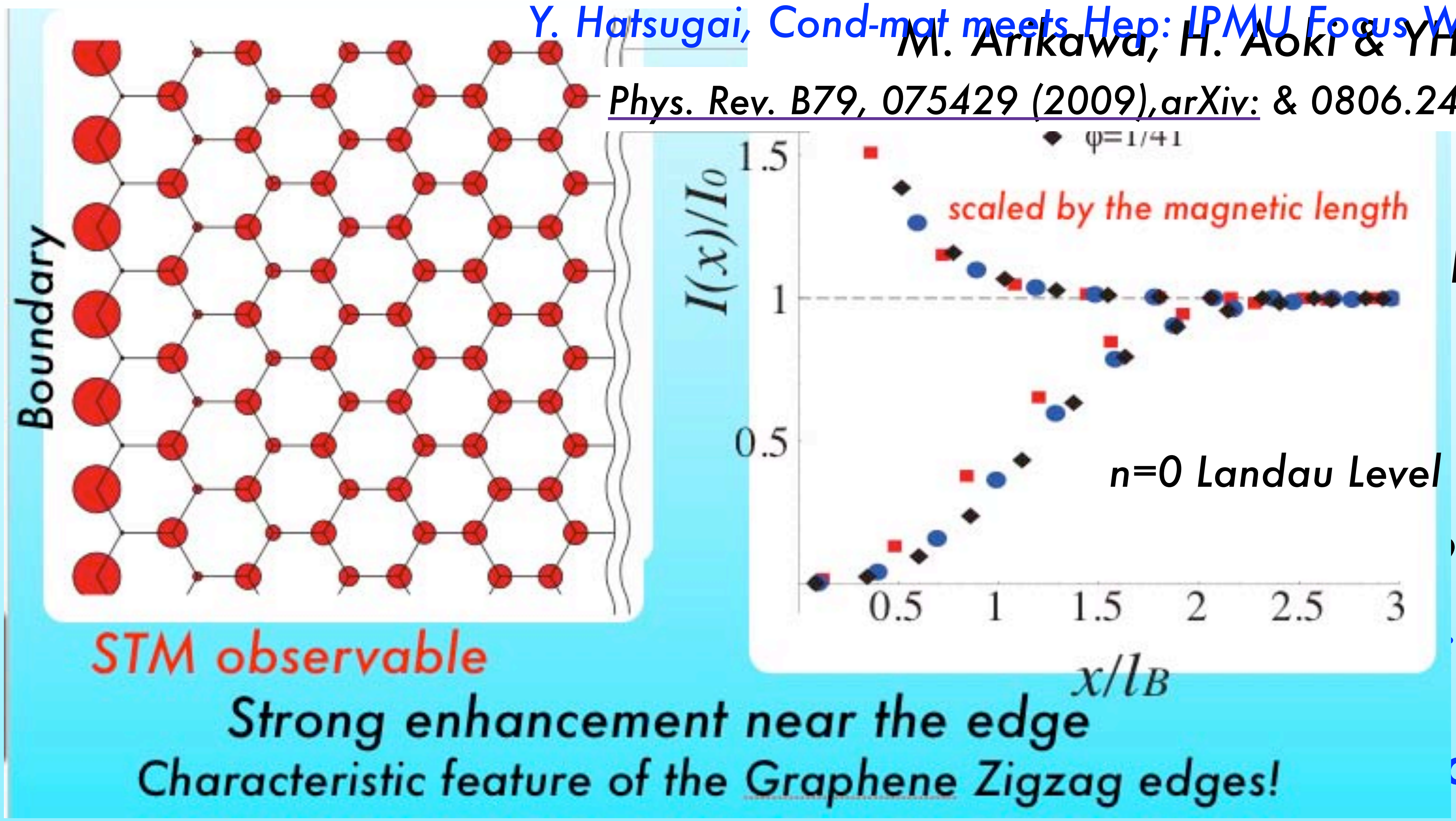


Kobayashi et al,
Phys. Rev. B71, 193406 (2005)

l. Mat.

M. Arikawa, H. Aoki & YH

Phys. Rev. B79, 075429 (2009),arXiv: & 0806.2429

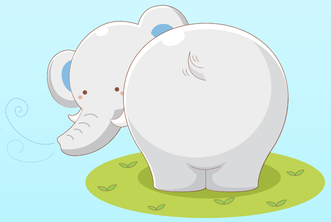


STM observable

Strong enhancement near the edge

Characteristic feature of the Graphene Zigzag edges!

l's sum rule



'3

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- ☆ Zero energy localized states of graphene Fujita et al.'96 Ryu-YH'02
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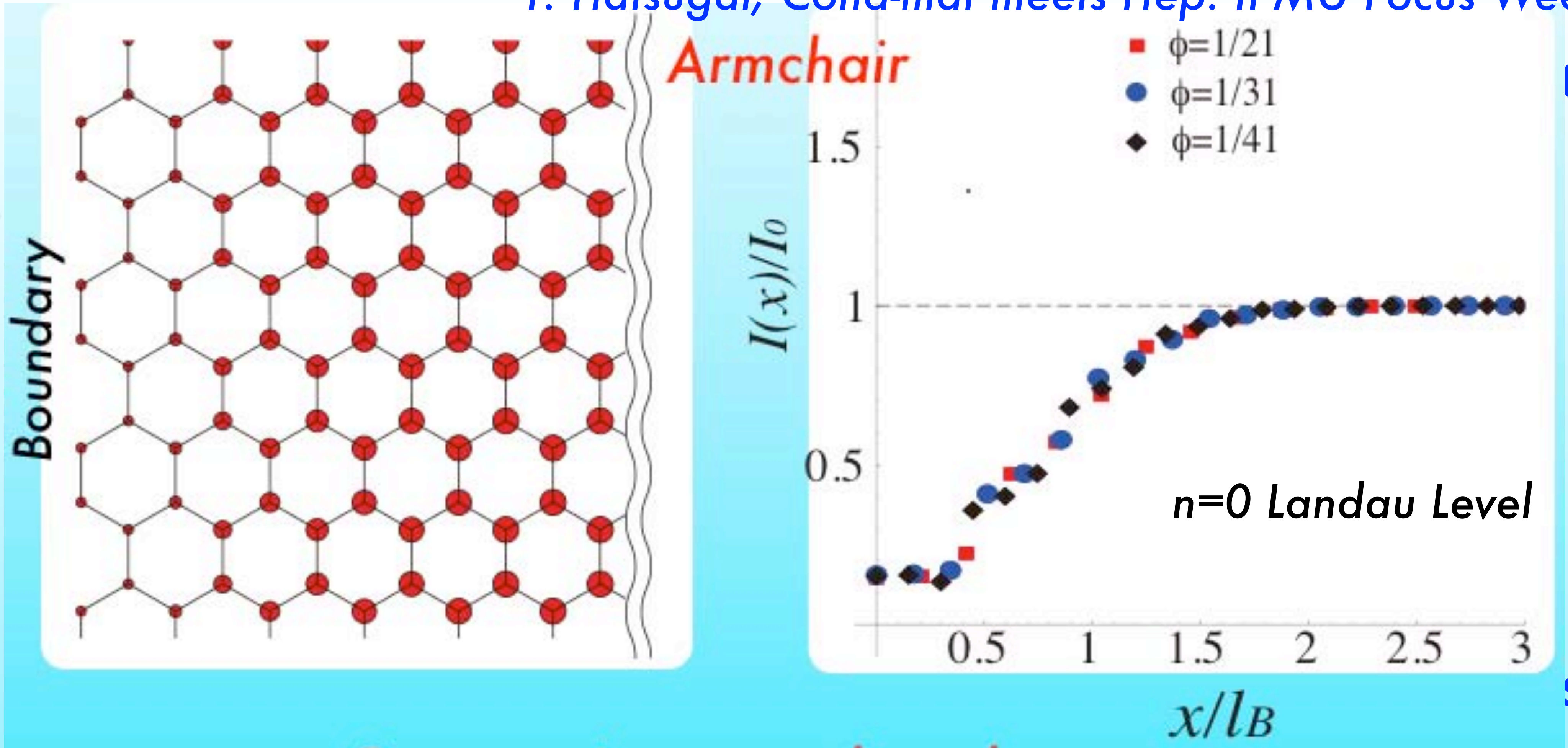
- ☆ Edge states in 2D cold atoms in optical lattice Scarola-Das Sarma., PRL 98, 210403 '07
- ☆ One-way edge modes in gyromagnetic photonic crystals Wang et al., '08, '09
- ☆ Spin Ladder with ring exchanges Arikawa-Tanaya-Maruyama, YH '09

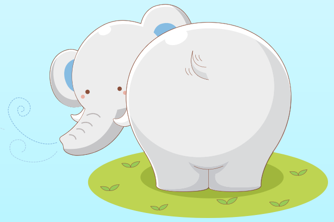
d. Mat.



Suppression near the edge

Standard behavior due to edge potential

sum rule



s Kennedy '90

uctors Hu, '94

- ☆ Zero energy localized states of graphene Fujita et al.'96 Ryu-YH'02
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Universality of Zero Energy Edge States

'02–'04 S. Ryu & YH

Boundary magnetic moments

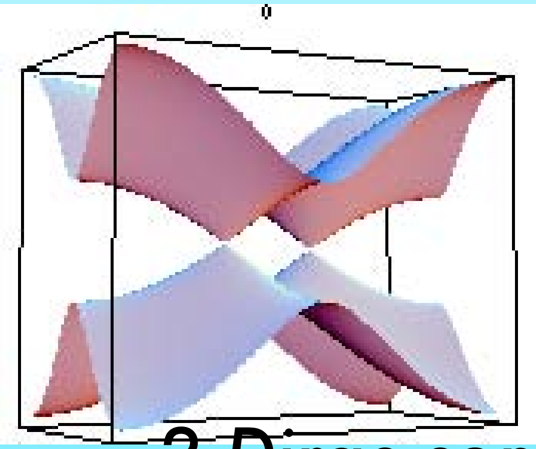
Spontaneous breaking of these chiral symmetries : Peierls instabilities of Flat (edge) bands

Spontaneous local flux generation near defects

Zero bia

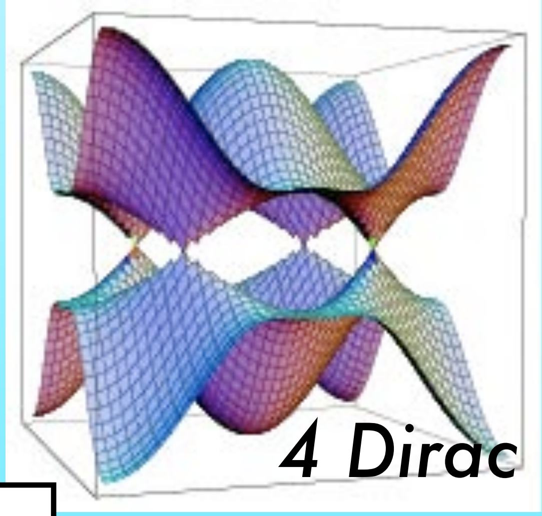
graphene

d-wave superconductor



2 Dirac cones

These 2 systems are topologically equivalent



4 Dirac cones

Γ :Bipartite

(A-B sublattice symmetry)

