Topological phases in quantum Hall and beyond: a pedagogical lecture

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Outline

- Wavefunctions for quantum Hall states
- p+ip superconductor and Majorana fermions
- Remarks on topological phases and gauge theory

Charged particle in a magnetic field in xy plane

$$H_1 = \frac{1}{2m}(-i\hbar\nabla - \mathbf{A})^2$$

$$\nabla \times \mathbf{A} = \mathbf{B}$$

Symmetric gauge $$\mathbf{A} = \frac{1}{2}\mathbf{r} \times \mathbf{B}$$

Energy eigenvalues $E_n = \hbar\omega_c(n + 1/2); \qquad n = 0,\, 1,\, 2,\, \ldots$

Cyclotron frequency $$\omega_c = B/m$$

Magnetic length $$\ell_B^2 = \hbar/B$$

Set $\hbar = \ell_B = 1$

Lowest (n=0) Landau level (LLL) eigenfunctions---**analytic** times Gaussian:

$$u_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m e^{-\frac{1}{4}|z|^2}, \quad m = 0,\, 1,\, 2,\, \ldots (z = x + iy)$$

Many identical particles in the LLL (because interactions are weak and temperature is low): analytic functions in N variables, (anti-) symmetric under permutation of particles (times Gaussian, often dropped)

Remaining Hamiltonian: Coulomb interaction projected to LLL
---highly degenerate perturbation theory, no small parameter

So consider “trial” states. Earliest/simplest:

$$\widetilde{\Psi}(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^Q$$

Laughlin 1983

Q=1, 2, 3, . . . ; Q **odd** for fermions, **even** for bosons

Filling factor: dimensionless density in interior of droplet

$$\nu=2\pi\ell_B^2\overline{n}=1/Q.$$

Wavefunction with “quasiholes” at $w_1, \ldots, w_n$:

$$\Psi(w_1,\ldots,w_n;z_1,\ldots,z_N) =$$
$$\prod_{k<l}(w_k - w_l)^{1/Q} \cdot \prod_{i,k}(z_i - w_k) \cdot \prod_{i<j}(z_i - z_j)^Q \cdot$$
$$\times\, e^{-\frac{1}{4Q}\sum_k |w_k|^2 - \frac{1}{4}\sum_i |z_i|^2}$$

Laughlin 1983, Halperin 1984

Plasma mapping:

$$\mathcal{Z}(w_1,\ldots,w_n) = \| |\Psi(w_1,\ldots,w_n)\rangle \|^2 =$$
$$\int \prod_i d^2 z_i \, \exp \frac{1}{Q}\left[\sum_{k<l} \ln|w_k - w_l|^2 + Q\sum_{i,k} \ln|z_i - w_k|^2 \right.$$
$$\left. + Q^2 \sum_{i<j} \ln|z_i - z_j|^2 - \frac{1}{2}\sum_k |w_k|^2 - \frac{Q}{2}\sum_i |z_i|^2 \right]$$

is the partition function of a Coulomb gas at temp Q with neutralizing background and impurities at $w_1, \ldots, w_n$

The plasma is in a **screening phase** for Q not too large.
Hence:

1. Density is uniform away from qholes, correlations decay exply

2. Quasiholes carry fractional charge -1/Q relative to background

3. Adiabatic statistics calculation: qholes are anyons with exchange statistics

$$e^{i\theta} = e^{i\pi/Q}$$

Arovas, Schrieffer, Wilczek 1984

Pseudopotentials: two-body ints in LLL described by one parameter for each relative angular momentum m
Exponent Q Laughlin state is unique lowest degree function of zero energy that has ang mom $\geq Q$ for all pairs of particles

i.e. annihilated by corresponding positive projection operator (or delta function) Ham Haldane 1983

(Expect gap in the spectrum of this, but not proven)

Moore-Read “Pfaffian” wavefunction   Moore and NR 1991

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\prod_{i<j}(z_i-z_j)^Q \cdot e^{-\frac{1}{4}\sum_i |z_i|^2}$$

where now Q is **even** for fermions, **odd** for bosons

Filling factor $\nu = 1/Q$

Special Hamiltonian: for Q=1, wfn vanishes when three coordinates are equal, so annihilated by three-body delta-function int (projected to LLL)

$$\sum_{i<j<k}\delta(z_i-z_j)\delta(z_j-z_k)$$

Greiter, Wen, Wilczek 1992

Unique such at lowest degree (highest density)
Generalizes to three- (plus two-) body int for Q>1.

Quasiholes? Statistics?

CFT relation:

Moore and NR, 1991

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \langle \psi_e(z_1)\cdots\psi_e(z_N)\rangle_{\mathrm{CFT}}$$

where

$$\psi_e(z) = e^{i\sqrt{Q}\phi(z)}\psi(z)$$

and ϕ is a free chiral scalar, and ψ is a free Majorana chiral field in **two** dimensions. (Also, a neutralizing background.)

