

Tightness and computing distances in the curve complex

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We address the lack of local-finiteness in Harvey’s curve complex by computational means, computing bounds on certain intersection numbers among curves lying on a natural family of geodesics. We give a finite time algorithm for constructing all of these geodesics between any two curves, and treat stable lengths.

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1 Introduction

A borderless, connected and orientable finite type topological surface is *non-exceptional* if and only if it is not a sphere with at most four holes or a torus with at most one hole.

In (13) Harvey associates to such a surface Σ its curve complex, a simplicial complex whose 1-skeleton we call the *curve graph* and denote by $\mathcal{G}(\Sigma)$. By encoding some of the asymptotic geometry of Teichmüller’s metric, the curve graph plays a central role in the celebrated proof of Minsky and his collaborators of Thurston’s ending lamination conjecture (25; 5). A key step is a theorem of Masur-Minsky’s (23; 3) that every curve graph is hyperbolic in the sense of Gromov. The curve graph is however nowhere locally finite and often two vertices are connected by infinitely many geodesics. The classical theory of locally compact hyperbolic spaces does not apply.

To overcome this issue Masur-Minsky (24) identified the first *tight geodesics*, a natural class of geodesic paths invariant under the action of the mapping class group, and proved any two vertices of the curve graph are connected by at least one and only finitely many tight geodesics. The arguments of Masur-Minsky (24) and of Bowditch (4) are not constructive as they involve taking limits of sequences. The argument of Bowditch yields constant but non-computed bounds. We propose a constructive alternative offering computable but non-constant bounds.

Theorem 1.1 *Let Σ be any non-exceptional surface. There exists an explicit function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any tight geodesic $(\alpha_0, \dots, \alpha_n)$ in $\mathcal{G}(\Sigma)$, both $\iota(\alpha_0, \alpha_j)$ and $\iota(\alpha_j, \alpha_n)$ are at most $F(\iota(\alpha_0, \alpha_n))$ for each index j .*

From this we recover Masur-Minsky’s finiteness theorem.

Theorem 1.2 (Masur-Minsky) *Let Σ be any non-exceptional surface. Between any two vertices of $\mathcal{G}(\Sigma)$ there are only finitely many tight multigeodesics.*

Geodesic paths are examples of multigeodesics, as indicated in Section 2.15. A “multigeodesic” is a sequence of sets of pairwise disjoint curves such that successively choosing a single curve arbitrarily from each always determines a geodesic path. See Sections 2.12–2.17 for formal definitions.

In Section 3 we concentrate on proving the following key statement.

Proposition 1.3 *Let Σ be any non-exceptional surface. There exists an explicit increasing polynomial function $F_1 : \mathbb{N} \rightarrow \mathbb{N}$ such that $k \leq F_1(k)$ for any non-negative integer $k \in \mathbb{N}$ and such that, for any multigeodesic (ν_0, \dots, ν_n) in $\mathcal{G}(\Sigma)$ of length at least 2 and tight at ν_1 , we have $\iota(\nu_1, \nu_n) \leq F_1(\iota(\nu_0, \nu_n))$.*

The degree of F_1 depends only on the topology of the surface. An inductive argument given in Section 4 yields the following statement, and Theorem 1.1.

Corollary 1.4 *Let Σ be any non-exceptional surface. There exists an explicit increasing function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that for any tight multigeodesic (ν_0, \dots, ν_n) in $\mathcal{G}(\Sigma)$, both $\iota(\nu_0, \nu_j)$ and $\iota(\nu_j, \nu_n)$ are at most $F(\iota(\nu_0, \nu_n))$ for each index j .*

The curves found on tight geodesics connecting two given curves belong to a computable set whose cardinality can be given an explicit upper bound. From this we can construct in explicitly bounded time all of the tight geodesics connecting any two curves.

Theorem 1.5 *There exists an explicit finite time algorithm which takes as input a non-exceptional surface Σ and two curves α and β on Σ , and returns all the tight geodesics in $\mathcal{G}(\Sigma)$ between α and β .*

The algorithm is admittedly impractical, with astronomical running time even for fairly small distances. Nevertheless, reading-off the length of any geodesic path found this way does prove the following statement.

Corollary 1.6 *There exists an explicit finite time algorithm which takes as input a non-exceptional surface Σ and two curves α and β on Σ , and returns $d(\alpha, \beta)$.*

Remark A version of Corollary 1.6 for closed surfaces of genus at least 2 is given in the unpublished thesis of Jason Leasure, see Corollary 3.2.6 of (19) where Leasure finds in a search space of curves a single geodesic connecting two given curves. We learned of this from Richard P. Kent IV (17) after submitting for publication.

Remark In giving a proof to Proposition 1.3 it will transpire that assuming our multipaths are geodesic along their whole length is far stronger than we need. We do not exploit this here, instead exploring such matters in (28) to study the action of the mapping class group on the curve graph from an entirely computational perspective.

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2 Background

We supply all the definitions and background needed both to understand the statements of our results and their proofs.

2.1 Surfaces

A *surface* Σ is a borderless, connected and orientable topological 2-manifold with finitely generated fundamental group, of genus $g(\Sigma)$ with $n(\Sigma)$ punctures. These surfaces are uniquely determined up to homeomorphism by their genus and number of punctures (27; 26). We define $\xi(\Sigma) := \max\{3g(\Sigma) + n(\Sigma) - 3, 0\}$, a quantity commonly referred to as the *complexity* of Σ . The surface is *exceptional* if and only if $\xi(\Sigma) \leq 1$ and *non-exceptional* otherwise. We sometimes speak of a subset of a surface, such as a circle or an interval, as “being on” the surface.