Symmetry protected Zero modes of Dirac fermions :1D Flat Band of edge states

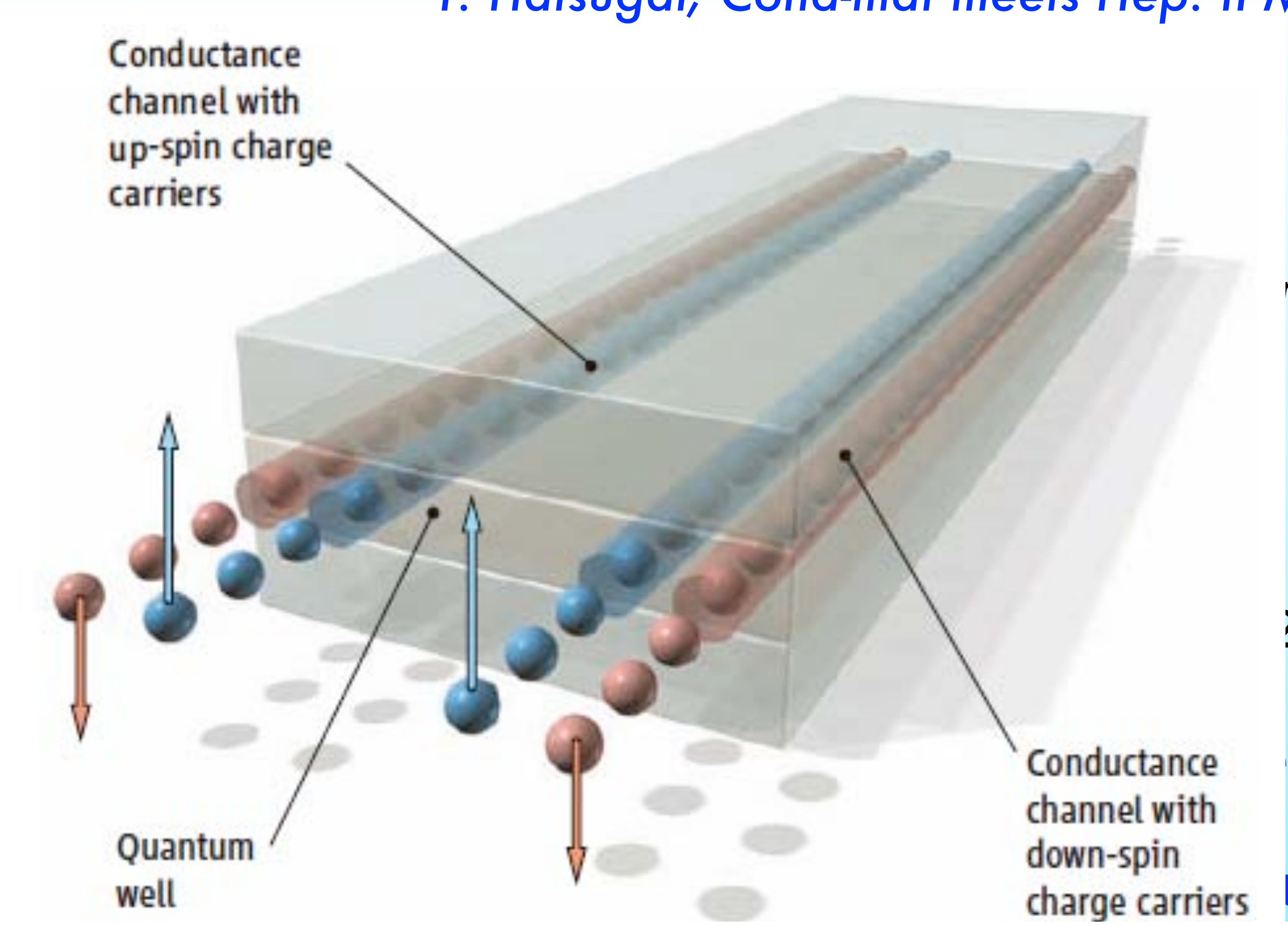
Γ:Time Reversal

(Real Order parameter)

$^{\exists}\Gamma$ chiral symmetry

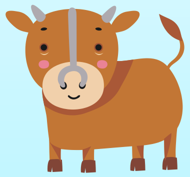
$$\{\Gamma, H\} = \Gamma H + H\Gamma = 0\ ,\ \Gamma^2 = 1$$

Z in Cond. Mat.



Konig, Wiedmann, Brüne, Roth, Hartmut Buhmann, Molenkamp,Qi and Zhang, Science 318, 776 (2007)

- Bou rem, Friedel's sum rule
- Sur
- Soli



- Edg 2 YH '93
- Loc impurities Kennedy '90
- Zer uperconductors Hu, '94
- Zer 6 Ryu-YH'02
- Quantum Spin Hall Edge states Kane-Mele'05 Bernevig-Hughes-Zhang '06
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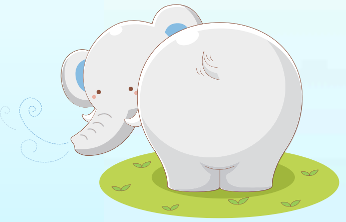
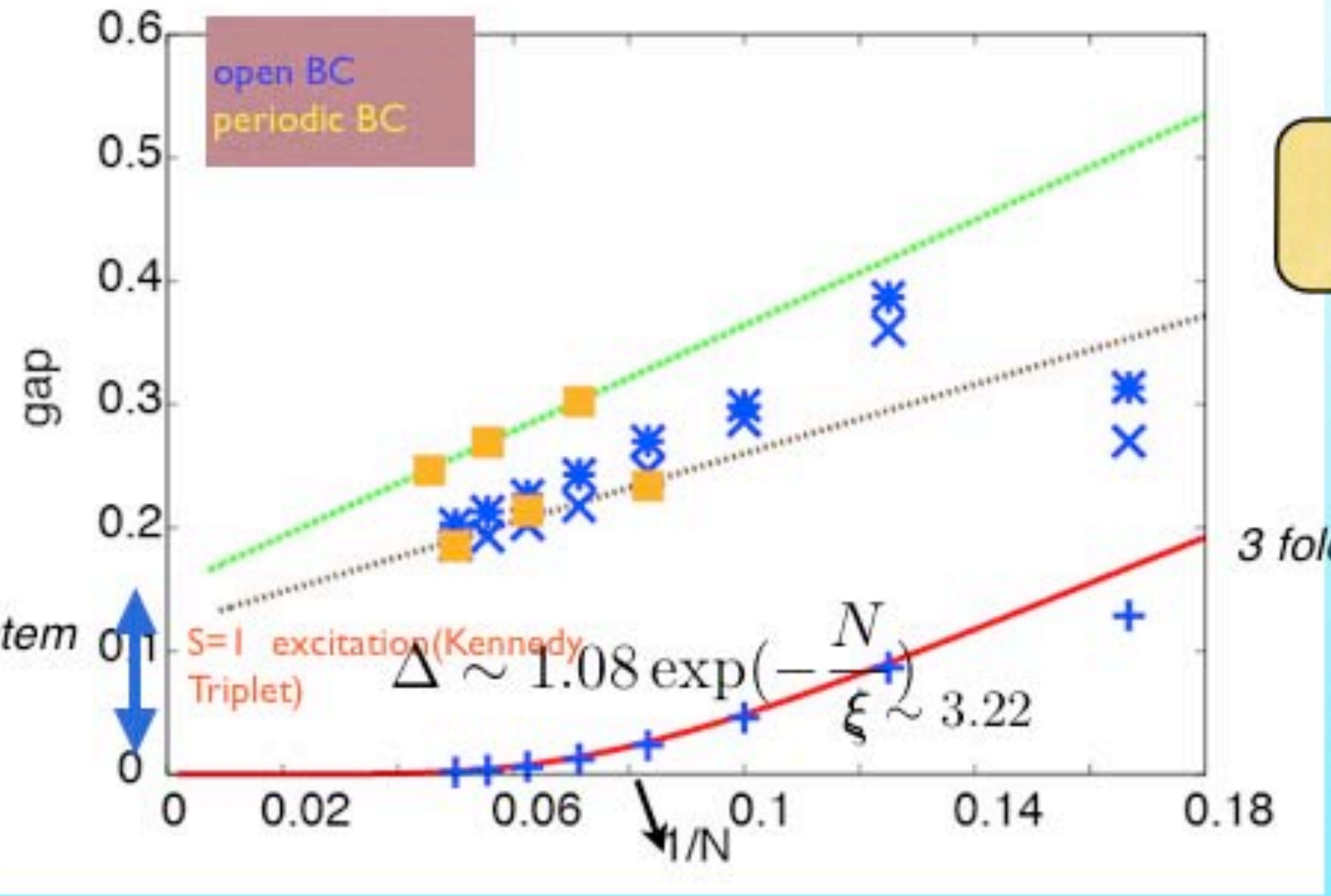
Energy spectrum with boundaries (diagonal)

Mat.



M. Arikawa, S. Tanaya, I. Maruyama, YH, Phys.Rev.B79, 205107 (2009)

gap
for periodic system



3 fold degenerate

ule

Edge states

nedy '90

Hu, '94

'02

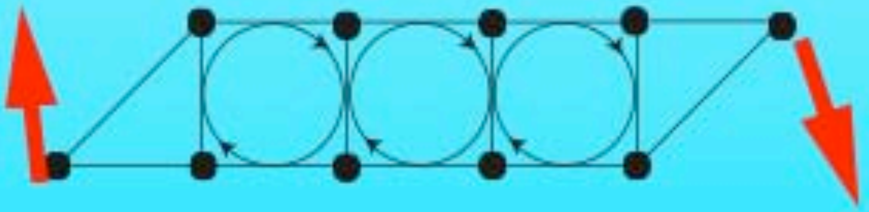
Kennedy '90
M. Hagiwara, K. Katsumata, I. Affleck, and B. Halperin, '90

Ex. for Haldane spin chain

ng '06

$$\mathcal{H} = \sum_i \{J_r \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i} + J_l(\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{1,i+1} + \boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{2,i+1}) + K(P_i + P_i^{-1})\}$$

Interaction between effective boundary spins



10403 '07

$$H_{eff} = \Delta \boldsymbol{S}_R \cdot \boldsymbol{S}_L$$

'08, '09

- ☆ Spin Ladder with ring exchanges

Arikawa-Tanaya-Maruyama, YH '09

Why the Edge States are there??

Accidental ?

NO !

Inevitable reasons

QH Effective field theory
X.G.Wen, '90

Universal Structures behind:

On Lattice: Hofstadter
YH '93

Bulk determines the edges : Bulk-Edge Correspondence

Extended states : unnormalized

localized states (edge states) : normalized

clear distinction in a macroscopic system

Energy gap: extended states can not be there

As for the topologically non trivial bulk,

localized states in the gap can not be destroyed

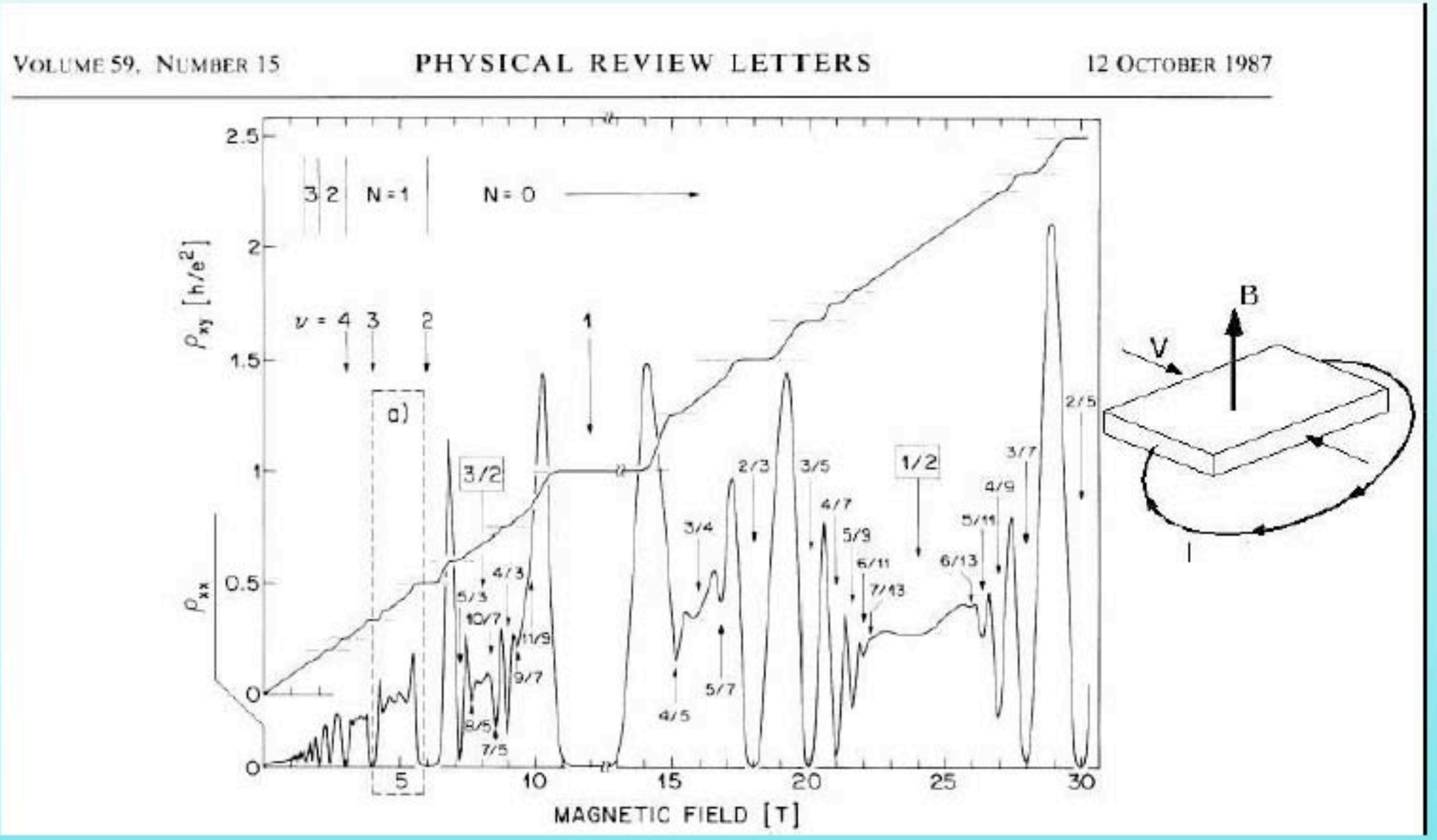
without collapsing the bulk gap

topological Stability of Edge states

A lucky example:
Integer quantum Hall effect (also Graphene)

