The Supersphere

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arXiv:0801.0444,....,0809.1046,1001.1344 [hep-th]

Superspace Sigma Models

Aim: Study non-linear sigma models with target space supersymmetry **not world-sheet**

- **Strings in AdS backgrounds [pure spinor]**
- **Cond mat systems w. random disorder**

Focus on scale invariant QFT, i.e. 2D CFT

Properties: Weird: logarithmic conformal field theories !

Remarkable: Many families with cont. varying exponents

Examples: Super-Cosets

$\text{Families}_R \leftarrow$ compact symmetric Sspaces

Radius ↗

only metric, no dilaton, B-field,H-flux

$$\frac{OSP(2S+2|2S)}{OSP(2S+1|2S)} \rightarrow S^{2S+1|2S}$$

$c = 1$

[Read,Saleur]

.

.

[Sethi][Schwarz]

$$\frac{GL(N|N)}{GL(N-1|N)} \rightarrow CP^{N-1|N}$$

:

$c = -2$

Calabi-Yau superspace

- note: c^v (GL(N|N)) = 0 = c^v (OSP(2S+2|2S))
- extension to non-compact known; G/G^{Z_4} ?

The Supersphere $S^{2S+1|2S}$

$$S^{2S+1|2S} = \{C := \sum_{i=1}^{2S+2} x_i^2 + 2\sum_{a=1}^{S} \eta_{2a-1}\eta_{2a} = 1\}$$

Family of CFTs with continuously varying exp.

parameter R

$$X = (x, \eta)$$

$$\mathcal{S}_R \sim R^2 \int d^2z \partial X_a \bar{\partial} X^a$$

+ constraint

$$C(X_a) = 1$$

cp. PCM on S^3 → massive flow

Solving constraints → non-linear action:

$$\mathcal{S}_R \sim R^2 \int d^2z (1 - 2\eta_1\eta_2)\left(\partial\varphi_1\bar{\partial}\varphi_1 + \cos^2\varphi_1\,\partial\varphi_2\bar{\partial}\varphi_2 + \sin^2\varphi_1\,\partial\varphi_3\bar{\partial}\varphi_3\right) + \ldots$$

Main Results & Plan of Talk

Explicit formula for $Z^R(S^{3|2})$ w. Neumann BC

Application: $S^{3|2}$ dual to Gross-Neveu Model

conjectured by [Candu, Saleur]

extends $S^1 \leftrightarrow$ massless Thirring

Plan:

- Cohomological Reduction of Sspheres
- The exact spectrum of the Ssphere $S^{3|2}$
- Application: The duality with GN-Model
- Conclusion and some Open Problems

I.1 Cohomological Reduction

$S^{1|0} \leftrightarrow$ compactified free boson

Consider Q in h = Lie H $\subset$ g = Lie G w. $Q^2 = 0$

$$\mathcal{S}_{G/H} = \mathcal{S}_{G'/H'} + Q\mathcal{K}$$

[Candu,Creutzig, Mitev,VS]

$$g' = H_Q(g) \qquad H_Q(h) = h'$$

and for observables $H_Q(\mathcal{A}_{G/H}) = \mathcal{A}_{G'/H'}$

e.g. $S^1 \to S^{3|2} \to S^{5|4} \to \cdots \to S^{2N+1|2N}$

let's check for zero modes

II.2 Reduction of Representations

Representation V of g $\longrightarrow$ cohomology $H_Q(V)$

- $H_Q(V)$ is a representation of $H_Q(g) = g'$
- *sdim* V = *dim* V_0 – *dim* V_1 = *sdim* $H_Q(V)$
- $H_Q(V) = 0$ when V is projective module

long or very special combination of short multiplets

Examples:

$H_Q(osp(4|2)) = o(2)$ — ***sdim* g = 9 – 8 = 1 = *sdim* g'**

$H_Q(osp(3|2)) = 0$ — ***sdim* h = 6 – 6 = 0**

$H_Q(F^{4|2}) = F^{2|0}$ — ***sdim* $F^{4|2}$ = 2 = *sdim* $F^{2|0}$**

Reduction for zero modes

↔ non-derivative fields

$$Z_0(z_1,z_2,z_3) = \lim_{t\to 1}(1-t^2)\frac{(1+z_1^{1/2}t)(1+z_1^{-1/2}t)}{(1-z_2^{1/2}z_3^{1/2}t)\cdots(1-z_2^{-1/2}z_3^{-1/2}t)}$$