2.2 Homotopy versus isotopy

Two subsets C_0 and C_1 of Σ are *freely homotopic* if there exists a continuous map $H : C_0 \times [0, 1] \rightarrow \Sigma$ such that $H(C_0 \times \{i\}) = C_i$ for $i \in \{0, 1\}$ and *freely isotopic*

if in addition the homotopy H is an *isotopy*, so that $H|_{C_0 \times \{t\}} : C_0 \times \{t\} \rightarrow \Sigma$ is a homeomorphism onto its image for all $t \in [0, 1]$. Given a third subset B , the sets C_0 and C_1 are *freely homotopic* (resp. *freely isotopic*) *relative to* B if there exists a homotopy (resp. isotopy) H such that $H^{-1}(B) = (C_0 \cap B) \times [0, 1]$.

2.3 Loops and curves

A *loop* on a surface is a homeomorphic image of the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Thus by definition all loops are simple and unoriented.

A loop is said to be *trivial* if it is the image of the restriction to \mathbb{S}^1 of a continuous map from the closed disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. A loop is said to be *peripheral* if it is the image of the restriction to \mathbb{S}^1 of a continuous map from the punctured disc $\mathbb{D} \setminus \{0\}$ whose image is not contained in any compact set. A loop is said to be *essential* if it is neither trivial nor peripheral.

On a surface two essential loops are freely homotopic if and only if they are freely isotopic (see (20), Proposition B.4.6 of (1) or Proposition 1.7 of (7)). We denote the free homotopy equivalence class of an essential loop c by $[c]$ and say that c *represents* the class $[c]$. Each such class is called a *curve*. We denote the set of all curves on Σ by $X(\Sigma)$. A surface Σ is non-exceptional if and only if $X(\Sigma)$ is not empty.

2.4 Intersection numbers

Associated to any pair of curves α and β is their *intersection number* $\iota(\alpha, \beta)$ defined equal to $\min\{|a \cap b| : a \in \alpha, b \in \beta\}$. Any curve can be represented by two disjoint loops and thus has zero intersection number with itself. A pair of curves of intersection number zero is said to be *disjoint* and is otherwise said to *intersect*.

We recall the following fact concerning intersection numbers.

Lemma 2.1 *Let Σ be any non-exceptional surface and let a and b be two essential loops on Σ such that $a \cap b$ is a finite set. Then $|a \cap b| = \iota([a], [b])$ if and only if there exist no closed discs $D \subset \Sigma$ such that ∂D is the union of a compact interval contained in a and a compact interval contained in b .*

A proof of Lemma 2.1 can be found in Section 1.2 of (7).

2.5 Multiloops and multicurves

A *multiloop* is a non-empty union of pairwise disjoint and pairwise non-homotopic essential loops. A *multicurve* is a non-empty set of pairwise distinct and pairwise disjoint curves, also viewed as the free homotopy class of a multiloop. We may view a loop as a multiloop with one component, and a curve as a multicurve with one element. We denote the free homotopy class of a multiloop ν by $[\nu]$. The intersection number of two multicurves or a curve and a multicurve is defined additively.

A *pants decomposition* is a multicurve maximal subject to inclusion. All pants decompositions of Σ have cardinality $\xi(\Sigma)$.

2.6 Hyperbolic geometry

The Klein-Poincaré uniformization theorem implies that a surface of negative Euler characteristic such as a non-exceptional surface admits a complete hyperbolic Riemannian metric of constant negative curvature -1 . Any curve (resp. multicurve) is uniquely represented by a simple closed geodesic (resp. union of pairwise disjoint simple closed geodesics) in such a metric. (See Lemma 2.3 of (6), Proposition B.4.7 of (1), Proposition 1.5 of (7) or Lemma 2.4.4 of (16) for instance.) Given two curves or multicurves μ and ν on Σ , if $u \in \mu$ and $v \in \nu$ denote their unique geodesic representatives in a hyperbolic metric then $\iota(\mu, \nu) = |u \cap v|$. (See Lemma 2.6 of (6) or Lemma 1.4 of (7) for instance.)

We recall the following fact concerning the lifts of two loops to the universal cover.

Lemma 2.2 *Let Σ be any non-exceptional surface and let a and b be two essential loops on Σ such that $|a \cap b| = \iota([a], [b])$. Then, for any universal covering map $\pi : \mathbb{H}^2 \rightarrow \Sigma$, the intersection of any component of the preimage $\pi^{-1}(a)$ and any component of the preimage $\pi^{-1}(b)$ is either empty or a single point.*

A proof of Lemma 2.2 can be found in Section 1.2 of (7).

2.7 Arcs

An *arc* on a surface is a homeomorphic image of the compact interval $[0, 1]$ of real numbers. All arcs are therefore simple and unoriented. On a surface two arcs with common endpoints are homotopic relative to their endpoints if and only if they are isotopic relative to their endpoints (9). An arc properly contained in a given arc, loop or multiloop ν is referred to as a *subarc* of ν .

2.8 Properly embedded arcs

For any subset $A \subseteq \Sigma$ we say that an arc h is *properly embedded in* (Σ, A) if $h \cap A = \partial h$ and there are no closed discs whose boundary is the union of a compact subinterval of h and a compact interval contained in A . More generally, for two subsets $A, B \subseteq \Sigma$ we say that an arc h is *properly embedded in* $(\Sigma, \{A, B\})$ if $\partial h \cap (A \cup B) = \partial h$, the intersection $h \cap A \cap B$ is empty, and there are no closed discs whose boundary is the union of a compact subinterval of h and a compact interval entirely contained in at least one of A and B . An arc is properly embedded in (Σ, A) if and only if it is properly embedded in $(\Sigma, \{A, \emptyset\})$.

2.9 Parallel arcs

Two properly embedded arcs h_0 and h_1 are *parallel in* (Σ, A) if there exists a continuous map $H : h_0 \times [0, 1] \rightarrow \Sigma$ such that $H^{-1}(A) = (h_0 \cap A) \times [0, 1]$ and $H(h_0 \times \{i\}) = h_i$ for $i \in \{0, 1\}$. More generally, two arcs h_0 and h_1 such that $\partial h_0 \subseteq A$ and $\partial h_1 \subseteq A$ are *parallel in* (Σ, A) if there exist two sets $\{h_0^1, \dots, h_0^n\}$ and $\{h_1^1, \dots, h_1^n\}$ of arcs properly embedded in (Σ, A) such that $h_0 = \bigcup_1^n h_0^i$ and $h_1 = \bigcup_1^n h_1^i$, such that both $h_0^i \cap h_0^{i+1}$ and $h_1^i \cap h_1^{i+1}$ are connected sets for each index $i \in \{1, \dots, n-1\}$, and such that h_0^i and h_1^i are parallel for each index $i \in \{1, \dots, n\}$. If h_0 and h_1 are parallel in (Σ, A) and h_1 and h_2 are parallel in (Σ, A) then h_0 and h_2 are parallel in (Σ, A) .