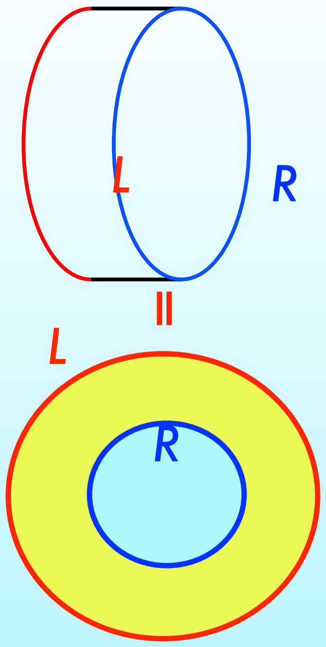
Bulk-Edge correspondence (explicitly shown)

Quantization of the Hall conductance σ_{xy} with anomalous accuracy: $I = \sigma_{xy} V$


VOLUME 59, NUMBER 15 PHYSICAL REVIEW LETTERS 12 OCTOBER 1987

R. Willet et al. PRL 59, 1776 (1987)

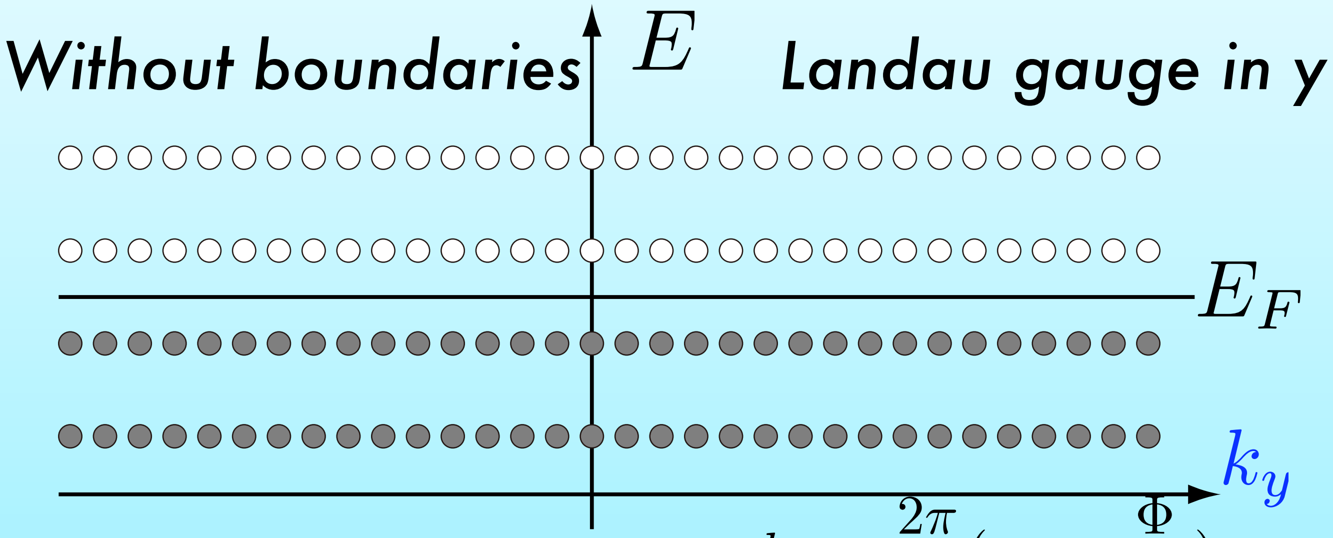
Quantum Hall Effects by edge states



☆ Edge states and Hall conductance σ_{xy} Halperin '82

Without boundaries

Landau gauge in y



$$H_{2D} = \sum_{k_y} H_{1D}(k_y)$$

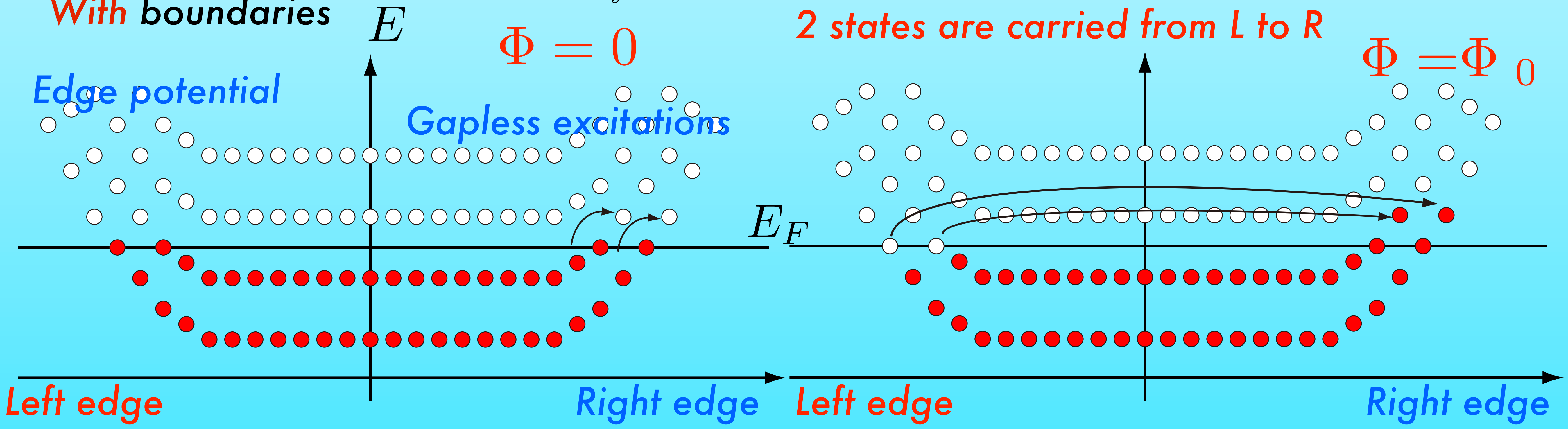
$H_{1D}(k_y)$

: harmonic osc. centered at

$$\langle x\rangle \sim \ell_B^2 k_y$$

$$k_y = \frac{2\pi}{L_y}(n_y + \frac{\Phi}{\Phi_0}) \quad n_y = 0, \pm 1, \pm 2, \cdots$$

With boundaries

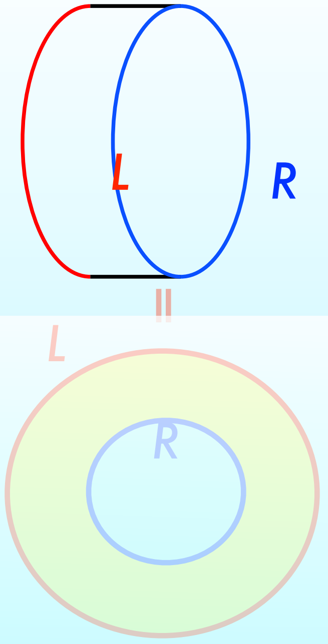


Laughlin's undetermined n : # of Landau Levels below E_F

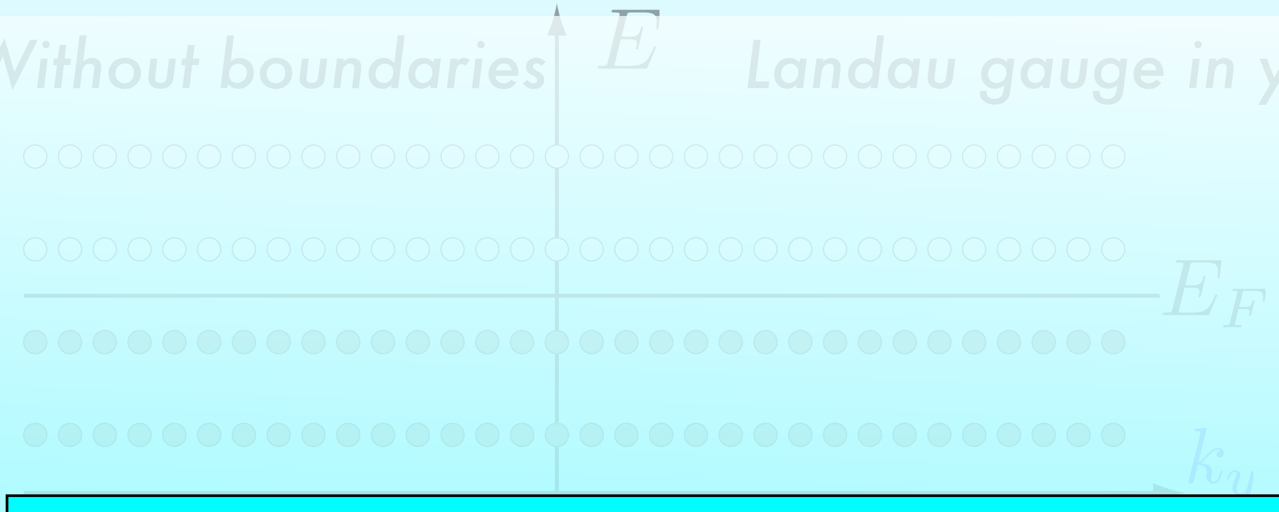
Edge states are essential in the QHE !

Quantum Hall Effects by edge states

☆ Edge states and Hall conductance σ_{xy} Halperin '82



Without boundaries

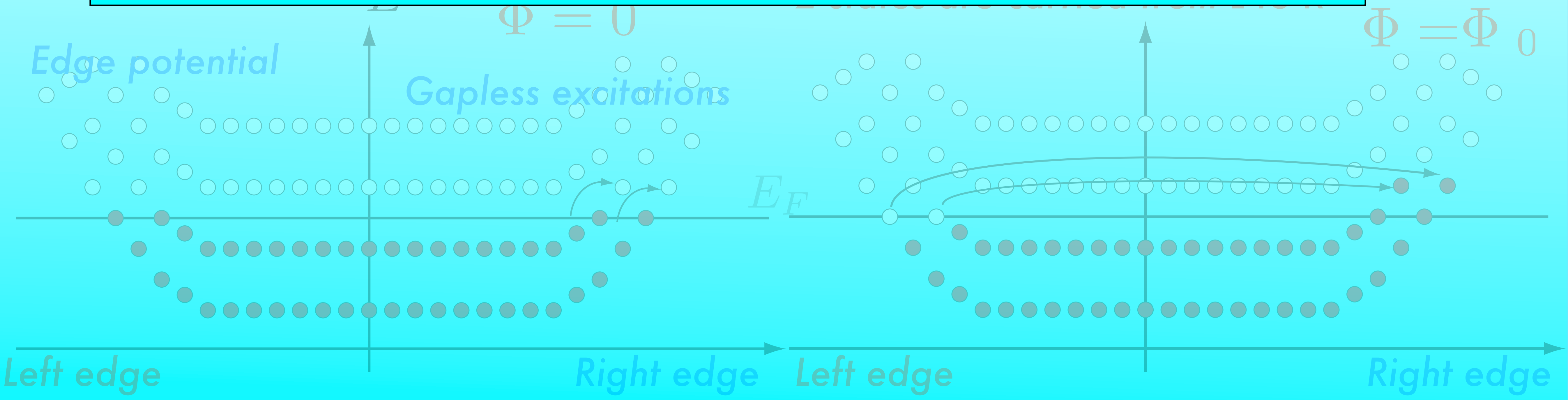


Landau gauge in y

$$H_{2D}=\sum_{k_y}H_{1D}(k_y)$$

$H_{1D}(k_y)$: harmonic osc. centered at $\langle x\rangle\sim\ell_B^2k_y$

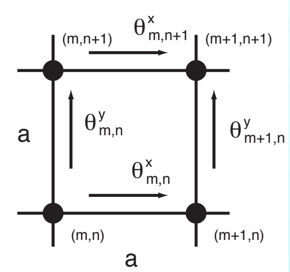
Edge states are essential in the QHE !



