Fractionalization and Hidden Symmetry in Topological Orders

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In collaboration with

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• MS, M.Kohmoto, Y.-S.Wu, “Braid Group, Gauge Invariance, and Topological Order”, Phys. Rev. Lett. 97, 010601, (2006).
• MS, “ Topological discrete algebra, ground state degeneracy, and quark confinement”, Phys. Rev. D77, 045013, (2008).

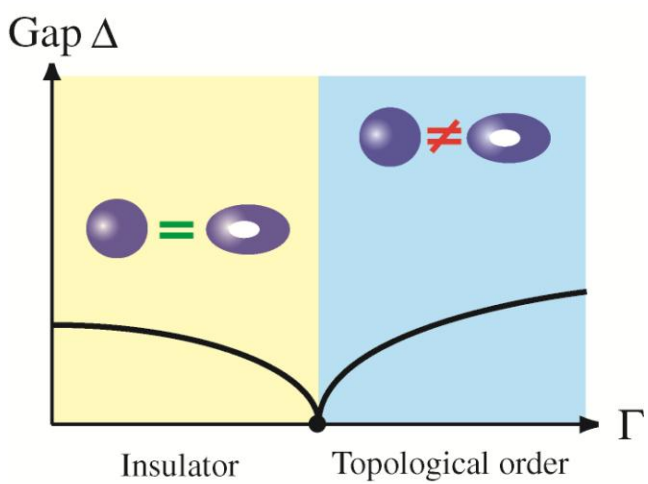
Outline

1. What is topological order ?
2. Fractionalization (charge, statistics, quantum Hall effect) and hidden symmetry
3. Topological degeneracy
4. Hidden symmetry in quark unconfinement phase
5. Summary

What is topological order?

Wen '90

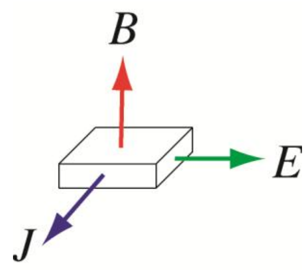
Gapped quantum phase with ground state degeneracy depending on topology of the system (topological degeneracy)

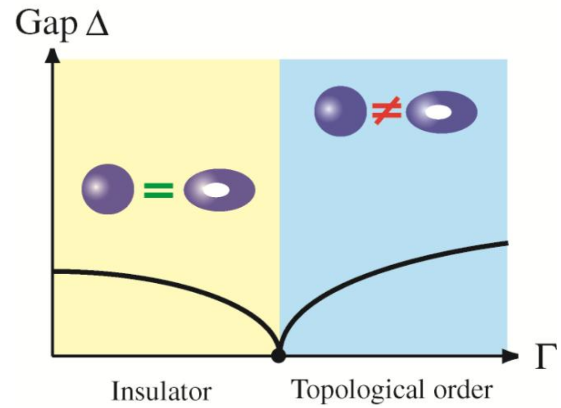


Topological orders cannot be understood by a conventional spontaneously symmetry breaking

Ex.) fractional quantum Hall state

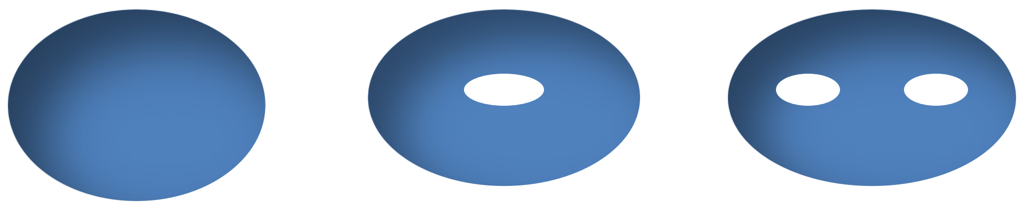
$$\sigma_{xy} \equiv \frac{J}{E} = \frac{e^2}{h}\nu$$





Laughlin state $\nu = \frac{1}{q}$

topological degeneracy



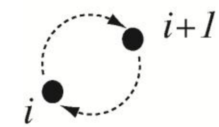
1 q q^2 • • •

Wen-Niu ‘90

- FQH states do not break symmetry of the Hamiltonian
- They can be characterized by topological degeneracy.