2.10 Disjoint arcs

Two arcs h_0 and h_1 in (Σ, A) are *homotopically disjoint* if there exists a continuous map $H : h_0 \times [0, 1] \rightarrow \Sigma$ such that $H(h_0 \times \{0\}) = h_0$, $H^{-1}(A) = (h_0 \cap A) \times [0, 1]$ and $H(h_0 \times \{1\}) \cap h_1$ is empty.

2.11 Mapping class group

The *mapping class group* $\text{Map}(\Sigma)$ is defined as the group of all homeomorphisms of Σ modulo the normal subgroup of those homeomorphisms homotopic to the identity. The mapping class group acts naturally on $X(\Sigma)$ via the group action of homeomorphisms on the set of all loops. These actions have been studied by numerous authors in numerous ways; see (2; 4; 12; 28) and references contained therein.

A mapping class $\phi \in \text{Map}(\Sigma)$ is said to be *pseudo-Anosov* if no non-zero power of ϕ fixes a curve, equivalently if $\iota(\alpha, \phi^n \beta)$ grows exponentially in n for any two curves $\alpha, \beta \in X(\Sigma)$ (see (8)). A mapping class ϕ is said to be *partially pseudo-Anosov* if no non-zero power of ϕ fixes a pants decomposition, equivalently if $\iota(\mu, \phi^n \nu)$ grows exponentially in n for any two pants decompositions μ and ν .

2.12 Harvey's curve graph

In (13) Harvey associates to a surface Σ a flag simplicial complex of dimension $\xi(\Sigma) - 1$ called the curve complex. We study the 1-skeleton of this complex, denoted $\mathcal{G}(\Sigma)$ and referred to as the *curve graph*. This is defined independently by taking the vertex set to be $X(\Sigma)$ and declaring two vertices to be joined by an edge if and only if they are distinct and, as curves, they are disjoint.

If $\xi(\Sigma) \geq 2$ the curve graph is connected and can be endowed with the canonical path-metric d by realising $\mathcal{G}(\Sigma)$ as a length space with each edge of length 1. The distance between two curves is the length of any shortest path between them. As the action of the mapping class group preserves intersection number, the action is by isometries. With only a few exceptions, the isometry group of the curve graph is in fact isomorphic to the mapping class group (15; 18; 22). For any two curves $\alpha, \beta \in X(\Sigma)$ and $\phi \in \text{Map}(\Sigma)$ a pseudo-Anosov mapping class, the distance $d(\alpha, \phi^n \beta)$ grows linearly in n (24).

The curve graphs all have infinite diameter (23; 14) but it is a simple matter to effectively characterise the first three distances: Two curves are at distance 0 if and only if they are equal; are at distance 1 if and only if they are distinct and disjoint, and are at distance 2 if and only if they intersect essentially and there exists a curve disjoint from both. Two curves α and β are at distance at least 3 if and only if every curve intersects at least one of α and β essentially, so that $\{\alpha, \beta\}$ *fills* the surface.

The distance between a pair of curves and their intersection number are related in the following well-known lemma.

Lemma 2.3 *If $\xi(\Sigma) \geq 2$, we have $d(\alpha, \beta) \leq \lfloor 2 \log_2(\iota(\alpha, \beta)) \rfloor + 2$ for all $\alpha, \beta \in X(\Sigma)$.*

Upper bounds on distance in terms of intersection number have been circulating for some time, though the earliest in print appears to be Lemma 2.1 of (23) or Lemma 2.1 of (14). Their proofs typically use an argument given by Lickorish (21). In general one can do no better than logarithmic given the existence pseudo-Anosov mapping classes.

We say that two multicurves μ and ν are of *distance* n , and write $d(\mu, \nu) = n$, if $d(\gamma, \delta) = n$ for each $\gamma \in \mu$ and $\delta \in \nu$. For two multicurves μ and ν of distance

$d(\mu, \nu) \geq 1$, if $u \in \mu$ and $v \in \nu$ are their geodesic representatives in a hyperbolic metric on the surface then $\iota(\mu, \nu) = |u \cap v| = |u \pitchfork v|$.

2.13 Intervals

An *interval of integers* is a non-empty subset of \mathbb{Z} of the form $\{n \in \mathbb{Z} : i \leq n \leq j\}$ for some $i, j \in \mathbb{Z} \sqcup \{\pm\infty\}$ with $i \leq j$.

A *partition* of a compact interval J into *compact intervals* is a set \mathcal{J} of compact intervals such that $J = \bigcup_{\mathcal{J}} I$ and such that, for all I_1 and I_2 in \mathcal{J} , the intersection $I_1 \cap I_2$ is infinite if and only if $I_1 = I_2$.

2.14 Paths and multipaths

We regard a *path* in the curve graph as a sequences of curves $(\gamma_i)_i$ indexed by an interval of integers such that consecutive curves are distinct and disjoint, equivalently they span an edge in the curve graph. A *multipath* is a sequence of multicurves $(\nu_i)_i$ indexed by an interval of integers such that, for each $\gamma_i \in \nu_i$ and each i , the sequence $(\gamma_i)_i$ is a path. If $(\gamma_i)_i$ is a path then $(\{\gamma_i\})_i$ is a multipath.

If a given path or multipath has a finite minimal or maximal index we refer to such an index and the corresponding curve or multicurve as *terminal*. We refer to all the other indices and their corresponding curves or multicurves as *non-terminal*.