Laughlin's undetermined n : # of Landau Levels below E_F

Hall Conductance has a Topological meaning

- Discussion by the Bloch electrons (Peierls substitution)
  - preserve U(1) gauge symmetry
  - without cutoff ambiguity
  - recover continuum theory by scaling limit (weak field limit)



$$H = \sum_{\langle ij \rangle} c_i^\dagger e^{i\theta_{ij}} c_j \qquad 2\pi\phi = \sum_{\langle ij \rangle \in P} \theta_{ij} \qquad \phi = \frac{Ba^2}{\Phi_0}$$

P : plaquette

When E_F is in the j-th gap

Two topological quantities

★ $$\sigma_{xy}^{\text{bulk}} = \frac{e^2}{h} \sum_{\ell : \epsilon_\ell(k) < E_F} C_\ell$$

Sum of the First Chern numbers below E_F

Thouless-Kohmoto-Nightingale-den Nijs '82

with randomness Aoki-Ando '86

★ $$\sigma_{xy}^{\text{edge}} = \frac{e^2}{h} I(\alpha_j, C^j)$$

Winding number of the edge state in the complex energy surface

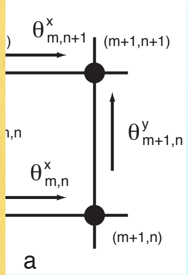
Hatsugai '93a

Bulk — Edge Correspondence Hatsugai '93b

$$\boxed{\sigma_{xy}^{\text{bulk}} = \sigma_{xy}^{\text{edge}}}$$

Hall Conductance has a Topological meaning

- Discuss stitution)
  - pres
  - with
  - reco weak field limit)



$H = \sum_{\langle i}$

ette

Direct application for the Graphene

When E_F is in the j-th gap

Two topological quantities

$$\sigma_{xy}^{\text{bulk}} = \frac{e^2}{h} \sum_{\ell:\epsilon_\ell(k)<E_F} C_\ell$$

Sum of the First Chern numbers below E_F

Thouless-Kohmoto-Nightingale-den Nijs '82

with randomness Aoki-Ando '86

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Winding number of the edge state in the complex energy surface

Hatsugai '93a

Bulk — Edge Correspondence

Hatsugai '93b

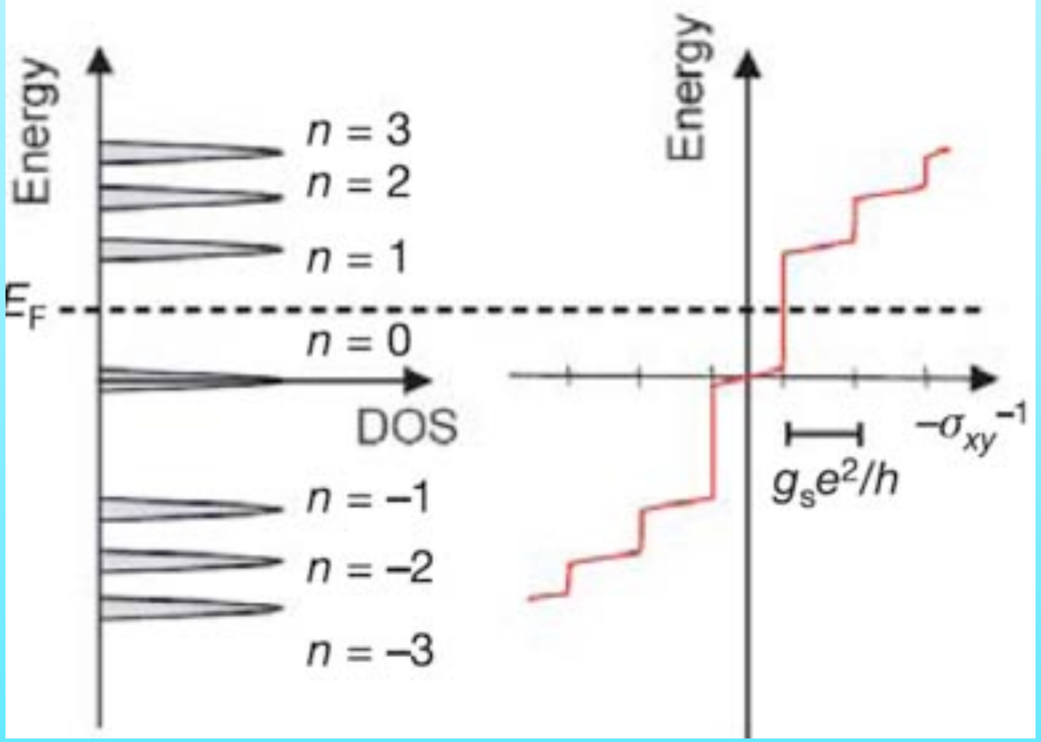
$$\boxed{\sigma_{xy}^{\text{bulk}} = \sigma_{xy}^{\text{edge}}}$$

Observation of Anomalous QHE in Graphene

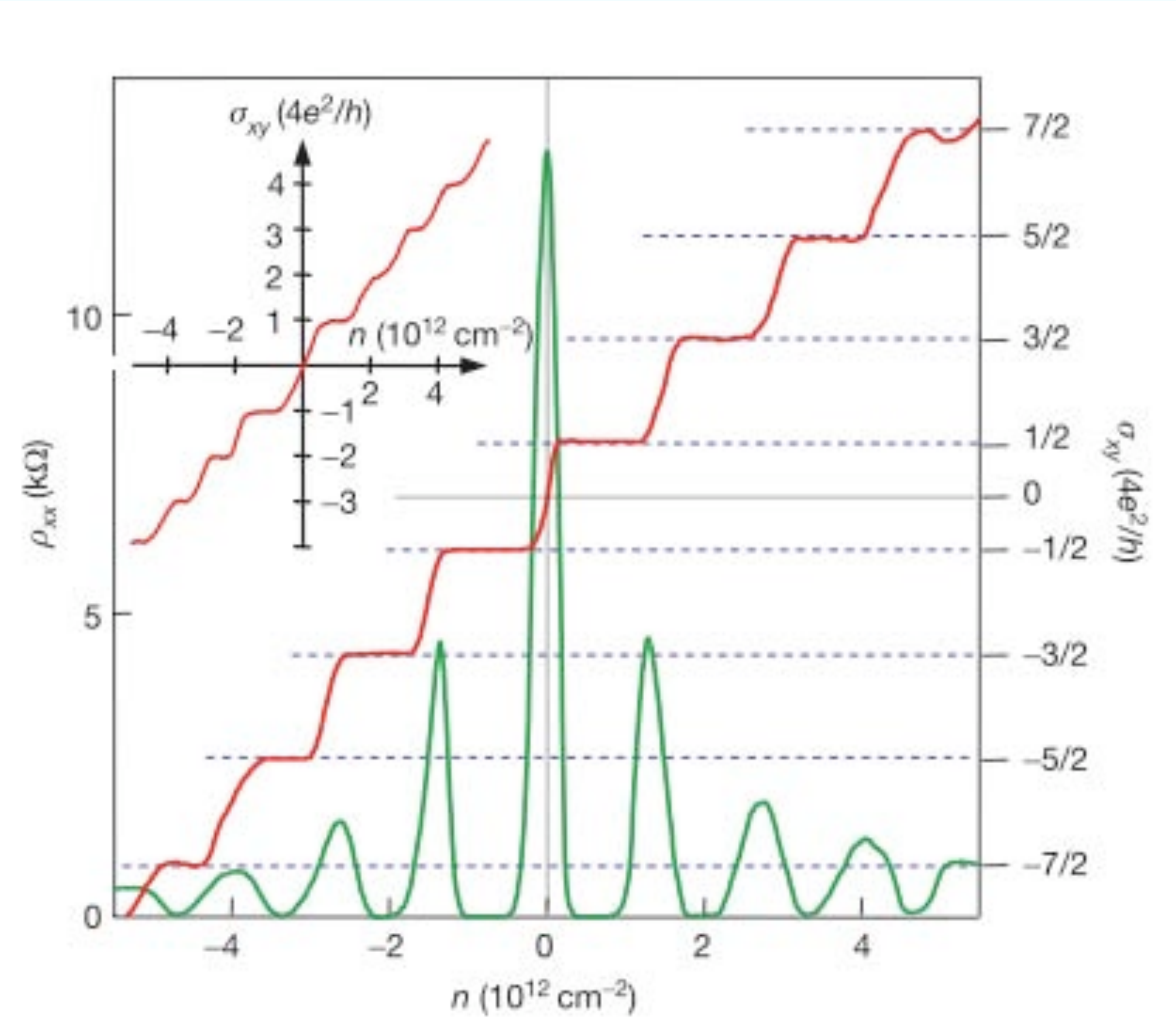
- Anomalous QHE of gapless Dirac Fermions

$$\sigma_{xy} = \frac{e^2}{h}(2n+1),\ \ n = 0, \pm 1, \pm 2, \cdots$$

$$= 2\frac{e^2}{h}(n+\frac{1}{2})$$



Zhang et al. Nature 2005



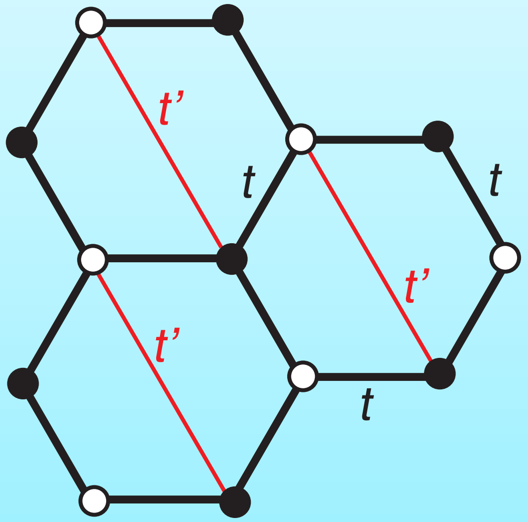
Novoselov et al. Nature 2005

Dirac Cones are Stable!

- ☆ The Dirac Cornes are not accidental
- ☆ It has topological stability

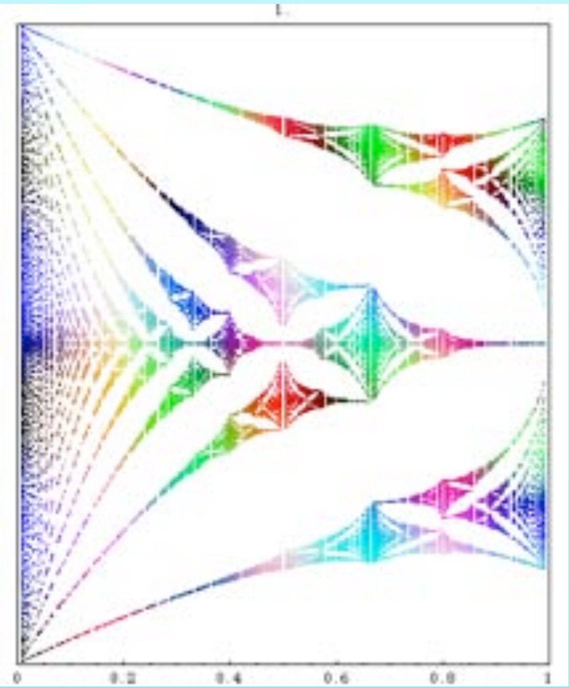
Chiral Symmetry

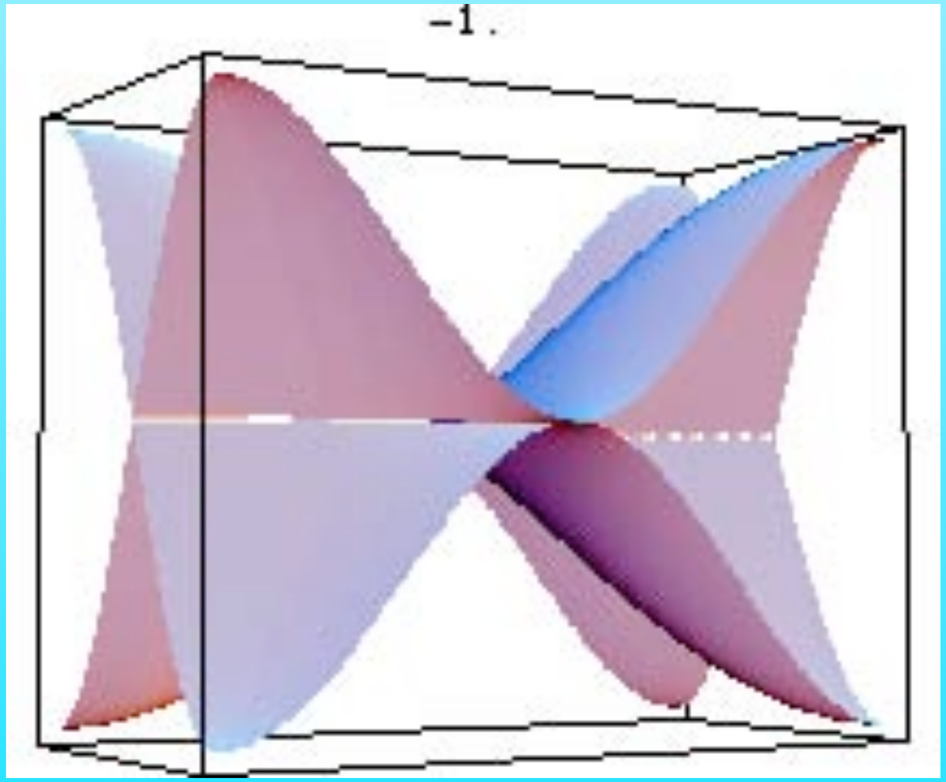
$$\{H, \Gamma\} = 0$$



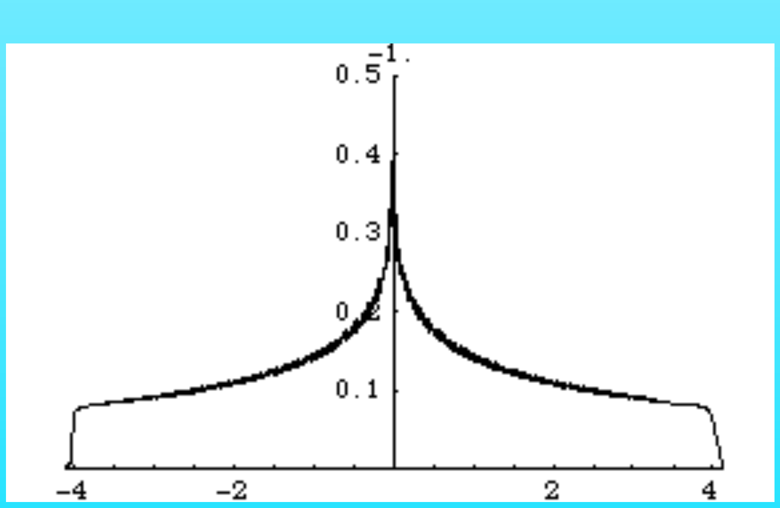
$$-3 < \frac{t'}{t} < 1$$ Doubled Dirac Cones