In the following, I present a simple connection between topological orders and fractionalization.

- charge fractionalization $$e^* = \frac{p}{q} e$$
- fractional anyon statistics $$\sigma_i = e^{i\theta} 1$$



- fractional quantum Hall effect $$\sigma_{xy} \equiv \frac{J}{E} = \frac{e^2}{h} \nu$$

Ex.) Laughlin state

$$e^* = \frac{1}{q} e \qquad \theta = \frac{\pi}{q} \qquad \nu = \frac{1}{q}$$

D.Arovas, J.Schrieffer, F.Wilczek, '84

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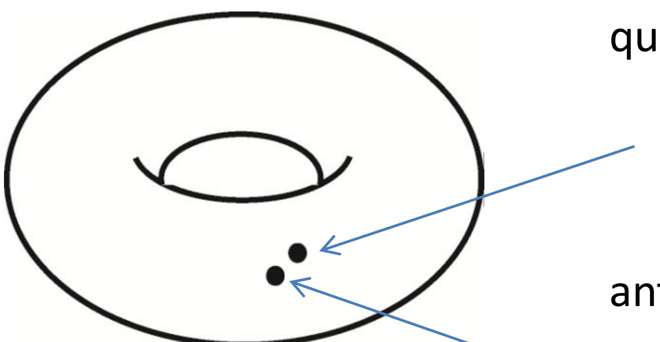
Fractionalization and hidden symmetry

[Einarsson (90), Wen-Niu (90), Wen-Dagotto-Fradkin (90), Wu-Hatsugai-Kohmoto (91), Oshikawa-Senthil (06), MS-Kohmoto-Wu (06)]

We consider the following 2+1d system

① The system is on a torus.
② The system is gapped.
③ Charge fractionalization occurs.



quasiparticle excitation

$$e^* = \frac{p}{q} e$$

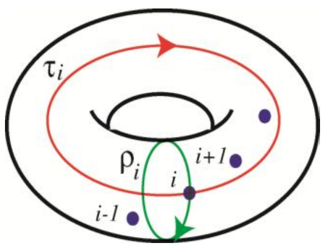
anti-quasiparticle excitation

$$-e^* = -\frac{p}{q} e$$

(e : charge of the constituent particle)

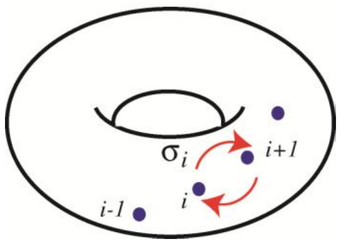
We consider the following three adiabatic processes on a torus

A) movement of i-th quasi-particle excitation along loops of torus



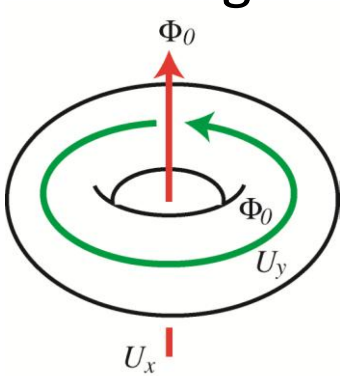
$\rho_i, \ \tau_i$ loop op.

B) exchange between i-th and (i+1)-th quasi-particle excitations



σ_i exchange op.