2.15 Geodesics and multigeodesics

A *geodesic* is a path $(\gamma_i)_i$ such that $d(\gamma_i, \gamma_j) = |i - j|$ for any two finite indices i and j . A multipath $(\nu_i)_i$ is a *multigeodesic* if for each $\gamma_i \in \nu_i$ and each i the sequence $(\gamma_i)_i$ is a geodesic path. If $(\gamma_i)_i$ is a geodesic then $(\{\gamma_i\})_i$ is a multigeodesic.

2.16 Relative boundary

To any pair of multicurves ν_0 and ν_2 at distance 2, Masur-Minsky (24) associate a multicurve called its *relative boundary* $\partial(\nu_0, \nu_2)$ as follows. First, choose $v_0 \in \nu_0$ and $v_2 \in \nu_2$ so that $|v_0 \cap v_2| = \iota(\nu_0, \nu_2)$ and take an open set U that admits a neighbourhood deformation retraction onto the union $\nu_0 \cup \nu_2$ and whose boundary is a disjoint union of finitely many loops. Attach to U all the closed disc and once-punctured closed

disc components of $\Sigma \setminus U$ and denote the resulting subsurface of Σ by U^* . The free homotopy class of any multiloop maximal subject to inclusion and contained in ∂U^* is a non-empty and well-defined multicurve associated to the pair (ν_0, ν_2) , its relative boundary.

2.17 Tight geodesics

Definition 2.4 (Masur-Minsky) A multigeodesic $(\nu_i)_i$ is *strongly tight at ν_j* for some non-terminal index j if $\nu_j = \partial(\nu_{j-1}, \nu_{j+1})$. We say $(\nu_i)_i$ is *strongly tight* if it is strongly tight at each non-terminal ν_j .¹

The existence of a strongly tight multigeodesic between any two curves is established in the proof of Lemma 4.5 from (24) as follows: Let us suppose $(\nu_0, \nu_1, \nu_2, \nu_3)$ is a multigeodesic strongly tight at ν_2 . We replace ν_1 with the relative boundary $\partial(\nu_0, \nu_2)$ of ν_0 and ν_2 . It can be shown that this does not affect the tightness at ν_2 , that is $(\partial(\nu_0, \nu_2), \nu_2, \nu_3)$ is strongly tight. This tightening operation is therefore stable and can be applied to the vertices of a geodesic path in any order to produce a strongly tight multigeodesic with the same endpoints. Between any two curves at distance 2 there is only one strongly tight multigeodesic. In general it is conceivable that tightening a given geodesic or multigeodesic in a different order may produce a different multipath.

An important observation concerning the relative boundary is the following.

Lemma 2.5 Suppose ν_0 and ν_2 are two multicurves at distance 2. If $\delta \in X(\Sigma)$ is any curve such that $\iota(\delta, \partial(\nu_0, \nu_2)) > 0$, then $\iota(\delta, \nu_0) > 0$ or $\iota(\delta, \nu_2) > 0$.

This brings us to the following definition.

Definition 2.6 (Bowditch) A multigeodesic $(\nu_i)_i$ is *tight at ν_j* for some non-terminal index j if for any curve $\delta \in X(\Sigma)$ whenever $\iota(\delta, \nu_j) > 0$ we have $\iota(\delta, \nu_{j-1}) > 0$ or $\iota(\delta, \nu_{j+1}) > 0$. We say $(\nu_i)_i$ is *tight* if it is tight at each non-terminal ν_j .

We note that any strongly tight multigeodesic is tight, and that if (ν_0, ν_1, ν_2) is a tight multigeodesic then $\nu_1 \subseteq \partial(\nu_0, \nu_2)$.

Definition 2.7 (Bowditch) A geodesic $(\alpha_i)_i$ is *tight* if there exists a tight multigeodesic $(\nu_i)_i$ such that $\alpha_i \in \nu_i$ for each index i .

We note that in the curve graph of the 5-holed sphere and the 2-holed torus every geodesic path is tight for the topology of the two surfaces leaves no alternative.

¹The term ‘‘tight multigeodesic’’ is used in (24).

2.18 Stable lengths

Given a metric space (X, d) and an isometry $\phi : X \rightarrow X$, the *stable length* $\|\phi\|$ of ϕ is defined equal to $\lim_{n \rightarrow \infty} d(x, \phi^n x)/n$, for any $x \in X$. It can be verified that $\|\phi\|$ is always finite and, by twice applying the triangle inequality, that $\|\phi\|$ does not depend on the choice of x .

3 Proof of Proposition 1.3

3.1 Curve surgery

We first take a pair of intersecting curves and return a new curve with useful properties.

Lemma 3.1 *Let γ and β be any two curves on Σ , and let $c \in \gamma, b \in \beta$ be loops such that $|c \cap b| = \iota(\gamma, \beta)$. Let J be any compact non-empty subinterval of b . Suppose there exists a subarc h of c such that $|h \cap J| = 3$. Then, there exists a curve $\delta \in X(\Sigma)$ represented by a loop in every open subset of Σ containing $h \cup J$ such that $\iota(\delta, \gamma) > 0$ and such that $\iota(\delta, \beta) > 0$.*

Proof We construct a parameterised loop $p : [0, 1] \rightarrow \Sigma$ as follows. Let x_1, x_2 and x_3 denote the three points of $h \cap J$ where x_2 separates x_1 and x_3 along h , and let h_1 denote the subarc of h connecting x_1 and x_2 and let h_2 denote the subarc of h connecting x_2 and x_3 . The construction of p falls into three types (see the left half of Figure 1).

- (Type 1) If there exists $i \in \{1, 2\}$ such that h_i is incident to J from both sides of b we form p by traversing h_i and then the subinterval of J spanned by ∂h_i .
- (Type 2) Otherwise, if x_2 does not separate x_1 and x_3 along J we form p by traversing h from x_1 and then the subinterval of J spanned by x_3 and x_1 .
- (Type 3) If instead x_2 separates x_1 and x_3 along J we form p by traversing h_1 from x_1 to x_2 , J from x_2 to x_3 , h_2 from x_3 to x_2 , and then J from x_2 to x_1 .