t'/t = 1 : Square Lattice
t'/t =0 :Honeycomb Lattice
t'/t=-1 : π Flux State





Density of states



Bulk Hall conductance of graphene

☆ Hall conductance by Chern number

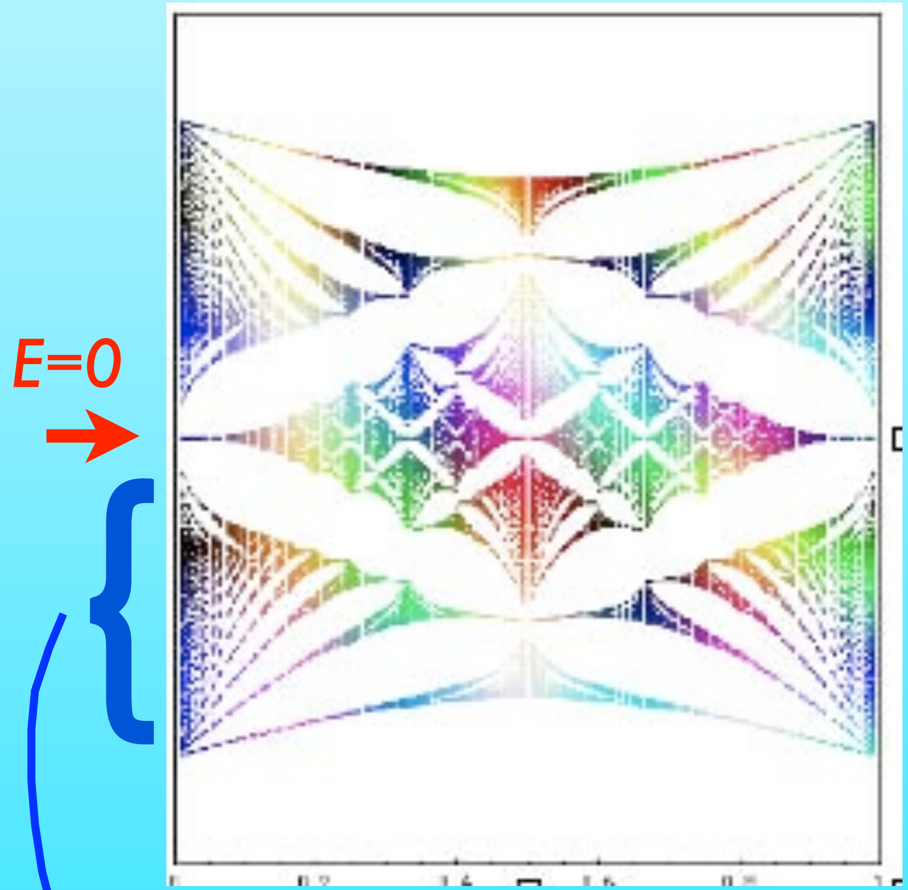
Counting vortices in the band

$$\sigma_{xy}^{j} = \frac{e^2}{h}\sum_{\ell=1}^{j} C_\ell, \quad C_\ell = \frac{1}{2\pi i}\int_{BZ} dA_\ell, \quad A_\ell = \langle \psi_\ell | d\psi_\ell \rangle$$

$\epsilon_\ell(k) < \mu_F,\ \ell = 1,\cdots,j$

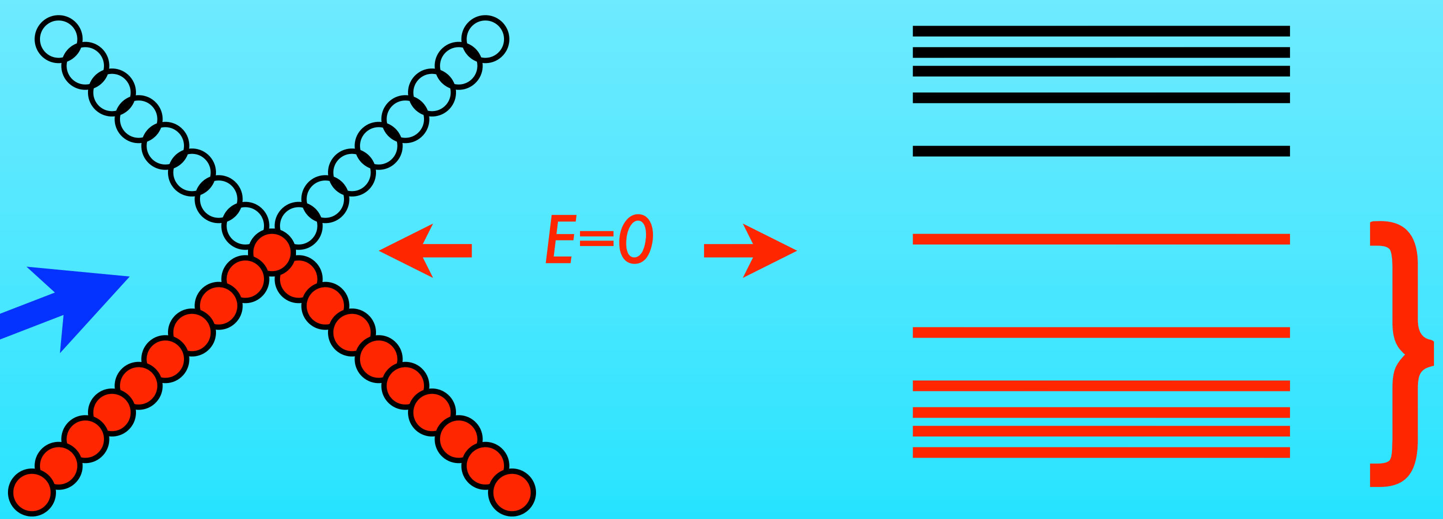
Thouless-Kohmoto-Nightingale-den Nijs 1982
with randomness Aoki-Ando 1986

graphene



Sum over the filled bands
Need to sum many bands until E=0

Numerical difficulty for the weak field
(experimental situation)
Need to fill negative energy Dirac sea



Need to sum over them

Bulk σ_{xy} of the Filled Fermi sea & Dirac Sea

- Integration of the NonAbelian Berry Connection of the filled “Fermi Sea” & “Dirac Sea”

Technical advantage for graphene

$$H_j(k)|\psi_j(k)\rangle = \epsilon_j(k)|\psi_j(k)\rangle$$

$$|\Psi\rangle = (c^\dagger\psi_1)\cdots(c^\dagger\psi_M)|0\rangle$$

$c^\dagger = (c_1^\dagger, \cdots, c_N^\dagger)$, N : number of sites

$$\mathcal{A} = \langle\Psi|d\Psi\rangle$$

Many body

$$\Psi = (\ |\psi_1\rangle, \cdots, |\psi_M\rangle)$$

Collect M states below the Fermi level

non Abelian one body

$$A_{FS} \equiv \Psi^\dagger d\Psi = \begin{pmatrix} \langle\psi_1^\dagger|d\psi_1\rangle & \cdots & \langle\psi_1^\dagger|d\psi_M\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi_M^\dagger|d\psi_1\rangle & \cdots & \langle\psi_M^\dagger|d\psi_M\rangle \end{pmatrix}$$

One body to Many body

$$\boxed{\mathcal{A} = \mathrm{Tr}\ A_{FS}}$$

Matrix vector potential of the Fermi (Dirac) Sea

Non Abelian extension for the Chern numbers

$$\sigma_{xy} \underset{\uparrow}{=} \frac{e^2}{h}\frac{1}{2\pi i}\int_{T^2} d\mathcal{A} = \frac{e^2}{h}\frac{1}{2\pi i}\int_{T^2} \mathrm{Tr}_M\ dA_{FS}$$

Tao-Thouless-Wu

YH ‘04

Y. Hatsugai, Cond-mat meets Hep: IPMU Focus Week, Feb 9, 2010

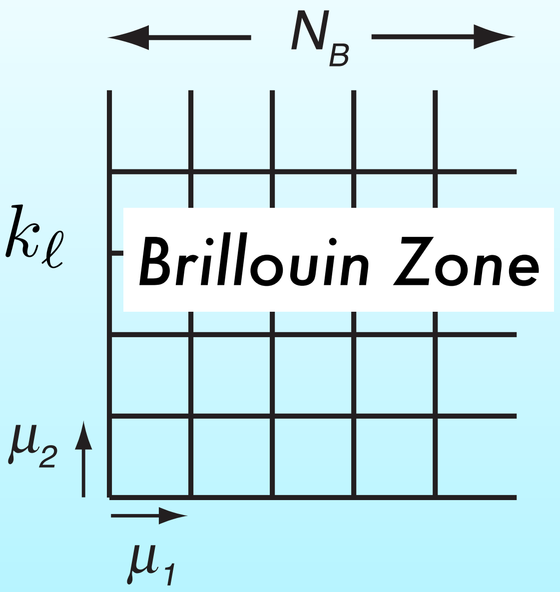
Luscher '82, '99

Numerical Technique from the Lattice gauge theory

☆ Topological Invariant on Discretized Lattice

Lattice in k space (discretization for the integral)

Technical Advantage for large Chern Numbers



Without gauge fixing

$\sigma_{xy} =$ h F_{1234} *gauge invariant*

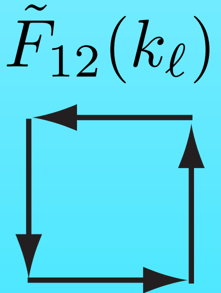
$F_{1234} = \mathrm{Im}$ U_{41}

$U_{mn} = \det_j \Psi_m^\dagger \Psi_n,$ $), \cdots, \psi_j(k_n))$

i Sea of j filled bands

$$U_\mu(k_\ell) \equiv \langle n(k_\ell) | n(k_\ell + \hat\mu) \rangle / \mathcal{N}_\mu(k_\ell)$$

$$\mathcal{N}_\mu(k_\ell) = |\langle n(k_\ell) | n(k_\ell + \hat\mu) \rangle|$$

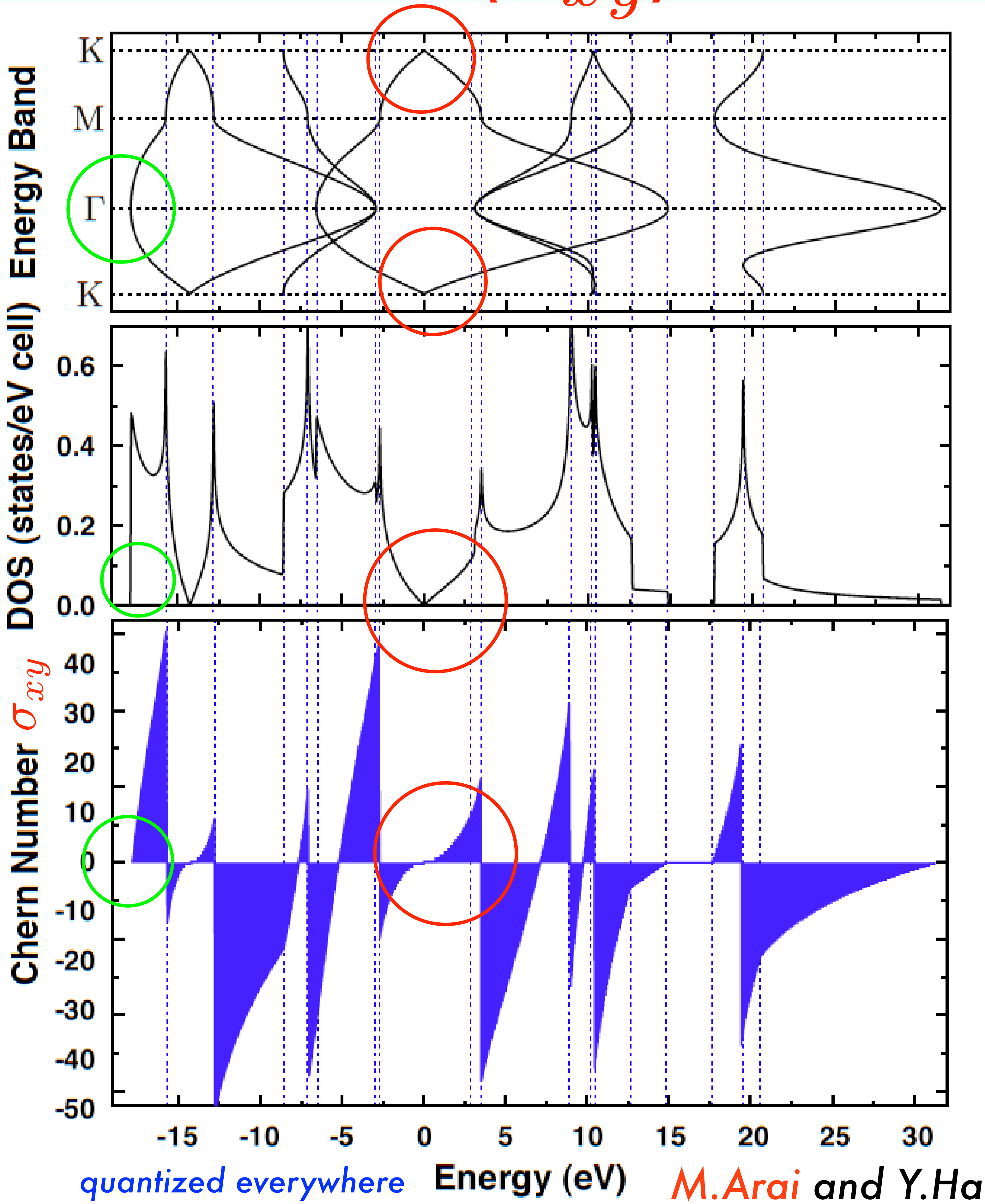


$$\tilde F_{12}(k_\ell) \equiv \ln U_1(k_\ell) U_2(k_\ell + \hat 1) U_1(k_\ell + \hat 2)^{-1} U_2(k_\ell)^{-1}$$

