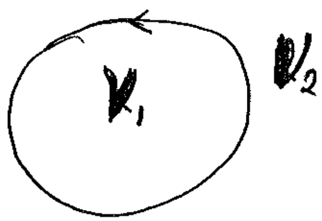
Top. phases for free fermions

1) Introduction

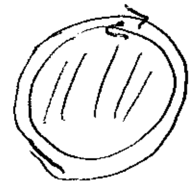
IQHE

$\nu \in \mathbb{Z}$



Top. insulator

$\nu \in \mathbb{Z}_2$

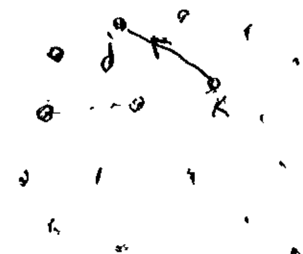


edge modes are protected by T-symmetry

Classification depends on symmetry and dimension

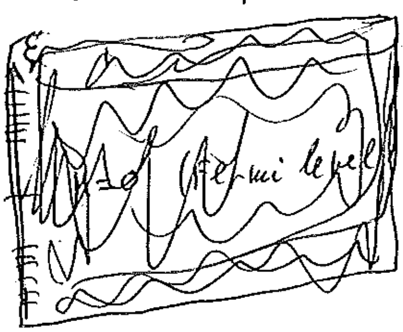
2) Microscopically:

$$\hat{H} = \sum \underbrace{H_{jk}}_{\text{hopping matrix}} \hat{a}_j^\dagger \hat{a}_k \quad (+\hat{a}_j^\dagger \hat{a}_k^\dagger)$$



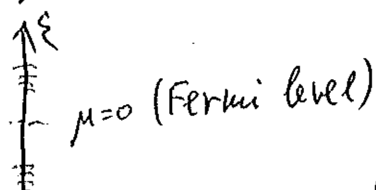
Mode space $\mathcal{M} = \mathbb{C}^n$ or $\mathbb{R}^n$

Fock space



1) r- local : $H_{jk} = 0$ if $|\vec{r}_j - \vec{r}_k| > r$

2) α- gapped : $\alpha \leq \varepsilon_m^2 \leq \alpha^{-1}$



α - gapped condition can be tested locally (in each disk of radius λ)

$\alpha = 1-\delta \approx 1 \quad \|H^2 - I\| \leq \delta$

$\|X\| \quad X \leftrightarrow \Pi_D X \Pi_D$

$\forall D \quad \underbrace{\|\Pi_D X \Pi_D\| \leq \delta}_{\text{locally gapped}} \Rightarrow \|X\| \leq \delta(1+d$

3) What kind of math. results to expect?

$\nu_H(x_1) = \nu_H(x_2)$

Expected theorem

If H is locally gapped then $\nu_H(x)$ ~~does not depend on x.~~ is locally constant

Example Unitary local matrix in 1d

$\Pi_- \quad k \quad x \quad j \quad \Pi_+$

$$f_{jk} = |U_{jk}|^2 - |U_{kj}|^2$$

unitarity $\Rightarrow \sum_k f_{jk} = 0$

Global flow $F_U(x) = \sum_{j \geq x} \sum_{k < x} f_{jk}$ (translation by ℓ)

$$F_U(x) = \mathrm{Tr}\left(U^+ \Pi_+ U \Pi_- - U^+ \Pi_- U \Pi_+\right) = \mathrm{Tr}\left(U^+ \Pi_+ U - \Pi_+\right) \in \mathbb{Z}$$