We extend p to a periodic map $p : \mathbb{R} \rightarrow \Sigma$ such that $p(t+1) = p(t)$ for all $t \in \mathbb{R}$. Let $\pi : \mathbb{H}^2 \rightarrow \Sigma$ denote a universal covering map and \tilde{p} any π -lift of p . We denote $\tilde{p}(i)$ by z_i , the π -lift of b containing z_i by \tilde{b}_i , and the π -lift of c containing z_i by \tilde{c}_i for each integer $i \in \mathbb{Z}$. The union of all \tilde{b}_i will typically be a proper subset of the preimage $\pi^{-1}(b)$, and similarly the union of all \tilde{c}_i will typically be a proper subset of $\pi^{-1}(c)$.

We note $(\tilde{b}_i)_i$ and $(\tilde{c}_i)_i$ are both sequences of uniform simple quasi-geodesics, and that \tilde{b}_{i-1} and \tilde{b}_{i+1} are separated in \mathbb{H}^2 by \tilde{b}_i and that \tilde{c}_{i-1} and \tilde{c}_{i+1} are separated in \mathbb{H}^2 by \tilde{c}_i for every $i \in \mathbb{Z}$ regardless of the type defining p . As $|c \cap b| = \iota(\gamma, \beta)$, according to Lemma 2.2 we know \tilde{b}_i intersects \tilde{c}_j in at most one point for each $i, j \in \mathbb{Z}$. In particular we note that $z_0 \neq z_2$, so $p(\mathbb{R})$ is not homotopically trivial, and that $\tilde{p}(\mathbb{R})$ intersects every \tilde{b}_i and every \tilde{c}_i in a non-empty connected set. (See the right half of Figure 1.)

Regardless of the type defining p there exists a positive real number $\epsilon_0 > 0$ such that, for any positive real number ϵ with $0 < \epsilon < \epsilon_0$, at least one component \tilde{e} of the boundary of the open ϵ -neighbourhood of the image $\tilde{p}(\mathbb{R})$ in \mathbb{H}^2 projects to a loop $\pi(\tilde{e})$ under π . We denote this loop by e . Since \tilde{e} is homotopic to $\tilde{p}(\mathbb{R})$ we note e is not a trivial loop. Moreover, \tilde{e} intersects each \tilde{b}_i and each \tilde{c}_j in a single point. It follows e is also not peripheral and must be essential, defining a curve δ whose intersection number with γ and with β is positive. \square

3.2 Realising curves

From now on (ν_0, \dots, ν_n) is a multi-geodesic tight at ν_1 , c_n is the component of $\nu_n \in \nu_n$ containing an arbitrary compact subinterval J of $\nu_n \setminus \nu_0$, and the multiloops ν_i together satisfy the following conditions:

- (1) the components of ν_i are pairwise disjoint for each i ;
- (2) ν_i and ν_j intersect transversely for $i \leq j - 2$;
- (3) $\nu_i \cap \nu_{i+1} = \emptyset$ for each $i \leq n - 1$;
- (4) $|\nu_i \cap \nu_n| = \iota(\nu_i, \nu_n)$ for each $i \leq n - 2$, and
- (5) $\nu_i \cap \nu_j \cap \nu_n = \emptyset$ for $i + 1 < j < n - 1$.

We could endow Σ with a fixed complete hyperbolic metric and represent each ν_i uniquely by a union of geodesics to give (1)–(4). Perturbing these geodesics if need be, in addition we then have (5). Lastly, we draw the reader's attention to the notational distinction between the multiloop ν_i and its multicurve $\nu_i = [\nu_i]$.

3.3 Minor intersection

Lemma 3.2 *Suppose $|c_1 \cap J| \geq 3$ for some component c_1 of ν_1 . Then $\nu_2 \cap J$ is non-empty.*

Proof Let h denote any subarc of c_1 such that $|h \cap J| = 3$. According to Lemma 3.1, there exists a curve δ represented by a loop in any open set containing $h \cup J$ and of positive intersection number with ν_1 . The tightness of the multigeodesic at ν_1 implies δ has positive intersection number with at least one of ν_0 and ν_2 . However ν_0 is disjoint from ν_1 and J , and ν_2 is disjoint from ν_1 . It follows $\nu_2 \cap J$ must be non-empty. \square

Corollary 3.3 *Suppose $|\nu_1 \cap J| \geq 2\xi(\Sigma) + 1$. Then $\nu_2 \cap J$ is non-empty.*

Proof There exists a component c_1 of ν_1 such that $|c_1 \cap J| \geq 3$. The claim now follows from Lemma 3.2. \square

3.4 Nested intersection

Lemma 3.4 *For $m \geq 2$ suppose there exists a continuous map $H : [0, 1] \times [0, 1] \rightarrow \Sigma$ such that:*

- (1) $H([0, 1] \times \{0, \frac{1}{2}, 1\}) \subseteq J$;
- (2) $H(\{0, 1\} \times [0, 1]) \subseteq \nu_1$, and
- (3) $H(\{\frac{1}{2}\} \times [0, 1]) \subseteq \nu_m$.

Let $J_0 \subseteq J$ be any compact subinterval containing the image $H(\{\frac{1}{2}\} \times \{0, \frac{1}{2}, 1\})$. Then, the set $\nu_{m+1} \cap J_0$ is non-empty.

Proof The image $H(\{\frac{1}{2}\} \times [0, 1])$ is necessarily an arc, for otherwise it is a component of ν_m disjoint from each component of ν_0 and this is absurd. Let h denote any subarc of $H(\{\frac{1}{2}\} \times [0, 1])$ such that $|h \cap J_0| = 3$, and let c_m denote the component of ν_m containing h . (See Figure 2.) We apply Lemma 3.1, with $\gamma = [c_m], \beta = [c_n], c = c_m, b = c_n$ and to h along J_0 , to find a curve δ represented by a loop in any open set containing $h \cup J_0$. Since $\nu_0 \cap h = \emptyset$, $\nu_0 \cap J_0 = \emptyset$ and the topology of a surface is regular there exists an open set U containing $h \cup J_0$ such that $U \cap \nu_0 = \emptyset$. For any loop $e_m \in \delta$ such that $e_m \subset U$ it follows $e_m \cap \nu_0 = \emptyset$. We conclude $\iota(\delta, \nu_0) = 0$.