C) unit flux insertions through holes of torus



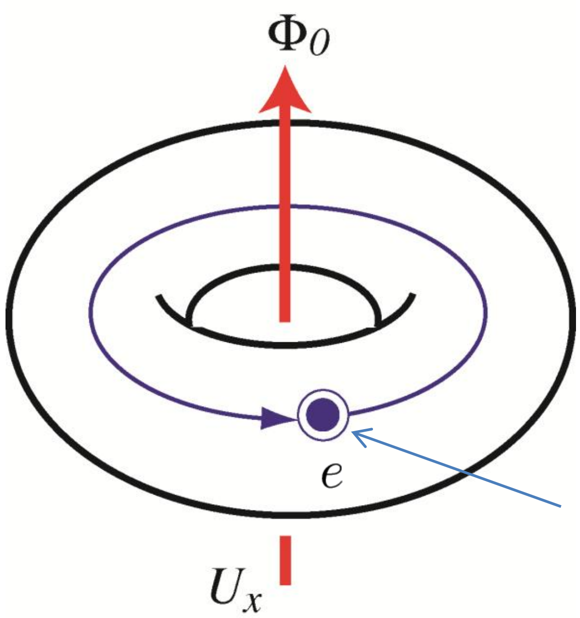
$U_x, \ U_y$ flux op.

$$\Phi_0 = \frac{2\pi}{e}$$ a unit flux

The spectrum of the system is invariant under these processes (in the thermodynamic limit).

For c), this is because the constituent particle cannot detect the inserted unit flux.

Laughlin (81)



$$\Phi_0 = \frac{2\pi}{e}$$

AB phase

$$e^{ie\Phi_0} = 1$$

We can show that these processes satisfy the following algebra

MS.M.Kohmoto, Y.S.Wu (06)

① Braid Group

σ_i, exchange op.

ρ_i, τ_i, loop op.

② Aharanov-Bohm effect

$$U_x\tau_i = e^{-2\pi ip/q}\tau_i U_x, \quad U_y\rho_i = e^{-2\pi ip/q}\rho_i U_y$$
$$U_x\rho_i = \rho_i U_x, \quad U_x\sigma_i = \sigma_i U_x,$$
$$U_y\tau_i = \tau_i U_y, \quad U_y\sigma_i = \sigma_i U_y.$$

③ Schur's lemma

$$U_x U_y = e^{2\pi ik/l} U_y U_x,$$

AB effect

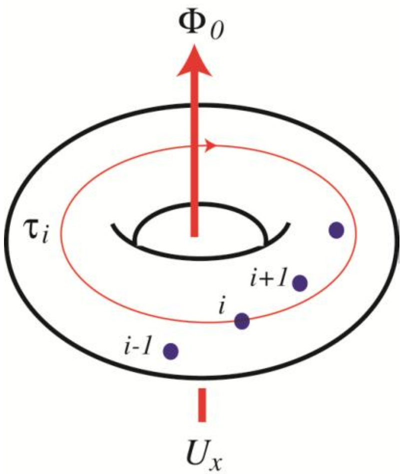
A quasiparticle with a fractional charge can detect the inserted unit flux.

Loop op. after flux insertion give rise to non-trivial AB phase

$$U_x \tau_i = e^{-2\pi i p/q} \tau_i U_x \qquad e^* = \frac{p}{q} e$$

AB phase

loop op. after flux insertion



In a similar manner, we obtain

$$U_y \rho_i = e^{-2\pi i p/q} \rho_i U_y$$

$$U_x \rho_i = \rho_i U_x, \quad U_x \sigma_i = \sigma_i U_x,$$
$$U_y \tau_i = \tau_i U_y, \quad U_y \sigma_i = \sigma_i U_y.$$

These processes satisfy the following algebra

① **Braid Group**

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$$U_x\tau_i = e^{-2\pi ip/q}\tau_i U_x, \quad U_y\rho_i = e^{-2\pi ip/q}\rho_i U_y$$
$$U_x\rho_i = \rho_i U_x, \quad U_x\sigma_i = \sigma_i U_x,$$
$$U_y\tau_i = \tau_i U_y, \quad U_y\sigma_i = \sigma_i U_y.$$

③ Schur's lemma

$$U_x U_y = e^{2\pi ik/l} U_y U_x,$$

By taking into account the AB effect,

$U_x U_y U_x^{-1} U_y^{-1}$ commutes with all the braid group operators

Schur's lemma

$$\Rightarrow \quad U_x U_y U_x^{-1} U_y^{-1} = \text{const} = e^{2\pi i \lambda}$$

In a similar manner, we can show

$$U_x^q = \text{const}, \; U_y^q = \text{const} \quad \Rightarrow \quad [U_x^q, U_y] = [U_y^q, U_x] = 0$$

To obtain consistency between them,

$$\lambda = \frac{k}{l} \qquad (l : \text{a divisor of } q)$$

From these considerations, we completely determine the algebra satisfied by these operators

① **Braid Group**

$\sigma_i,$ exchange op.