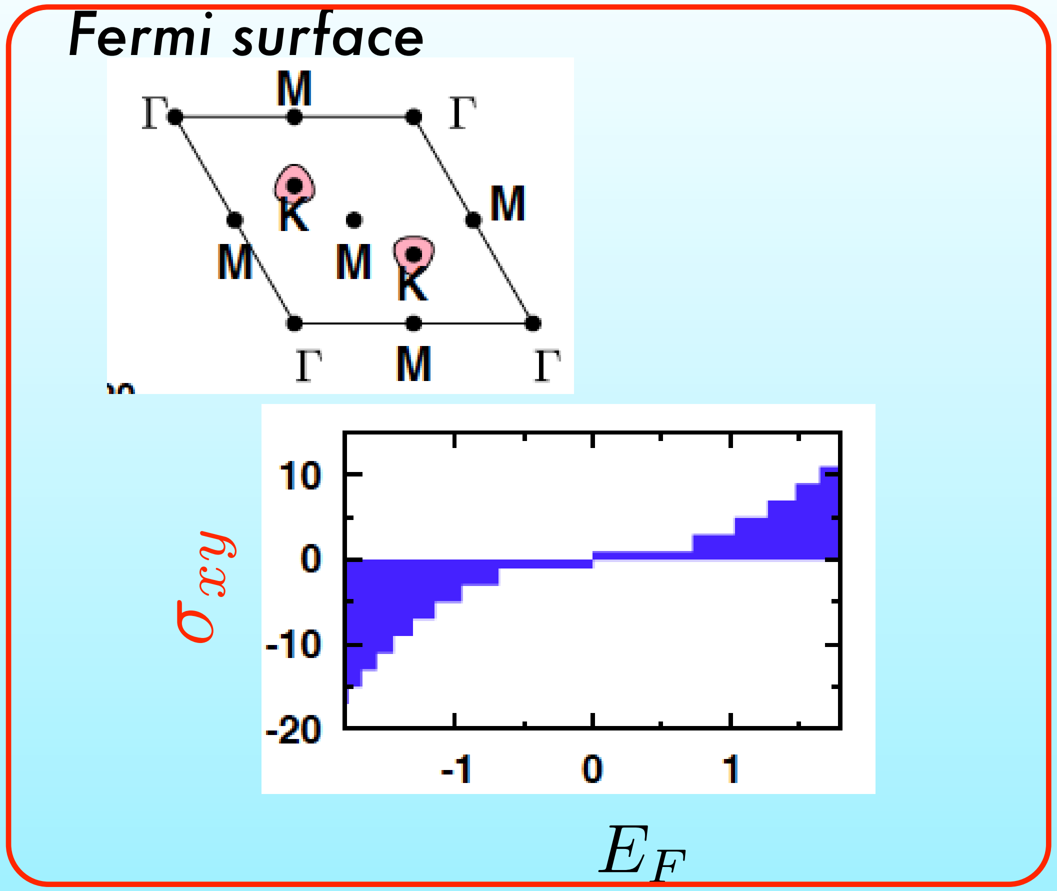
$$-\pi < \tilde F_{12}(k_\ell)/i \leq \pi$$ (principal value)

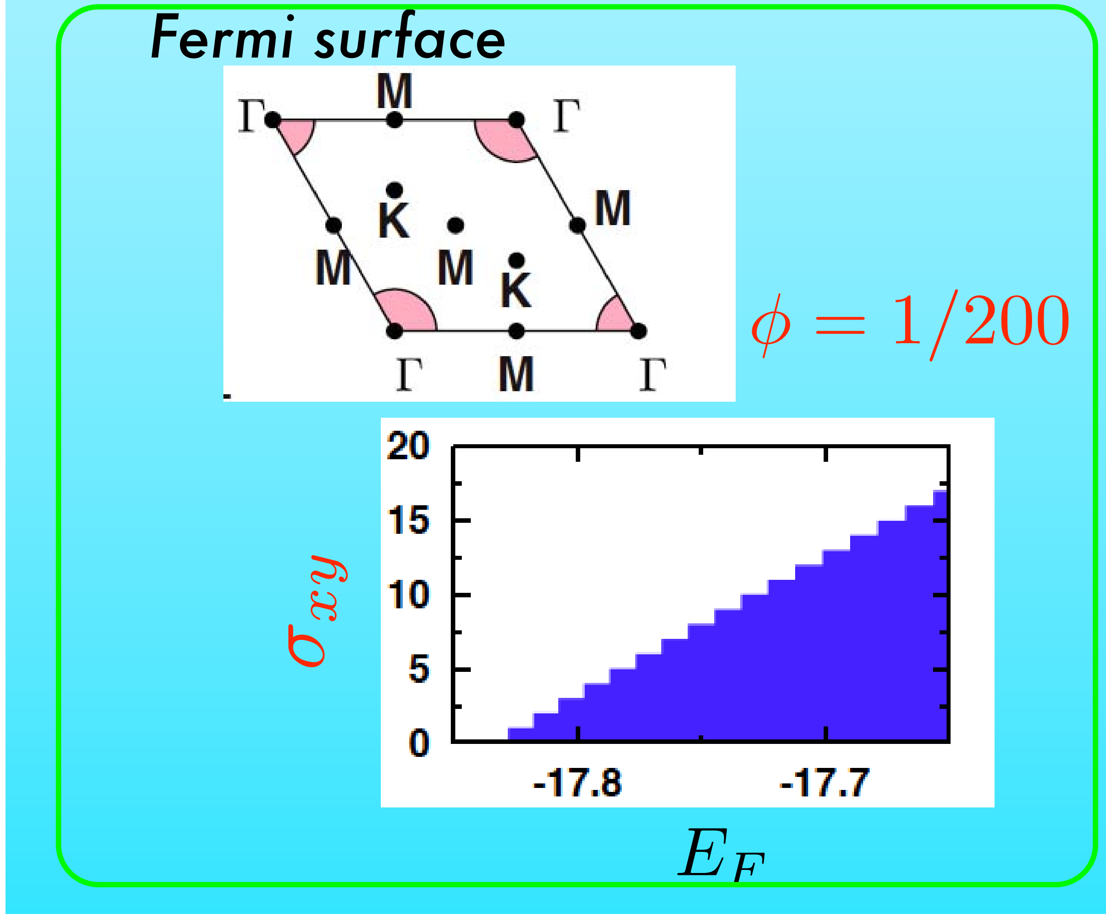
Fukui-Hatsugai-Suzuki 2005

Chern numbers (σ_{xy}) based on Realistic Band Calc.



quantized everywhere





M.Arai and Y.Hatsugai, Phys.Rev. B79, 075429 (2009)