Recall $c_m \cap \nu_{m+1} = \emptyset$. If $\nu_{m+1} \cap J_0 = \emptyset$ then, after replacing both U and e_m if need be, e_m is disjoint from ν_{m+1} . It follows that $\iota(\delta, \nu_{m+1}) = 0$. Moreover, since $\iota(\delta, \nu_0)$ is zero, δ is distinct from each component of ν_{m+1} . We therefore have a multipath $(\nu_0, \{\delta\}, \nu_{m+1})$ of length 2 and so $d(\nu_0, \nu_{m+1}) \leq 2$. This is absurd, since $d(\nu_0, \nu_{m+1}) = m + 1 \geq 2 + 1 = 3 > 2$. We deduce $\nu_{m+1} \cap J_0$ must be non-empty, and with this we complete a proof of Lemma 3.4. \square

3.5 Parallel arcs

Lemma 3.5 *Let s be a non-negative integer. There exists an explicit constant $K_0(s)$ such that, for any compact interval I on Σ , in any set of at least $K_0(s)$ pairwise homotopically disjoint arcs properly embedded in (Σ, I) there are at least s arcs parallel in (Σ, I) .*

Proof A set of pairwise homotopically disjoint pairwise non-parallel arcs properly embedded in (Σ, I) has cardinality at most $6g(\Sigma) + 2n(\Sigma) - 3$. By the pigeonhole principle, we can define $K_0(s) := \max\{(6g(\Sigma) + 2n(\Sigma) - 3)(s - 1) + 1, 0\}$. \square

Corollary 3.6 *Let s be a non-negative integer. There exists an explicit constant $K(s)$ such that in any set \mathcal{A} of at least $K(s)$ pairwise homotopically disjoint arcs each intersecting a compact interval I on Σ only three times and whose boundaries are contained in I there are at least s arcs parallel in (Σ, I) .*

Proof A fairly crude constant $K(s)$ can be obtained as follows. Suppose \mathcal{A} contains at least $K_0(K_0(s))$ such arcs. Each of these arcs is expressed as the union of two arcs properly embedded in (Σ, I) . By Lemma 3.5, there exist $K_0(s)$ arcs in \mathcal{A} containing a properly embedded subarc parallel to a common arc. Among these arcs, Lemma 3.5 implies there exist s parallel arcs. We thus set $K(s) = K_0(K_0(s))$. \square

3.6 Large intersection

Lemma 3.7 *There exists an explicit, computable and montonic increasing exponential function $G : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds. For m a non-negative integer, if $|v_1 \cap J| \geq G(m)$ then there exists a sequence nested subintervals $J_1 \supset J_2 \supset \dots \supset J_m$ of J such that:*

- (1) $\partial J_i \subset v_i$ for each $i \in \{1, \dots, m\}$, and
- (2) $v_{m+1} \cap J_m \neq \emptyset$.

Proof We define $G(0) := 0$ and note by Corollary 3.3 that $G(1) := 2\xi(\Sigma) + 1$ satisfies our assertion for $m = 1$ when J_1 is defined to be the intersection of all subintervals of J containing a common set comprising $G(1)$ pairwise distinct points of $v_1 \cap J$. We regard $m = 1$ as the base case for an induction on m .

Suppose inductively we have found $G(m)$. If $|v_1 \cap J| \geq 3K(5)G(m)$ we can partition J into the union of at least $3K(5)$ compact subintervals J_i^j , for $i \in \{1, \dots, 3K(5)\}$,

each intersecting v_1 at least $G(m)$ times and such that $\partial J_1^i \subset v_1$ for each index i . The inductive hypothesis implies the existence of a nested sequence of compact intervals $J_1^i \supset J_2^i \supset \cdots \supset J_m^i$ such that $\partial J_j^i \subset v_j$ and $v_{m+1} \cap J_m^i \neq \emptyset$ for each index i and j .

Let \mathcal{A} denote a set of subarcs h of v_1 such that $|h \cap J| = 3$, $\partial h \subset J$ and $\partial h \cap \partial J^i$ is non-empty for some index i and such that each element of ∂J_1^i is contained in some element of \mathcal{A} for each index i . The set \mathcal{A} has cardinality at least $K(5)$. We apply Corollary 3.6 to this set \mathcal{A} and (Σ, J) to find at least five pairwise parallel arcs belonging to \mathcal{A} .

Among all of these parallel arcs two are related by a homotopy $H : [0, 1] \times [0, 1] \rightarrow \Sigma$ such that $H([0, 1] \times \{0\}) = J_1^i$ for some index i and such that $H([0, 1] \times \{0, \frac{1}{2}, 1\}) \subseteq J$. As $v_j \cap v_{j+1} = \emptyset$ for each $j \in \{1, \dots, m-1\}$ and $J_1^i \supset J_2^i \supset \cdots \supset J_m^i$ we may reparameterise H so that in addition

$$I_j := H\left(\left\{\frac{1}{2} \pm \frac{m+1-j}{2m}\right\} \times [0, 1]\right) \subseteq v_j$$

for each $j \in \{1, \dots, m+1\}$ and where the images I_j are pairwise disjoint. The homotopy H can be one ‘‘swept out’’ by $J_1^i \supset \cdots \supset J_m^i$ so to speak.