$\rho_i,\ \tau_i,$ loop op.

② **Aharanov-Bohm effect**

$$U_x\tau_i = e^{-2\pi ip/q}\tau_i U_x, \quad U_y\rho_i = e^{-2\pi ip/q}\rho_i U_y$$
$$U_x\rho_i = \rho_i U_x, \quad U_x\sigma_i = \sigma_i U_x,$$
$$U_y\tau_i = \tau_i U_y, \quad U_y\sigma_i = \sigma_i U_y.$$

③ **Schur's lemma**

$$U_x U_y = e^{2\pi ik/l} U_y U_x,$$

If we assume that the quasi-particle obeys abelian statistics, the algebra is simplified.

$$\sigma_i = e^{i\theta}\mathbf{1},\ \theta = \frac{m}{n}\pi,$$

(boson, fermion, fractional anyon)

The solution of braid group is

loop ops.

$$\tau_j = e^{-2i\pi m(j-1)/n} T_x, \qquad \rho_j = e^{2i\pi m(j-1)/n} T_y$$

$$T_x T_y = e^{-2i\pi m/n} T_y T_x,$$

$$T_x T_y = e^{-2\pi i m/n} T_y T_x, \qquad U_x U_y = e^{2\pi i k/l} U_y U_x,$$

$$U_x T_x = e^{-2\pi i p/q} T_x U_x, \qquad U_y T_x = T_x U_y,$$

$$U_x T_y = T_y U_x, \qquad U_y T_y = e^{-2\pi i p/q} T_y U_y$$

① Braid Group

② Aharanov-Bohm effect

③ Schur's lemma

MS, Kohmoto, Wu (06)

$$e^* = \frac{p}{q}e,\ \theta = \pi\frac{m}{n},\ \lambda = \frac{k}{l}$$

Hidden symmetry from fractionalization

From the hidden symmetry, we can obtain these two results.

- Topological degeneracy

- Existence of nonzero Chern (or Thouless-Kohmoto-Nightingale-den Nijs) numbers of the wave function

$$U_x U_y = e^{2\pi i k/l} U_y U_x,$$

→ nonzero Chern# of ground states

fractional quantum Hall effect

$$T_x T_y = e^{-2\pi i m/n} T_y T_x,$$

Niu-Thouless-Wu (84)

$$U_x T_x = e^{-2\pi i p/q} T_x U_x,$$

→ nonzero Chern # of quasiparticle states

→ nonzero Chern # of quasiparticle states

Phase with fractionalization ⇨ Phase with non-trivial topological #

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1. What is topological order ?