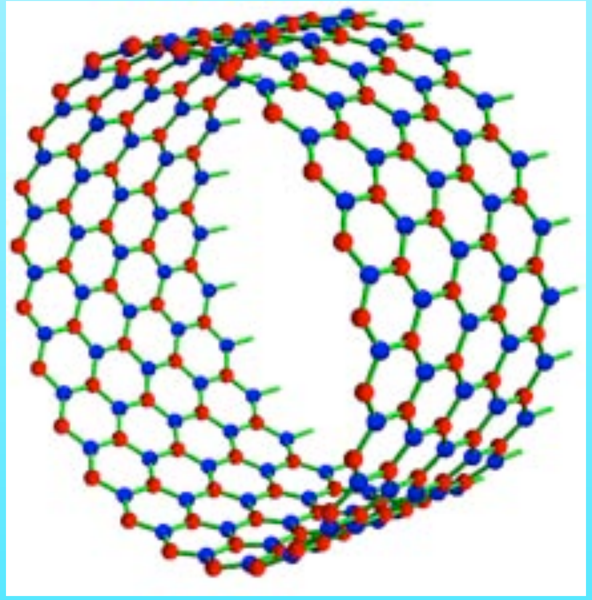
Edge States of Graphene

☆ Zigzag Edges

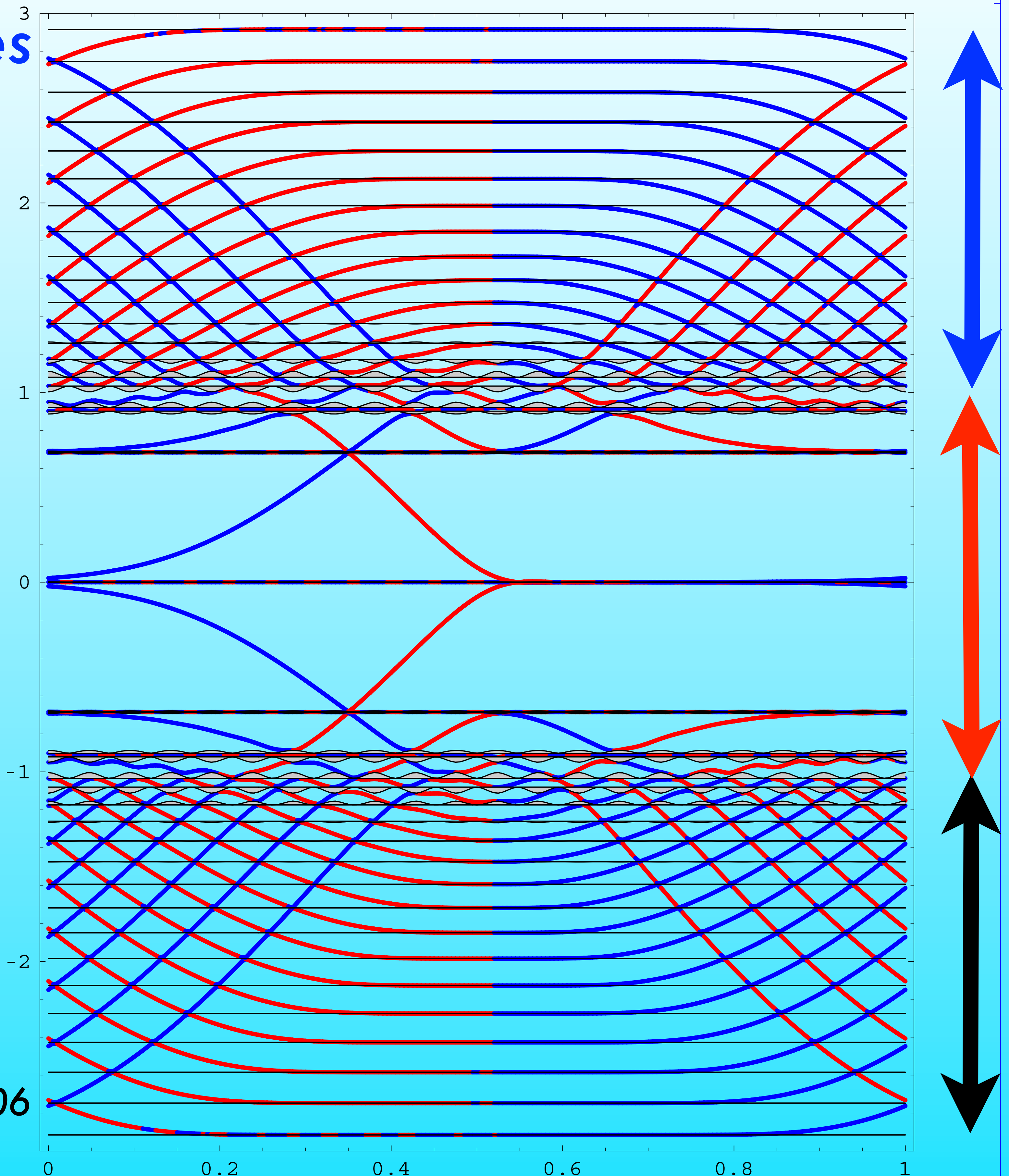
Full Spectrum

$\phi = 1/21$

Weak Field



YH, T. Fukui & H. Aoki, '06



Edge State of Holes

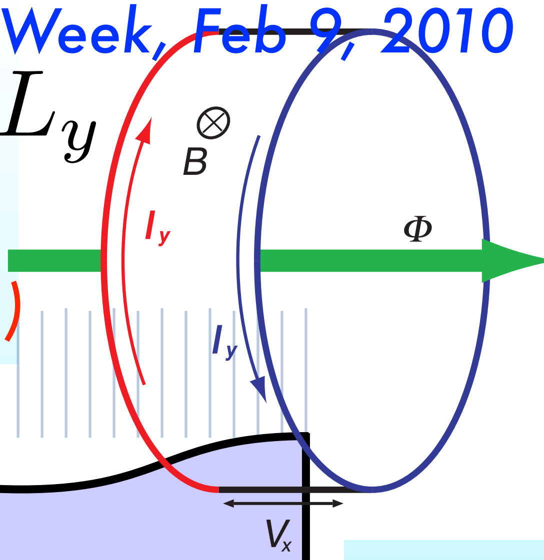
Edge State of Dirac Fermions

Edge State of Electrons

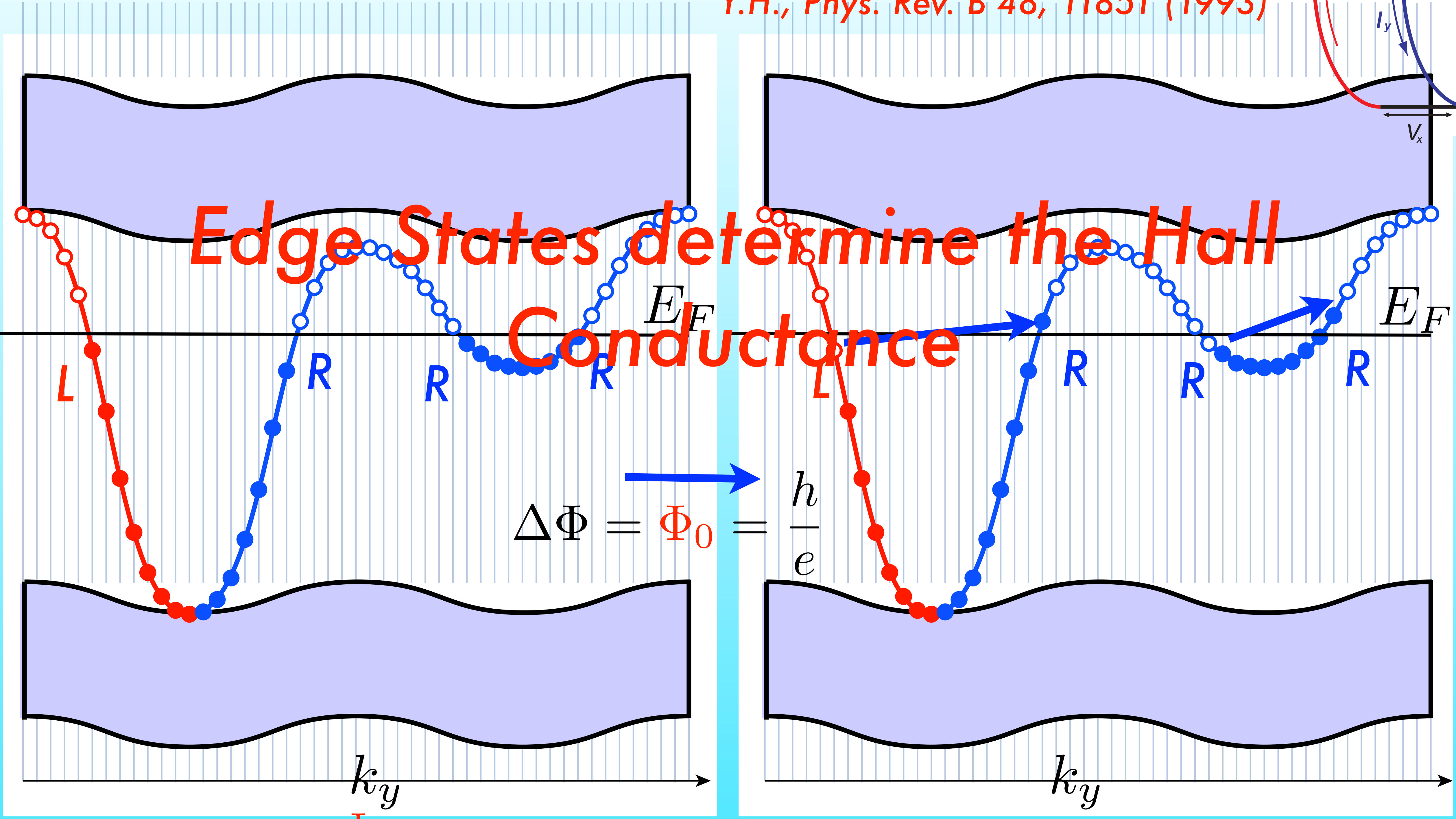
Laughlin's Argument & Edge States

- Adiabatic Charge Transfer

Y.H., Phys. Rev. B 48, 11851 (1993)



Edge States determine the Hall Conductance



$$\Delta\Phi = \Phi_0 = \frac{h}{e}$$

$$k_y = 2\pi \frac{n + \frac{\Phi}{\Phi_0}}{L_y}, \quad n: \quad \text{integers}$$

1 Electron is carried from the Left to the right in this case

$$\sigma_{xy} = \frac{e^2}{h} \cdot 1$$

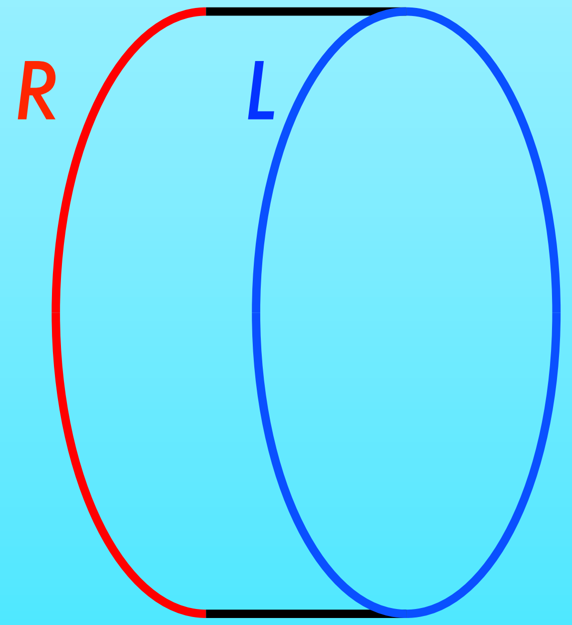
Graphene

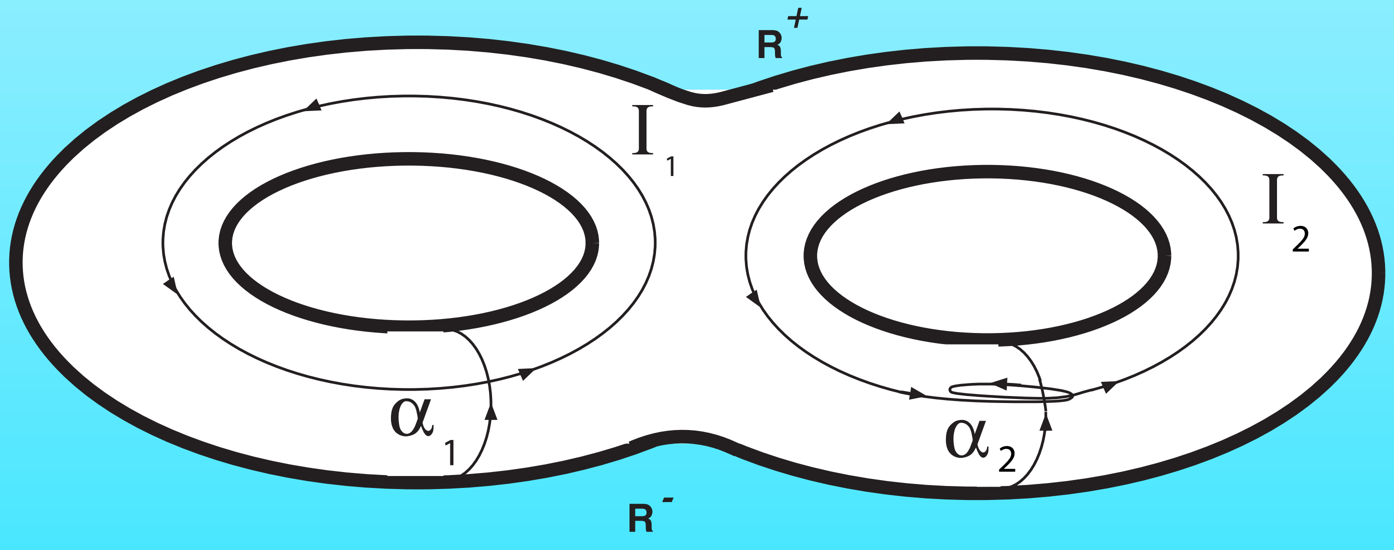
Analytical continuation of Bloch st. to the Edge state

YH, T. Fukui & H. Aoki, Phys. Rev. B74, 205414 (2006)

- ☆ *Followed by the discussion on a square lattice*

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)





Edge State and Bloch State

- Bloch electrons, 2D = $\sum$(1D Harper problem with parameter k_y)

As for the 1D Harper equation,

- Edge state : bound state
- Bloch state: scattering state

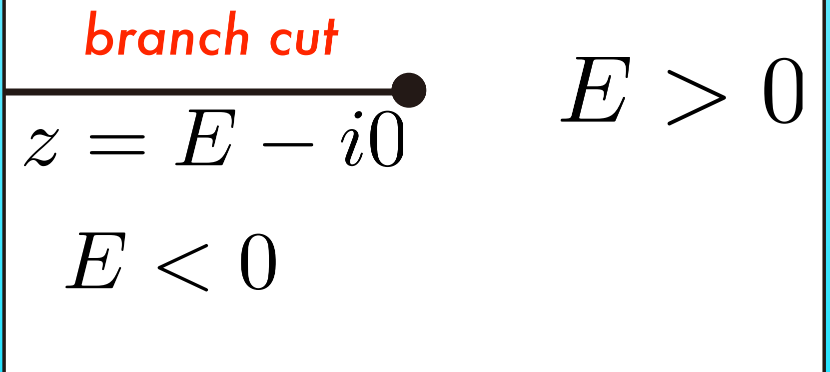
These two can be treated in a unified way by considering complex energy

In a quantum mechanics course, we learn

- bound state
- scattering state

$$E=\frac{\hbar^2k^2}{2m}\begin{cases} <\ 0 & k=i\kappa, \quad \psi\sim e^{-\kappa x} \\ >\ 0 & k\in\mathbb{R}, \quad \psi\sim e^{ikx}\end{cases}$$

E = z (complex energy)



unified description

$$\psi\sim e^{i\sqrt{2mE}\,x/\hbar}$$

energy of the bound state is in the gap region E<0

Analytic Continuation of the Bloch State

- The Edge State is obtained from the Bloch State by Analytical continuation

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

  - Energy of the Bloch state ψ_B is in the band
  - Energy of the edge state ψ_E is in the gap
- Complex energy surface : genus Q Riemann surface

ψ_B & ψ_E :Unified on Complex Energy surface

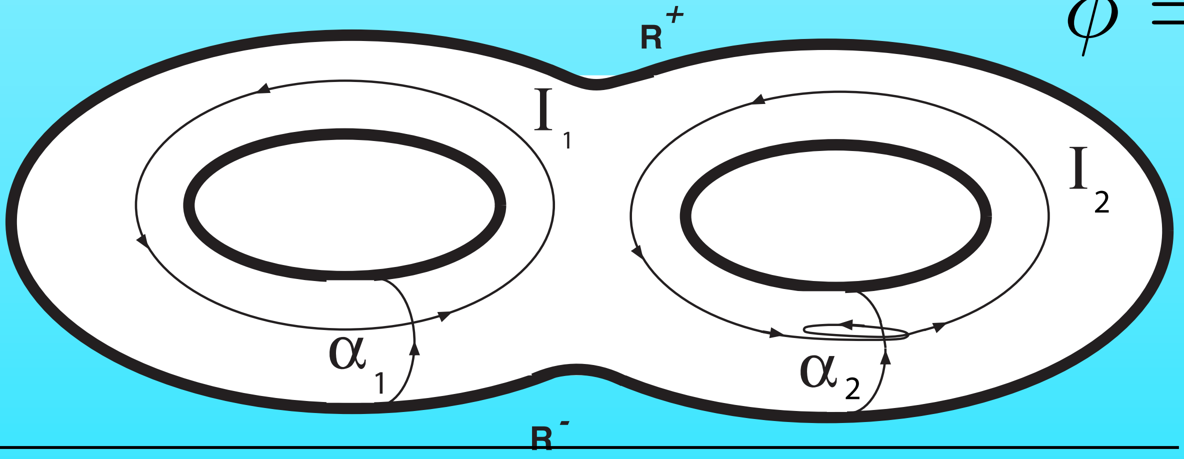
  - Energy bands : branch cuts, 2 Riemann sheets required
  - Q branch cuts
  - genus (number of holes) g=Q-1 Riemann surface
  - g : number of the energy gaps

$$\phi = P/Q$$

Flux per hexagon

Complex Energy surface
of Harper eq.