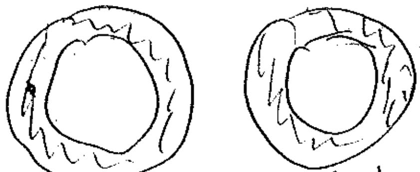
4) Another kind of local condition

$M_H(x)$ - operator on $\mathcal{H} \otimes \mathcal{F}$ ($\dim \mathcal{F} = 2^d$)

trivial outside the disk

Lemma $M_H(x)$ is continu

$M_H(x) \in S$



$\nu = 0 \qquad \nu = 1$

$\nu_H(x) \in \pi_0(S)$

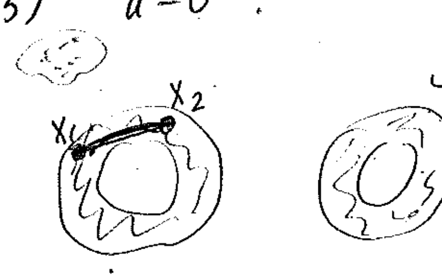
$\nu_H(x) =$ connected component of $M_H(x)$

Main theorem

~~Motivation~~ $\cancel{D = \mathbb{I}_d \otimes \left(\sum_{\mu=1}^{d} \gamma_\mu \partial_\mu\right) + M_H(x) \sim H}$ $\exists \left(\gamma_\mu \partial_\mu + M_H(x) \sim H\right)$

5) $d = 0$: the space of Hamiltonians

$$\alpha < H^2 < \alpha^{-1}$$



Example of a path:

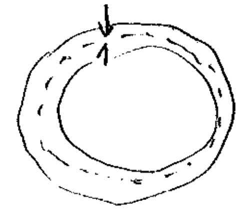
$X_0, X_1 \in S, \quad \{X_0, X_1\} = 0$

$$X(t) = \sqrt{1-t}\, X_0 + \sqrt{t}\, X_1$$

$$X(t)^2 = (1-t) X_0 + t X_1$$

$$\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \approx \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Retraction (spectral flattening)



$\tilde{H} = \operatorname{sgn} H$ $\qquad H = U\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_d\end{pmatrix}U^+$

$\tilde{\varepsilon}_s = \operatorname{sgn} \varepsilon_s$

$$\tilde{H} \in U(n)/U(\underset{\substack{\uparrow \\ \#\text{ negative e.v.}}}{k}) \times U(n-k)$$

Topology simplifies when $n \to \infty$, $k \to \infty$
But how do we take $k \to \infty$ and still keep track of the invariant.

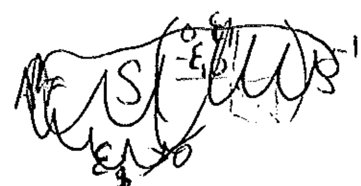
Solution: consider 2 matrices: $\mathrm{diff}(H', H'')$

$$k = k'' - k' = \text{const} \quad , k', k'' \to \infty$$

$\mathrm{diff}(H, H^{(\text{triv})})$ is classified by $\underline{\left(\lim_{n,m\to\infty} U(n+m)/U(n)\times U(m)\right)}$

6) No symmetry $\to$ Majorana formalism

$a_j^+ a_k^+$



$\hat{C}_{2l-1} = \hat{a}_l + \hat{a}_l^+$

$\hat{C}_{2l-1} = \dfrac{\hat{a}_l - a_l^+}{i}$ $\qquad H = \frac{i}{4}\sum \underbrace{A_{jk}}_{\text{real skew-symm.}} C_j C_k$

top inv. : $(-1)^\nu = \operatorname{sgn}(\mathrm{Pf}\, A) = \underline{\det S = \pm 1}$

Classifying space $- \to O(2n)/U(n)$

$$A = S\begin{pmatrix}\begin{matrix}0 & \varepsilon_1 \\ -\varepsilon_1 & 0\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}0 & \varepsilon_n \\ -\varepsilon_n & 0\end{matrix}\end{pmatrix}S^{-1}$$

$\varepsilon_s > 0$

$\tilde{\varepsilon}_s = 1$

$$\tilde{A} = S\begin{pmatrix}\begin{matrix}0 & 1 \\ -1 & 0\end{matrix} & \\ & \ddots\end{pmatrix}S^{-1}$$