For quasiholes, use

$$\langle \psi(z_1)\cdots\psi(z_N)\tau(w_1)\cdots\tau(w_n)\rangle_{\mathrm{CFT}} = \Psi_a(w_1,\ldots;z_1,\ldots)$$

where $\tau(w) = e^{i\frac{1}{2\sqrt{Q}}\phi(w)}\sigma(w)$, $\sigma(w)$ is (chiral) Ising spin field:

$$\psi(z)\sigma(0) \sim z^{-1/2}\sigma(0)$$

Exponential (scalar field) factors chosen so functions are single-valued in *z*'s---but **multivalued** in *w*'s. *a* labels basis in space of conformal blocks

Assume screening holds in charge sector as before.
Quasiholes carry charge -1/2Q. (Even more fractional!)
Suppose have a gap in spectrum of special Ham in thermo limit.

Moore-Read conjecture:

the adiabatic statistics of the quasiholes is same as monodromy of the conformal blocks

hence is non-Abelian here. This follows from the fusion rules of the Ising (Majorana) theory:

$$\begin{aligned} \psi \times \psi &= 1, \\ \psi \times \sigma &= \sigma, \\ \sigma \times \sigma &= 1 + \psi. \end{aligned}$$

These apply here to the excitations in 2+1 (combined with addition of charges)

Derivations of adiabatic stats:

1. Various partial and numerical works

2. Derivation using screening, but omitting charge sector NR 2009

3. One-dim derivation A. Seidel 2008

4. General arguments NR 2009

Read-Rezayi states:

NR, Rezayi 1999

1. Use parafermion field ψ_1 from Z_k parafermion CFT in place of Majorana ψ

2. Filling factors $\nu = \dfrac{k}{Mk+2}$ (M+2/k replaces Q in exponent of Laughlin factor)

k=1 is Laughlin, k=2 is MR state.

3. Special Hamiltonian is now k+1-body interaction

4. Effective field theory: for M=1, SU(2)$_k$ Chern-Simons theory
(Not for M>1) ----no known field-theoretic derivation for k>2

5. k=3 and k>4 are universal for topological quantum computation

Freedman, Larsen, Wang 2002

p+ip paired states and Majorana fermions

Composite or Chern-Simons boson/fermion theory: various authors 1980s–90s

---- singular gauge transformation, map particles in field B to bosons/fermions in field B coupled to U(1) Chern-Simons gauge field

----because particle density is non-zero, at mean field level for CS gauge field, the net effective magnetic field is zero.

E.g. for fermions at $\nu = 1/2$, can map to non-relativistic fermions at zero field (coupled to CS).

If the fermions Cooper pair to form a gapped BCS state, we have a QH state (topological phase).

Indeed, in the MR trial function, the Pfaffian factor is the real-space BCS wfn for p-ip pairing

Moore, NR 1991

BCS theory

$$K_{\rm eff}=\sum_{\mathbf{k}}\left[\xi_{\mathbf{k}}c_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}+\frac{1}{2}(\Delta_{\mathbf{k}}^{*}c_{-\mathbf{k}}c_{\mathbf{k}}+\Delta_{\mathbf{k}}c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger})\right],$$

Gap function $\Delta_{\mathbf{k}}\simeq\hat{\Delta}(k_x-ik_y),$

Ground state

$$\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\frac{1}{2^{N/2}(N/2)!}\sum_{P}\operatorname{sgn}P\times\prod_{i=1}^{N/2}g(\mathbf{r}_{P(2i-1)}-\mathbf{r}_{P(2i)}),$$

where $g(\mathbf{r})$ is the inverse Fourier transform of $g_{\mathbf{k}}$,

$$g(\mathbf{r})=L^{-2}\sum_{\mathbf{k}}e^{i\mathbf{k}.\mathbf{r}}g_{\mathbf{k}},\qquad g_{\mathbf{k}}=v_{\mathbf{k}}/u_{\mathbf{k}}.$$

$$g(\mathbf{r})\sim 1/z$$

in the “weak-pairing” phase --- MR form at large distances NR, Green 2000

Bogoliubov-de Gennes equation for eigenmodes

$$i\frac{\partial u}{\partial t} = -\mu u + \Delta^* i\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)v,$$

$$i\frac{\partial v}{\partial t} = \mu v + \Delta i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)u,$$

is Dirac eq, and $u(\mathbf{r},t) = v(\mathbf{r},t)^*$ makes these (massive) Majorana fermions (class D in symmetry classification)

On a vortex in superconducting order parameter, there is a Majorana zero mode. NR, Green 2000

On an edge, there is a 1+1 chiral Majorana mode.

These are the properties of the Ising part of the MR state --- charge sector is here a s.c., not QH. Otherwise, same topological phase. NR, Green 2000

Remarks on topological phases and gauge theory

(in wide sense, not only gapped phases)

Emergent gauge theories in strongly-correlated CM physics:

Kondo problem NR, Newns 1983 U(1) gauge theory, SU(N), N=6 or 8 flavour, 1/N exp

Antiferromagnets, t-J models etc NR, Sachdev 89, 91

Quantum Hall --- CS gauge thy Derivations?

Kitaev/Levin-Wen models ("string-nets" etc) lattice gauge thy, generalized to tensor cats

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"Long range entanglement" as central topic of study

-----techniques of tensor categories, K-theory, . . .

-----but also **gauge theory**, sufficiently generalized

Are all top phases gauge theories (generalized)?

What sorts can occur in condensed matter? Gravity? Classification?

Conclusion