Also
graphene



$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

Analytic Continuation of the Bloch State to the complex energy (Riemann surface)

$$C_j = I_j - I_{j-1}$$

Chern # = winding # Difference between the neighboring gaps

ψ_B

Bulk-Edge Correspondence
of the topological numbers

$$\sigma_{xy}^{\text{bulk}} = \sigma_{xy}^{\text{edge}}$$

Complex Energy surface
of Harper eq.

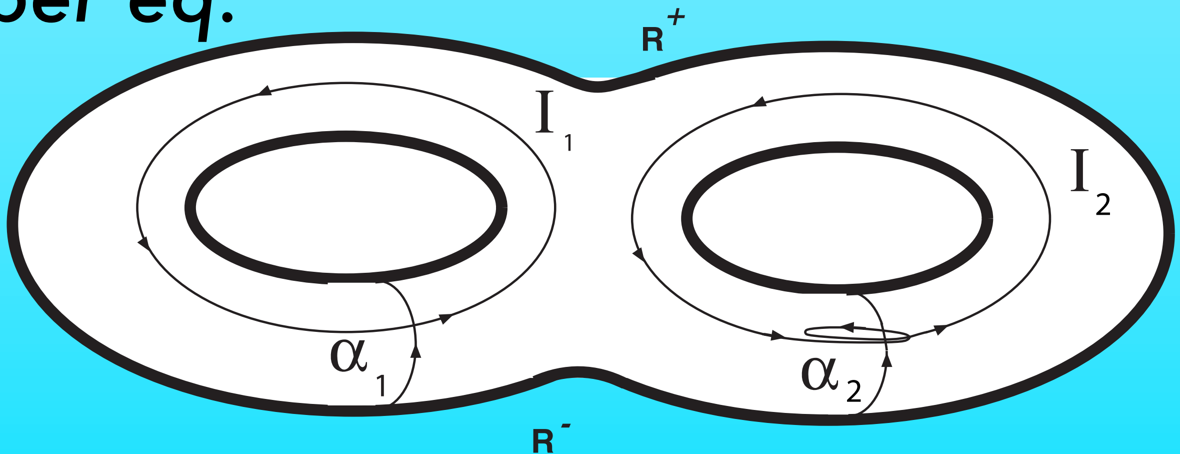
YH,'93

$\phi = P/Q$ graphene

YH, T. Fukui & H. Aoki, '06

genus g=q – 1:
number of the gaps

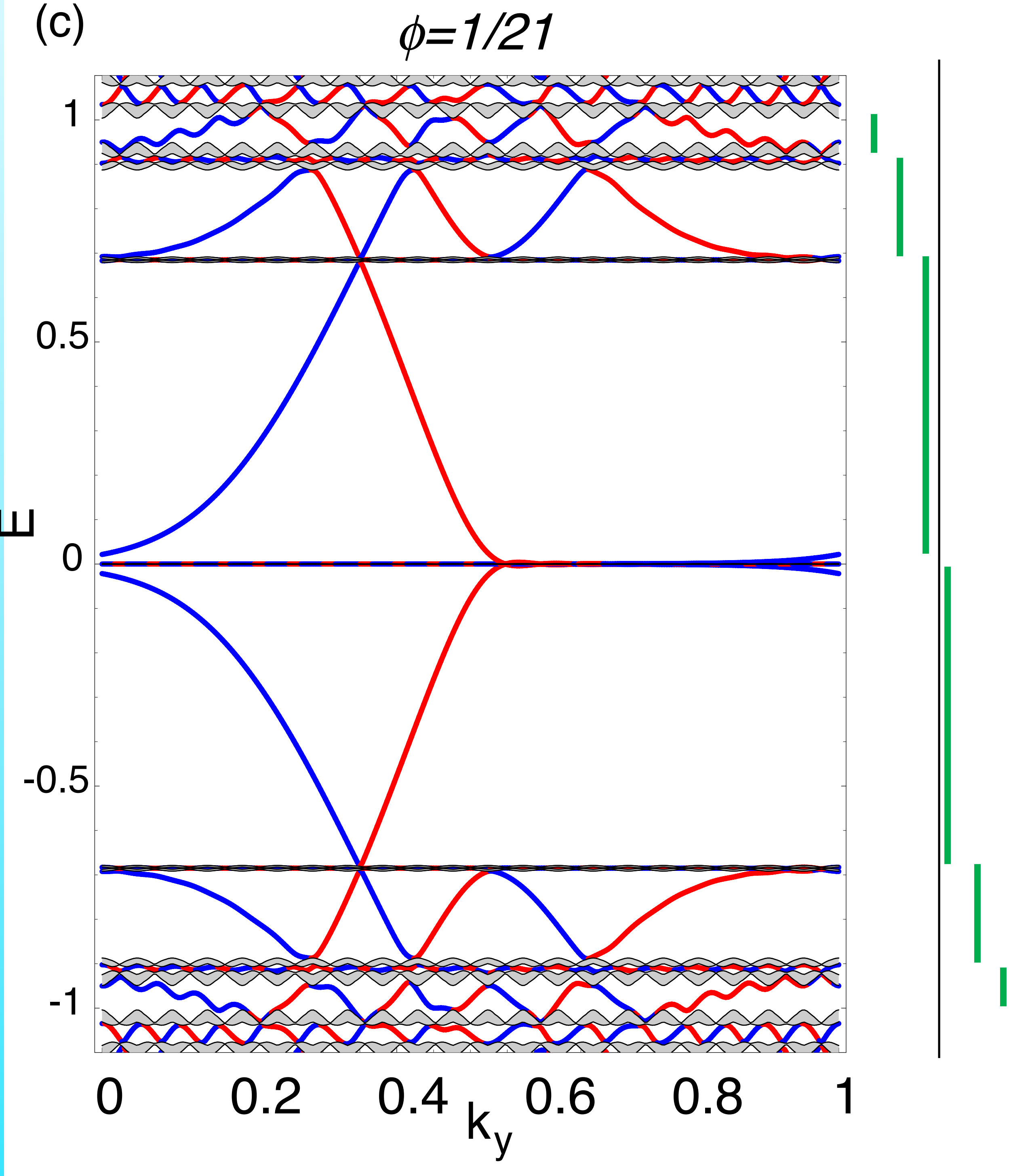
$$\phi = p/q$$

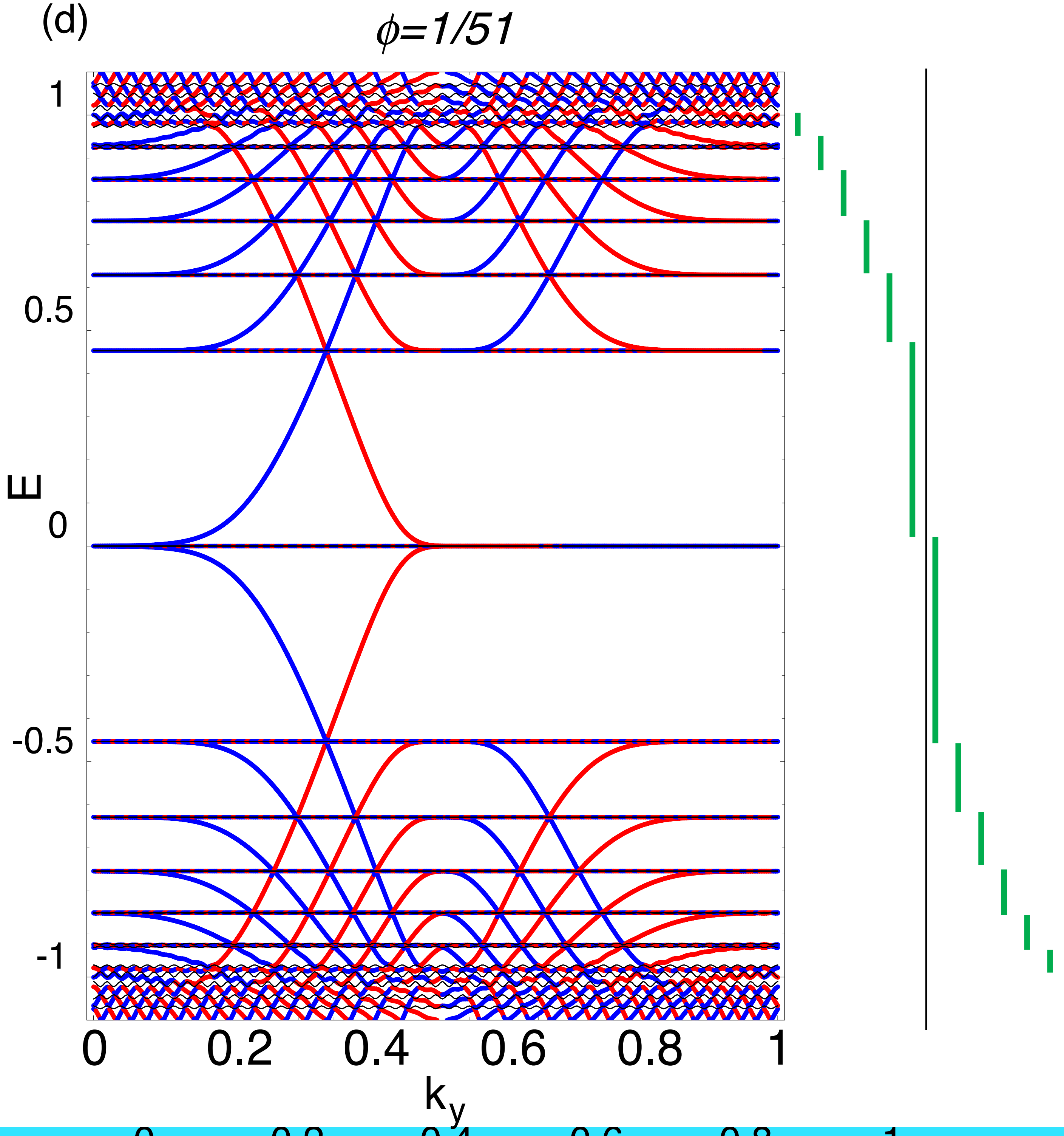


Bulk – Edge Correspondence

$$\sigma_{xy}{}^{\mathrm{bulk}} = \sigma_{xy}{}^{\mathrm{edge}}$$

Near Zero



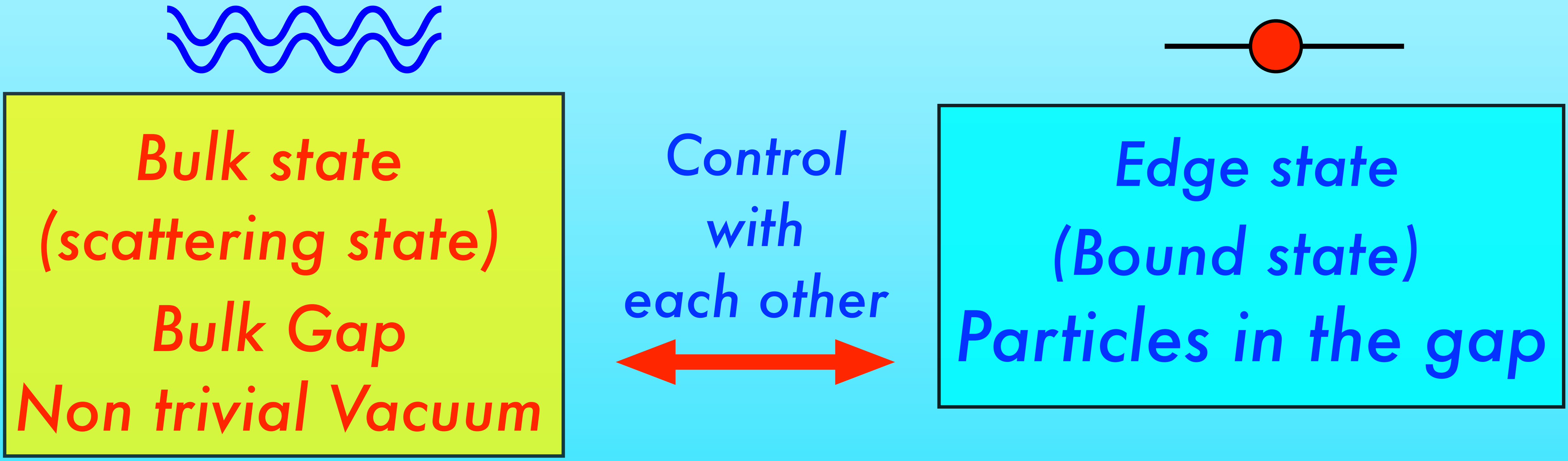


Summary

Looking around the Zoo of insulators
with

Bulk-Edge correspondence

Universality



Bulk state
(scattering state)
Bulk Gap
Non trivial Vacuum

Control
with
each other

Edge state
(Bound state)
Particles in the gap