$$= 1 + \sum_j \chi_{[\frac{1}{2},\frac{j}{2},\frac{j}{2}]}(z_1,z_2,z_3)$$

decomposition into harmonics

Representations [1/2,j/2,j/2] realized on $V^{k+2|k}$

$H_Q(V^{k+2|k}) = V^{2|0}$

short/atypical

$k = 2j^2$

$$H_Q(Fun(S^{3|2})) = Fun(S^1)$$

most derivative fields do not contribute to H_Q

I-II The free Boson - Summary

Neumann BC → Z

moduli dependence → $q^{\frac{1}{2R^2}m^2}$

$$Z \sim \sum_{m\in\mathbb{Z}} \frac{1}{\eta(q)}\; q^{\frac{1}{2R^2}m^2} z^m = \sum_{m\in\mathbb{Z}} \psi_m(q)\; q^{\frac{1}{2R^2}m^2} \chi_m(z)$$

$$\longrightarrow q^{-\frac{1}{24}} \phi(q) \lim_{t\to 1}(1-t^2) \prod_{n=0}^{\infty} \frac{1}{(1-z^{1/2}q^n)(1-z^{-1/2}q^n)}$$

$q^0 = t$

- R-dependence through universal fct $f \sim 1/2R^2$
- Branching fcts ψ_m can be computed at $R = \infty$
- Exponent fm^2 depends only on o(2) label m

II.0 Spectrum of σ-Model on $S^{3|2}$

[Mitev,Quella,VS]

$$Z^R(q;z_i) = \sum_\Lambda \psi_\Lambda(q)\; q^{\frac{1}{2}\frac{1}{R^2}C^{(2)}(\Lambda)}\; \chi_\Lambda(z_1,z_2,z_3)$$

- **Obtained by summing all order perturbative expansion**

  possible because of target space SUSY

- **Tested through extensive numerical lattice simulations**

II.1 Spectrum of σ-Model on $S^{3|2}$

[Mitev,Quella,VS]

$$Z^R(q; z_i) = \sum_\Lambda \psi_\Lambda(q)\ q^{\frac{1}{2}\frac{1}{R^2}C^{(2)}(\Lambda)}\ \chi_\Lambda(z_1, z_2, z_3)$$

$\Lambda = [j_1, j_2, j_3]$

many more Λ than in MSS!!

Character $\chi_\Lambda = \chi_\Lambda(z_1,z_2,z_3)$ of representation Λ of osp(4|2)

e.g. trivial rep 0 = [0,0,0], fund. rep F = [½,½,½]

$$\chi_0 = 1 \qquad \chi_f = z_1^{\frac{1}{2}} + z_1^{-\frac{1}{2}} + z_2^{\frac{1}{2}} z_3^{\frac{1}{2}} + \cdots + z_2^{-\frac{1}{2}} z_3^{-\frac{1}{2}}$$

II.2 Spectrum of σ-Model on $S^{3|2}$

[Mitev,Quella,VS]

$$Z^R(q;z_i) = \sum_\Lambda \psi_\Lambda(q)\ q^{\frac{1}{2}\frac{1}{R^2}C^{(2)}(\Lambda)}\ \chi_\Lambda(z_1,z_2,z_3)$$

Branching fcts at R = ∞ from decomposition of

$$Z^{R=\infty} = q^{-\frac{1}{24}}\phi(q)\lim_{t\to 1}(1-t^2)\prod_{n=0}^{\infty}\frac{(1+z_1^{1/2}q^n)(1+z_1^{-1/2}q^n)}{(1-z_2^{1/2}z_3^{1/2}q^n)\cdots(1-z_2^{-1/2}z_3^{-1/2}q^n)}$$

Euler fct

q^0 = t

II.2_1 The Branching functions

From following decomposition of Z^R at R = ∞

$$Z^{R=\infty}(q;z_i) = \sum_{\Lambda} \psi_\Lambda(q)\chi_\Lambda(z_1,z_2,z_3)$$

→ Branching functions **recall Λ = [j_1, j_2, j_3]**

$$\psi_{[j_i]}(q) = \frac{q^{-\frac{1}{2}C^{(2)}[j_i]}}{\eta(q)\phi(q)^3} \sum_{n,m=0}^{\infty} (-1)^{m+n} q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1}$$

$$\times \left(q^{(j_2-\frac{n}{2})^2} - q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2} - q^{(j_3+\frac{n}{2}+1)^2}\right)$$

replace ψ_m ~ 1/η for free boson

II.3 Spectrum of σ-Model on $S^{3|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^R(q;z_i) = \sum_\Lambda \psi_\Lambda(q)\; q^{\frac{1}{2}\frac{1}{R^2}C^{(2)}(\Lambda)}\; \chi_\Lambda(z_1,z_2,z_3)$$

Value of Quadratic Casimir in representation of osp(4|2)

$$C^{(2)}[j_1,j_2,j_3] = -4j_1(j_1-1)+2j_2(j_2+1)+2j_3(j_3+1)$$

can be positive and negative

Casimir evolution of weights for Ssphere may be established w. background field expansion