Let \widehat{J}_j denote the intersection of all subintervals of J containing $I_j \cap J$ for each $j \in \{1, \dots, m+1\}$. We note that $\widehat{J}_1 \supset \widehat{J}_2 \supset \cdots \supset \widehat{J}_{m+1}$ and that $\partial \widehat{J}_j \subset v_j$ for each such j . We also note that $H(\{\frac{1}{2}\} \times \{0, \frac{1}{2}, 1\}) \subset \widehat{J}_1$, and we summarise the other important properties of H :

- (1) $H([0, 1] \times \{0, \frac{1}{2}, 1\}) \subseteq J$;
- (2) $I_1 \subseteq v_1$, and
- (3) $I_{m+1} \subseteq v_{m+1}$.

We can thus apply Lemma 3.4 along $\widehat{J}_1 \subseteq J$ to deduce $v_{m+2} \cap \widehat{J}_{m+1} \neq \emptyset$. We may therefore define $G(m+1)$ equal to $3K(5)G(m)$ and then take $\widehat{J}_1 \supset \widehat{J}_2 \supset \cdots \supset \widehat{J}_{m+1}$ as the promised nested sequence of intervals, completing the induction. \square

Remark We may take $G(m) = (2\xi(\Sigma) + 1)(3K(5))^{m-1}$ for all positive integers m .

3.7 Intersection bounds

A proof of Proposition 1.3 can be completed as follows. We define a function $F_1 : \mathbb{N} \rightarrow \mathbb{N}$ by

$$F_1(n) := nG(\lfloor 2\log_2 n \rfloor)$$

for all $n \in \mathbb{N}_+$ and $F_1(0) := 0$. We note that $F_1(n)$ is polynomial in n with degree depending only on the topology of the surface, and that $n \leq F_1(n)$ for all non-negative integers $n \in \mathbb{N}$. We also note that $\iota(\nu_0, \nu_n)$ is non-zero as $d(\nu_0, \nu_n) \geq 2$ by assumption.

Suppose for contradiction $\iota(\nu_1, \nu_n) > F_1(\iota(\nu_0, \nu_n))$, noting n is then at least 3. The intersection number $\iota(\nu_1, \nu_n)$ can be expressed as the sum of $|v_1 \cap J_i|$ over each component J_i of $\nu_n \setminus \nu_0$, of which there are exactly $\iota(\nu_0, \nu_n)$, and so there exists a compact subinterval J of $\nu_n \setminus \nu_0$ such that

$$\iota(\nu_0, \nu_n)|v_1 \cap J| \geq \iota(\nu_1, \nu_n).$$

Combining the two inequalities, we find

$$\iota(\nu_0, \nu_n)|v_1 \cap J| > F_1(\iota(\nu_0, \nu_n)).$$

Substituting in the defining expression for F_1 yields

$$\iota(\nu_0, \nu_n)|v_1 \cap J| > \iota(\nu_0, \nu_n)G(\lfloor 2\log_2(\iota(\nu_0, \nu_n)) \rfloor)$$

and after dividing both sides by the non-zero integer $\iota(\nu_0, \nu_n)$ we have

$$|v_1 \cap J| > G(\lfloor 2\log_2(\iota(\nu_0, \nu_n)) \rfloor).$$

We recall $G(n)$ is monotonic increasing in n , and appealing to Lemma 2.3 we find

$$|v_1 \cap J| > G(n - 2).$$

However, condition (2) of Lemma 3.7 implies $\nu_{n-1} \cap \nu_n$ is not empty and this is absurd. With this we complete the contradiction, and a proof of Proposition 1.3.

4 Proof of Corollary 1.4

Given a function $\Theta : \mathbb{N} \rightarrow \mathbb{N}$ we define a sequence of functions $(\Theta^m)_{m \in \mathbb{N}}$ inductively by first defining Θ^0 to be the identity on \mathbb{N} and then defining $\Theta^{m+1} := \Theta \cdot \Theta^m$ for each non-negative integer $m \in \mathbb{N}$. We use the notation $\Theta^m(n)$ for the natural number obtained by evaluating the function Θ^m at n for each $m, n \in \mathbb{N}$. In general this is distinct from the value of $\Theta(n)$ raised to the power of m , denoted $\Theta(n)^m$.

For each index $m \in \{1, \dots, n - 2\}$, by Proposition 1.3 we have

$$\iota(\nu_m, \nu_n) \leq F_1(\iota(\nu_{m-1}, \nu_n)).$$

As $k \leq F_1(k)$ for all $k \in \mathbb{N}$, we have

$$F_1(\iota(\nu_{m-1}, \nu_n)) \leq F_1(F_1(\iota(\nu_{m-2}, \nu_n))) = F_1^2(\iota(\nu_{m-2}, \nu_n)) \leq \dots \leq F_1^m(\iota(\nu_0, \nu_n)).$$

If we define $F(n) = F_1^{\lfloor 2\log_2(n) \rfloor}(n)$ for all $n \in \mathbb{N}$, then by Lemma 2.3 we have

$$\iota(\nu_m, \nu_n) \leq F(\iota(\nu_0, \nu_n))$$

for all indices m . By reversing the multigeodesic and repeating this argument we find

$$\iota(\nu_0, \nu_m) \leq F(\iota(\nu_0, \nu_n))$$

for all indices m . This completes the proof of Theorem 1.4.

Remark Substituting in all the defining expressions we find our choice of $F(n)$ is explicitly bounded above by a constant multiple of a monomial in n of fixed degree, depending only on the topology of the surface, raised to the power $\log_2 n$ for all non-negative integers n .

5 Proof of Theorem 1.2

Lemma 5.1 *Suppose Σ is a non-exceptional surface. Let $\alpha, \beta \in X(\Sigma)$ be two curves such that $d(\alpha, \beta) \geq 3$ and let K be any non-negative integer. Then, the set $\{\gamma \in X(\Sigma) : \iota(\alpha, \gamma) + \iota(\gamma, \beta) \leq K\}$ is finite and has cardinality uniformly and explicitly bounded above in terms of $\iota(\alpha, \beta)$, $n(\Sigma)$ and K .*

Proof Let $a \in \alpha$ and $b \in \beta$ be representative loops such that $|a \cap b| = \iota(\alpha, \beta)$. As $d(\alpha, \beta) \geq 3$, the two curves together fill the surface. The complement of the union $a \cup b$ is therefore the disjoint union of open discs and once-punctured open discs. For any curve $\gamma \in X(\Sigma)$ we can find a loop $c \in \gamma$ such that $|c \cap a| = \iota(\gamma, \alpha)$, $|c \cap b| = \iota(\gamma, \beta)$ and $a \cap b \cap c = \emptyset$. We can thus express c as the union of at most $\iota(\alpha, \gamma) + \iota(\gamma, \beta)$ intervals each properly embedded in $(\Sigma, \{a, b\})$.