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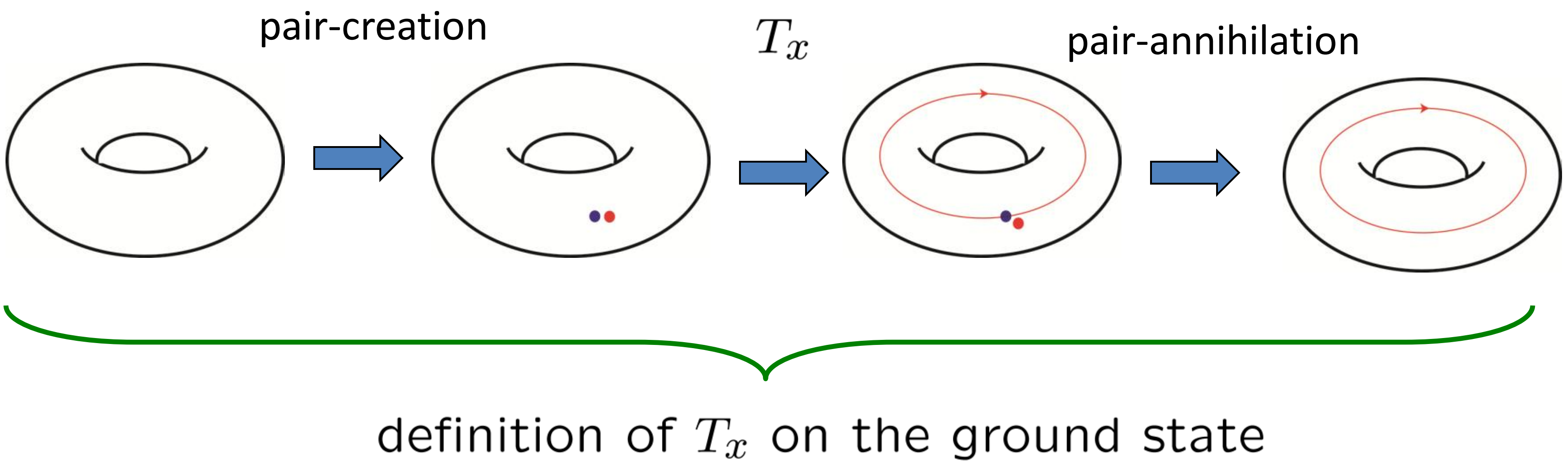
Topological degeneracy

To count ground state degeneracy, we define T_x on the ground state as follows



In a similar manner, we define T_y on the ground state

Even for this definition of Tx and Ty, we have the same hidden sym.

$$T_x T_y = e^{-2\pi i m/n} T_y T_x, \quad U_x U_y = e^{2\pi i k/l} U_y U_x,$$
$$U_x T_x = e^{-2\pi i p/q} T_x U_x, \quad U_y T_x = T_x U_y,$$
$$U_x T_y = T_y U_x, \quad U_y T_y = e^{-2\pi i p/q} T_y U_y$$

To be consistent with non-trivial hidden symmetry, the ground state must be degenerated.

The minimal ground state degeneracy is

MS,Kohmoto, Wu (06)

$n\mathcal{Q}^2$-fold ground state degeneracy

$(n/q = \mathcal{N}/\mathcal{Q}, \mathcal{N}, \mathcal{Q}$: co-prime integers)

specific to torus geometry

➡ **Topological degeneracy**

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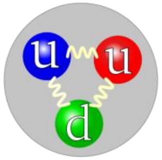
Hidden symmetry in quark unconfinement phase

Question: Is it possible to generalize hidden symmetry to non-abelian gauge theories such as QCD ?

Answer: **Yes**

However, fractionalization of the constituent particle is not necessary, but unconfinement of quarks (fractionalization of baryon #) is enough.



An important distinction from the previous argument is the existence of quatum fluctuations of **gluon**

$$\mathcal{L} = -\frac{1}{2}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}i\gamma^\mu(D_\mu - \underline{ie\Phi_0^{\rm B}\delta_{\mu,a}/L_a})\psi - m\bar{\psi}\psi + \cdots$$

insertion of flux coupled to baryon # $\Phi_0^{\rm B} = 2\pi/3e$

$$A_\mu \to \frac{i}{g}U\partial_\mu U^{-1} + UA_\mu U^{-1}$$

$$\psi \to Ue^{-i2\pi x_a/3L_a}\psi$$

center of SU(3)

$$U = \mathrm{diag}(e^{i2\pi x_a/3L_a}, e^{i2\pi x_a/3L_a}, e^{-i4\pi x_a/3L_a})$$