7) Symmetries

a) $\hat{Q} = \sum_l \left(a_l^+ a_l - \frac{1}{2}\right) = \frac{i}{4}\sum Q_{jk} C_j C_k$

b) $\hat{T}$ $\quad \hat{X} \mapsto \hat{T}\hat{X}\hat{T}^{-1}$

$a_{j\uparrow}\, a_{j\downarrow}$

$a_\uparrow \to a_\downarrow$

$a_\uparrow^+ \to a_\downarrow^+$

$a_\downarrow \to -a_\uparrow$

$a_\downarrow^+ \to -a_\uparrow^+$

$$\begin{matrix}\uparrow\{ \\ \downarrow\{\end{matrix}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix} \to \overset{\text{some real matrix}}{\Gamma}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix}$$

$T^2 = Q^2 = -1 \qquad TQ = -QT, \quad \hat{A}T = -T\hat{A}$

$e_1 = T \qquad e_j^2 = -1 \qquad e_j e_k = -e_k e_j \quad (j \neq k)$

$e_2 = QT$

$e_3 = \tilde{A}$

$Cl^{p,q}: \quad e_1^2 = \dots = e_p^2 = -1, \quad e_{p+1}^2 = \dots = e_{p+q}^2 = 1$

~~Problem~~

$$\boxed{\begin{array}{ll|l} Cl^{0,1} \cong \mathbb{C} & e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cong i & Cl^{1,1} \cong \mathbb{R}(2) \\ Cl^{1,0} \cong \mathbb{R}\oplus\mathbb{R} & e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \hline \multicolumn{3}{l}{Cl^{p+1,q+1} \cong Cl^{p,q} \otimes \mathbb{R}(2) \quad \text{- Morita equiv.}} \\ \multicolumn{3}{l}{Cl^{p+8,q} \sim Cl^{p,q}} \end{array}}$$

<u>Problem</u> Given a rep. of $Cl^{p,q}$, classify its extensions to $Cl^{p+1,q}$ $\quad$ ($e_i^2 = 1$ - symm. $\quad e_i^2 = -1$ - antisymm.)

$\boxed{Cl^{0,q} \text{ to } Cl^{0,q+1}}$ $\qquad \mathbb{Z}, \mathbb{Z}_2\, \mathbb{Z}_2\, 0\, \mathbb{Z}\, 0\, 0\, 0$

Standard cosets $\qquad$ Bott periodicity

8) $d = 1$ $\quad$ Majorana chain

$$\hat{H} = \frac{i}{2} \sum \left(u\, c_{2l-1} c_{2l} + v\, c_{2l} c_{2l+1} \right)$$

$u \approx v$

$$A = \begin{pmatrix} \partial & m(x) \\ m(x) & -\partial \end{pmatrix} \qquad \text{Jackiw-Rabbi soliton}$$

several modes in each direction

$$A = \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \partial}_{X} + M \qquad M = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$$

$\{X, M\} = 0$ - opens a gap

$(X+M)^2 = X^2 + M^2$

$$\begin{pmatrix} \cancel{M_1} & m \\ -m & \cancel{M_2} \end{pmatrix}$$

9) A-antisymmetric $\longrightarrow$ B - symmetric

$$\{B, e_j\} = 0 \qquad e_1^2 = e_2^2 = 1$$

$$e_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & A^t \\ -A & 0 \end{pmatrix}$$ skew symm.

10) $$\boxed{B = \gamma_\mu \partial_\mu + \underbrace{M}_{\text{symm.}} \qquad \{\gamma_\mu, M\} = 0}$$

$$\gamma_1^2 = \dots = \gamma_d^2 = -1 \qquad \{e_j, M\} = 0$$

$$e_1^2 = \dots = e_q^2 = 1$$

$$\boxed{Cl^{0,q} \text{ in dim } d \equiv \underbrace{Cl^{d,q}}_{\wr\, Cl^{0,q-d}} \text{ in dim } 0}$$