II.3_1 Casimir evolution of Weights

Free Boson: $$\Delta^R_\Phi = \Delta^0_\Phi + f(R) g^2_\Phi$$

In boundary theory

bulk more involved

at $R=R_0$ universal U(1) charge

<u>Prop.:</u> For boundary spectra of superspheres:

$$\Delta^R_\Phi = \Delta^0_\Phi + f(R) C^{(2)}_\Phi$$

quadratic

Casimir

Casimir evolution of the conformal weights Δ

[Bershadsky et al] <u>[Quella,VS,Creutzig]</u> [Candu, Saleur]

<u>Example:</u> mult. (x,η) $\Delta^R = \Delta^{R=\infty} + f(R)\, C_F = 0 + f(R)\, 1 = f(R)$

fund rep: $C_F = 1$

II.4 Spectrum of σ-Model on $S^{3|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^R(q;z_i) = \sum_{\Lambda} \psi_\Lambda(q)\; q^{\frac{1}{2}\frac{1}{R^2}C^{(2)}(\Lambda)}\; \chi_\Lambda(z_1,z_2,z_3)$$

Universal coefficient fct.

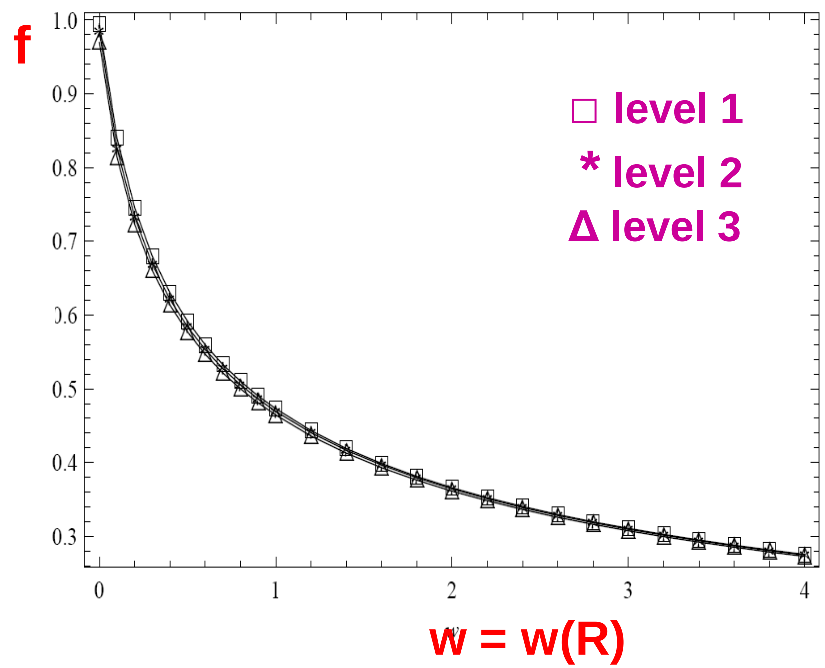
$f(R) = 1/2R^2$ (as for S^1)

← Cohomological reduction

$f(R) = \Delta^R(X)$

$H_Q(F)$ is fundamental of o(2)

$F = [½,½,½]$



II.4$_1$ Lattice Model for Supersphere

[Read, Saleur]
[Candu, Saleur]

I = | | P = X   E = 

$$H_\Delta = -\frac{1+\Delta}{2}\sum_{j=1}^{L-1}(I+P_j) - \frac{1-\Delta}{2}\sum_{j=1}^{L-1} E_j$$

extension of XXZ to S > 0   **acts on $V_F \times V_F \times \ldots\ldots \times V_F$**

Numerical studies ⟶ **not integrable: S≠0, Δ≠0**

Combinatorial PF agrees with $Z^R(S^{2S+1|2S})$!

$\Delta = \cos \pi/R^2$

Remark: $R^2 = 1 \leftrightarrow \Delta = -1$ i.e. no intersections

intersections generate deformation to large volume

III Duality w. Gross-Neveu Model

Compactified$_R$ free boson ↔ massless Thirring:

$R^2 = 1+g^2$

$$\mathcal{S}^{\text{Th}}_{m=0} \sim \int d^2z \sum_{i=1}^{2} \left[\psi_i \bar{\partial} \psi_i + cc + g^2 (\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2)^2\right]$$

Claim: [Candu,Saleur] Ssphere SM dual to GN:

2S+2 real fermions — S βγ-systems c=-1 — $h_\psi = h_\beta = h_\gamma = 1/2$

$$\mathcal{S}^{\text{GN}} \sim \int d^2z \left[\sum_i \psi_i \bar{\partial} \psi_i + \sum_a \beta_a \bar{\partial} \gamma_a\right] + cc$$

c=1 CFT with affine osp(2S+2|2S) ; k=1

$$+g^2 \int d^2z \left[\sum_i \psi_i \bar{\psi}_i + \sum_a (\gamma_a \bar{\beta}_a - \beta_a \bar{\gamma}_a)\right]^2 \sim J_\mu J^\mu$$

rule: x → ψ η → β,γ

[Mitev, Quella, VS]

III States in the free GN Model

Free GN has unique boundary state with J = $\bar{J}$

$$Z^{\mathrm{GN}}_{g^2=0} = \frac{\theta_3(q^2,z_2)\theta_3(q^2,z_3)+\theta_2(q^2,z_2)\theta_2(q^2,z_3)}{\eta(q)\theta_4(q,z_1^{1/2})}$$

sum of osp(4|2) characters at k=1

$$\sim\ q^{-\frac{1}{24}}\ \left(1+q^{\frac{1}{2}}\chi_f+q\ \chi_{\mathrm{ad}}+\dots\right)$$

How are Z^R and Z^{GN} related ?

III Main Result: SS/GN Duality

[Mitev, Quella, VS]

$$Z^{\rm SS}_{R=1}(z_i, q) = \sum_\Lambda \chi_\Lambda(z_i) q^{\frac{1}{2}C^{(2)}_\Lambda} \Psi_\Lambda(q) = Z^{\rm GN}_{g^2=0}$$

<u>Example:</u> fundamental multiplet X

$$\Delta^{R=1}(X) = f(R)|_{(R=1)} = ½ = \Delta\,(\psi)$$

IV.1 The Chiral Field on CP$^{N-1|N}$

$$CP^{N-1|N} = \left\{(Z_\alpha) = (z_\alpha, \eta_\alpha) | \varrho^2 = \bar{Z}_\alpha Z^\alpha = 1\right\} / U(1)$$

z_α , η_α →
ωz_α , $\omega \eta_\alpha$ = $U(N|N)/U(N-1|N) \times U(1)$

→ 2 parameter family of 2D CFTs;c = -2

$$\mathcal{S}_{\rm CP} = \frac{R^2}{2\pi} \int_{\varrho^2=1} d^2 z \left(D\bar{Z}_\alpha \bar{D} Z^\alpha + D Z_\alpha \bar{D} \bar{Z}^\alpha\right)$$

a - non-dynamical gauge field

D = ∂ - ia

$$+ \frac{i\theta}{2\pi} \int d^2 z \left(D\bar{a} - \bar{D} a\right)$$

Non-abelian ext. of symplectic fermion **bc ghost**

IV.2 The Spectrum of $CP^{1|2}$

For volume filling branes with $\mathcal{O}(k)$ line bundle

[Candu,Mitev,Quella,Saleur,VS]

Character $\chi_\Lambda = \chi_\Lambda(x,y,z)$ of representation Λ of u(2|2)

Branching fcts

$$Z^t_{k_1,k_2} = q^{\frac{1}{2}\lambda_t(\lambda_t-1)} \sum \psi^K_\Lambda(q)\, q^{\frac{\lambda_t}{2|k_1-k_2|}\delta C^{(2)}(\Lambda)}\, \chi^K_\Lambda(x,y,z)$$

$$\cos 2\pi\lambda_t(k_1,k_2) = \frac{(R^4+\Theta_1\Theta_2)^2 - (\Theta_1-\Theta_2)^2 R^4}{(R^4+\Theta_1\Theta_2)^2 + (\Theta_1-\Theta_2)^2 R^4}$$

$$t = R^2 + i\theta$$

$$\Theta_i = 2k_i + \theta/\pi$$

Conclusions and Open Problems

- Spectra of $S^{2S+1|2S}$ & $CP^{N-1|N}$ mod computed

  → **SS/GN duality!** **correlators ? ws SUSY ?**

- Is there WZ-point in mod space of $CP^{N-1|N}$?

  e.g. psu(N|N) at level k=1 **$CP^{0|1} = PSU(1|1)_{k=1}$**

  Systematics of dualities? **cp.``Phases of N=2 ..”**

- Extension to non-compact cosets & G/G^{Z_4}?

  quasi-ablian evolution in AdS_3 [Quella, VS, Creutzig]

- Massive deformation of the SS/GN duality?

  extend Sine-Gordon – massive Thirring duality

Some Speculation

Geometry (free Sigma Model) emerged from system of statistical mechanics (loop model)

Could AdS_5 geometry emerge in this way ?

non-intersecting loop model involving singleton representation of PSU(2,2|4)

Could repackaging of non-intersecting loop model give free N=4 Yang-Mills ?