↓

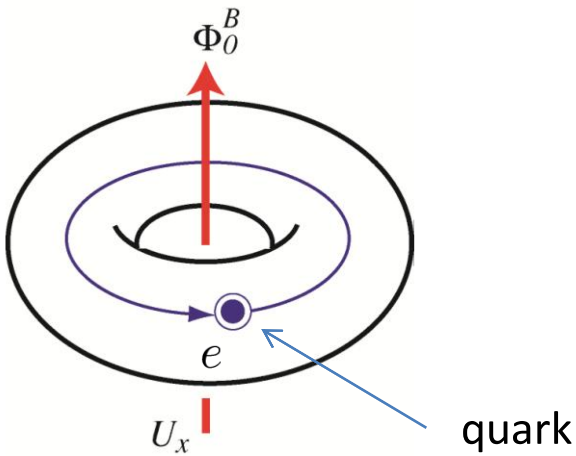
$$\mathcal{L} = -\frac{1}{2}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}i\gamma^\mu D_\mu\psi - m\bar{\psi}\psi + \cdots$$

The spectrum is invariant under flux insertion $\Phi_0^{\rm B} = 2\pi/3e$

~~$\Phi_0 = 2\pi/e$~~

Because the unit flux is reduced, the constituent particle can feel the inserted flux.

1) If quark is **unconfined**, we have the non-trivial AB phase



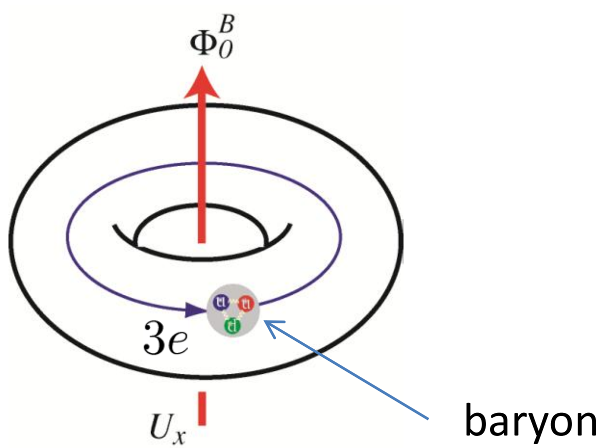
AB phase

$$e^{ie\Phi_0^{\mathrm{B}}} = e^{i2\pi/3}$$

$$U_x T_x = e^{-2\pi i/3} T_x U_x$$

hidden symmetry

2) On the other hand, if quark is **confined**, only trivial AB phase



AB phase

$$e^{i3e\Phi_0^{\mathrm{B}}} = 1$$

$$U_x T_x = T_x U_x$$ trivial

On D dimensional torus

hidden symmetry

T_a: quark loop op. U_a: flux op.

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$$

$$T_a T_b = T_b T_a \qquad U_a^3 = \text{const.}, \ (a = 1, \cdots, D)$$

MS (08)

the minimal topological degeneracy

3^D-fold ground state degeneracy

- On the other hand, no topological degeneracy if quark is confined

c.f.) Z_2 gauge theory
Senthil-Fisher (01), Kitaev(02),

● The topological degeneracy obtained from hidden symmetry is consistent with the Wilson's criterion of quark confinement in the static limit of quarks

In the static limit, QCD reduces to pure YM theory, so the system on a torus has a (spatial) center symmetry

$$W(C_a) \to e^{i2\pi/3} W(C_a)$$

① If quark is unconfined

perimeter law ➡ $\langle W(C_a) \rangle \neq 0$ breaking center symmetry

3^D-fold degeneracy

② If quark is confined

area law ➡ $\langle W(C_a) \rangle = 0$ preserving center symmetry

no degeneracy

Summary

● Hidden symmetry is derived from fractionalization (charge fractionalization, fractional anyon statistics, FQH effect) .

- ◆Completely characterize a generic 2D Abelian topological order
- ◆Formula for topological degeneracy on 2D torus
- ◆Non trivial topological # of wave function

● Hidden symmetry can be generalized to non-abelian gauge theories .

Topological degeneracy gives a gauge-invariant characterization of the quark (un)confinement phase (in the zero temperature)