PROBLEMS ON QUIVER VARIETIES

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1. Problems related to geometry/topology

(1) Study the class of hyper-Kähler manifolds which are hyper-Kähler reductions of finite dimensional quaternion vector spaces by products of unitary groups. (Probably it is better to assume that the action is linear.) Besides quiver varieties, hyper-Kähler toric varieties in the sense of Bielawski and Dancer [2] are such examples. When the quotients are nonsingular ? How much of geometric properties of quiver varieties can be generalized to hyper-Kähler manifolds in the class ?

(2) Let G be a compact Lie group. Moduli spaces of G-monopoles on $\mathbf{R}^3$ are identified with moduli spaces of Nahm's equations on the products of intervals, where each vertex of the Dynkin diagram of G gives an interval, and we impose the boundary conditions according to edges of the diagram. This result was shown when G is a classical group and the symmetry is maximally broken [6]. Extend this result to more generaly symmetry breaking cases. Then study moduli spaces of Nahm's equations associated with arbitrary Dynkin diagram, not necessarily classical, finite type. They are probably isomorphic to moduli spaces of rational curves into partial flag manifolds attached with the corresponding Kac-Moody Lie algebra. Study their geometries. Relate them to the corresponding representation theory and the theory of Gromov-Witten invariants. See [3] for the study in these directions.

(3) In [13, 14] I have given an algorithm to compute Betti numbers of quiver varieties using the torus action and virtual Hodge polynomials. But the algorithm has a recursive structure and it is practically hard to perform the computation. We need to study many quiver varieties at the same time. (Each quiver variety corresponds to a weight space of a representation of the Kac-Moody Lie algebra corresponding to the quiver. We fix a representation and need to study all quiver varieties corresponding to all weight spaces. So far, I have not succeeded the computation for two fundamental representations for $E8$.) It is desirable to give *closed* formula of Betti numbers. Such formula should exists also for quiver varieties of affine types, where having the algorithm are not a strong statment since we have infinitely many nonempty quiver varieties.

2. Problems related to quantum affine/toroidal algebras

(1) Extend the theory of quiver varities to *non*-symmetric Kac-Moody Lie algebras. For crystal bases, it can be done by using the folding of the quiver [18].

(2) Equivariant K-homology groups of quiver varieties of finite types are representations of quantum affine algebras [12]. In [15] it was shown that these representations are isomorphic to extremal weight modules introduced by Kashiwara [7, 8]. In particular, they have global crystal bases. The elements of the bases have the following characterization (up to sign): they are bar-invariant and almost orthonormal. The bar involution and the inner

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product are the same as ones defined geometrically by using the Grothendieck-Verdier duality and the intersection pairing together with the braid group operator associated with the longest element in the Weyl group [11, 17]. Give a geometric description of Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$. Note that Kashiwara-Saito [9] gave a geometric description of $\tilde{e}_i$, $\tilde{f}_i$ for integrable highest weight modules.

(3) Quiver varieties of affine types are moduli spaces of instantons on ALE spaces. Equivariant K-homology groups of them are representations of quantum toroidal algebras [12]. It is not possible to give geometric definitions of the bar involution and the inner product since the necessary ingredients the longest element for the affine Weyl group does not make sense. Also it is unclear how to define them algebraically since Drinfeld generators are not suitable for these purpose. However, it seems likely that equivariant K-homology groups have natural bases which have potential important applications to representation theory. I have the case of Hilbert schemes of points on the plane in my mind. The tautological bundles give natual bases and have very important in Haiman's work on Macdonald polynomials [5].

(4) Give a geometric construction of the Drinfeld-Jimbo quantum enveloping algebra in terms of equivariant constructible sheaves on the space of representations of the quiver. This space is not a quiver variety in my sense. It is the product of two copies of vector spaces used by Ringel/Lusztig (or probably, one need to introduce the framing). If this is possible, give a geometric construction of Kashiwara's extremal weight modules for the space associated with the affine quiver. If this is also possible, show that the derived category of equivariant coherent sheaves on the quiver varieties of finite type is equivalent to the derived category of equivariant constructible sheaves on the space of affine type. This is a quiver variety analog of the result [1].

(5) Give a purely algebraic characterization of the class of *small* representations introduced in [14]. Study their tensor product decompositions. This is probably a combinatorial problem.

3. Intersection theory on quiver varieties

This part is less understood since a development is very new [16]. I hope to add more in the symposium.

(1) Consider equivariant K-homology groups of quiver varieties. They are representations of the quantum affine (resp. toroidal) algebra when the underlying quiver is of type finite (resp. affine). The tensor products of tautological vector bundles (or more generally their associated bundles corresponding to irreducible representations of unitary groups) form commuting family of operators on equivariant K-groups. Write down them in terms of the quantum affine/toroidal algebra. This is probably a first step toward conceptual understanding of Nekrasov's conjecture solved in [16].

(2) By [4, 10] the (small) quantum cohomology groups of the flag manifold G/B is the quotient of the polynomial algebra modulo integral of motions of the Toda system of the Langlands dual G^L. A 'quantization' of this result was obtained recently by Braverman (talk at Cortona, 2003 June). An extension of his result to partial flag manifolds is probably related to the first question.

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