$q = 2$:
$d = 2,6 \pmod 8$ — $\mathbb{Z}$
$d = 0,1 \pmod 8$ — $\mathbb{Z}_2$

11) Hopping matrix $\longrightarrow$ Dirac-type operator

$(B \to M_B(x))$

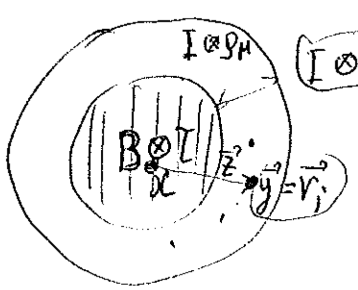
$$\mathcal{F} = \mathbb{C}^{2^d} \qquad \beta_\mu^2 = a_\mu^+ + a_\mu \qquad \beta_\mu^2 = 1$$

$$\gamma_\mu = a_\mu^+ - a_\mu \qquad \gamma_\mu^2 = -1$$

Grading operator: $\tau = \prod (\beta_\mu \gamma_\mu)$

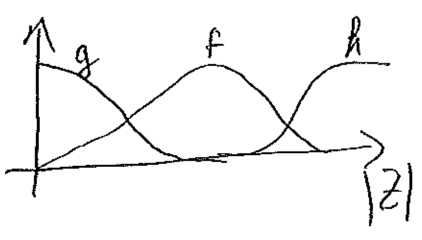
$$\gamma_\mu \partial_\mu = d + d^+$$
↑exterior derivative.

$$D = \underbrace{I_{\mathcal{M}}}_{\text{mode space}} \otimes \underbrace{\gamma_\mu \partial_\mu}_{\mathcal{F}} + \underbrace{M_B(x)}_{\mathcal{M} \otimes \mathcal{F}}$$



$$\varphi = \varphi(\vec{y})$$

$$(\varphi B)_{jk} = \frac{\varphi(\vec{r}_j) + \varphi(\vec{r}_k)}{2}$$

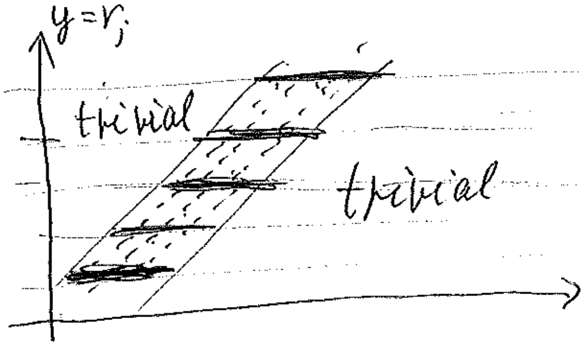


$$\vec{z} = \vec{y} - \vec{x}$$

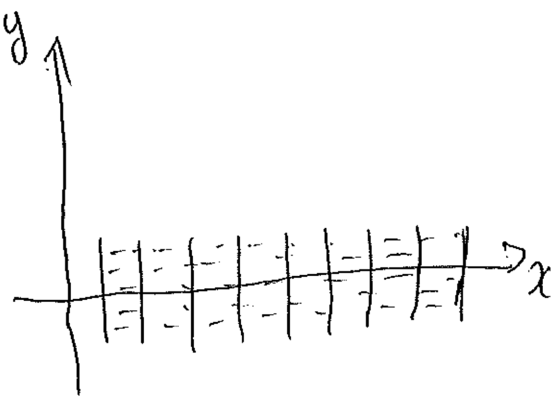
$$M_B(x) = g(|z|)\, B \otimes \tau + f(z) \frac{z_\mu}{|z|} I \otimes \beta_\mu + h(|z|)\, I \otimes \tau$$

Main theorem

$$D = I_\mu \otimes \gamma_\mu \partial_\mu + M_B(x) \sim B$$



(controls locality of D)



$$\tilde{D} = I \otimes \gamma_\mu \frac{\partial}{\partial y_\mu} + f(y) \frac{y_\mu}{y} I \otimes \rho_\mu$$
$$+ g(y)\, B \otimes \tau$$

~~$B = i\xi^a$, ... $d=4$~~

B is a number, $d = 1$

$$\boxed{D = \gamma \frac{\partial}{\partial y} + f\rho + gB\tau \sim B}$$