Let Λ denote the set of all loops c equal to the union of at most K compact intervals each properly embedded in $(\Sigma, \{a, b\})$. We say that two such loops c_0 and c_1 are *equivalent* if there exists a continuous map $H : c_0 \times [0, 1] \rightarrow \Sigma$ such that $H(c_0 \times \{i\}) = c_i$ for $i \in \{0, 1\}$ and such that if $I \subset c_0$ is a compact interval properly embedded in $(\Sigma, \{a, b\})$ then $H(I \times \{t\})$ is similarly a properly embedded interval for all $t \in [0, 1]$. This defines an equivalence relation on Λ , and the set of equivalence classes is not only finite but also has cardinality that can be uniformly and explicitly bounded in terms of K and $n(\Sigma)$. Each equivalence class determines a curve and the number of curves that can be represented this way while using at most K properly embedded intervals is similarly bounded. As every curve $\gamma \in X(\Sigma)$ such that $\iota(\alpha, \gamma) + \iota(\gamma, \beta) \leq K$ can be represented in this manner, we already have an upper bound on the number of such curves. \square

Consider any two curves $\alpha, \beta \in X(\Sigma)$. If $d(\alpha, \beta) \leq 2$ then there is only one strongly tight multi-geodesic connecting α and β and therefore only finitely many tight geodesics. When $d(\alpha, \beta) \geq 3$, by Theorem 1.4 the vertices of any tight multi-geodesic, and therefore any tight geodesic, connecting α and β have intersection number with α and intersection number with β at most $F(\iota(\alpha, \beta))$. It follows from Lemma 5.1 with $K = 2F(\iota(\alpha, \beta))$ that only finitely many curves can appear on tight geodesics connecting α and β . There are thus only finitely many tight geodesics connecting α and β . This completes the proof of Theorem 1.2.

Remark All the dependence on the genus $g(\Sigma)$ is subsumed by $\iota(\alpha, \beta)$ since $\{\alpha, \beta\}$ is assumed to fill the surface.

6 Proof of Theorem 1.5

An algorithm is given as follows. If $d(\alpha, \beta) \leq 2$ there is only one strongly tight multi-geodesic connecting α and β . We assume $d(\alpha, \beta) \geq 3$ so that α and β together fill the surface. By Lemma 5.1, the set of all curves γ found on tight geodesics connecting α and β and satisfying $\iota(\alpha, \gamma) + \iota(\gamma, \beta) \leq F(\iota(\alpha, \beta))$ has finite cardinality uniformly and explicitly bounded above in terms of $\iota(\alpha, \beta)$ and $n(\Sigma)$. We list all of the paths connecting α and β of length at most $\lfloor 2\log_2(\iota(\alpha, \beta)) \rfloor + 2$ that can be constructed using these curves. By Lemma 2.3 there exists at least one such path. The shortest paths on this list are all geodesics, and those that are tight account for all of the tight geodesics.

7 Stable lengths

In (3) hyperbolicity constants for $\mathcal{G}(\Sigma)$ in terms of the logarithm of the surface complexity are found. Theorem 1.4 of (4) asserts the existence of a positive integer N depending only on the topology of the surface such that, for each pseudo-Anosov mapping class ϕ , there exists a tight geodesic axis in $\mathcal{G}(\Sigma)$ invariant under the action of ϕ^N .

Proposition 7.1 *There exists an explicit finite time algorithm which takes as input a surface Σ , a positive integer N and a pseudo-Anosov mapping class ϕ and returns the stable length $\|\phi\|$.*

Proof We may fix a choice of k such that $\mathcal{G}(\Sigma)$ is k -hyperbolic (23), and choose an integer $M \geq 50k + 20$. If ϕ is not already the N th power of a pseudo-Anosov mapping class we may replace it with such and remember to divide the new stable length by N . Thus, without loss of generality, let us suppose that ϕ is already the N th power of some pseudo-Anosov mapping class. Then, ϕ is also a pseudo-Anosov mapping class and has a geodesic axis, denoted L , on which it acts by translation.

Choose any curve α and use Corollary 1.5 to construct a geodesic ρ from α to $\phi^M \alpha$ in $\mathcal{G}(\Sigma)$. In a k -hyperbolic geodesic metric space each geodesic rectangle is $8k$ -thin, so that any point on any one side of the rectangle is within distance $8k$ from the union of the other three; see (11; 10). There thus exists a vertex β of ρ which lies within $8k$ of L . On the one hand we have $\|\phi\| = \|\phi^M\|/M \leq d(\beta, \phi^M \beta)/M$, and on the other

$$\begin{aligned} d(\beta, \phi^M \beta)/M &\leq (\|\phi^M\| + 16k)/M \\ &= (M\|\phi\| + 16k)/M \\ &= \|\phi\| + 16k/M \\ &< \|\phi\| + 1. \end{aligned}$$

That is, $d(\beta, \phi^M \beta)/M - 1 < \|\phi\|$. Combining the two inequalities, we have

$$d(\beta, \phi^M \beta)/M - 1 < \|\phi\| \leq d(\beta, \phi^M \beta)/M$$

and this uniquely determines the integer $\|\phi\|$. To compute $\|\phi\|$ we now need only appeal to Corollary 1.6 and calculate $d(\beta, \phi^M \beta)$. \square

We have neither constructed an axis for ϕ^N nor have we seen how to compute appropriate values for N . It may interest the reader to find ways of doing so.

References

- [1] R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry* : Springer Universitext (1992).
- [2] M. Bestvina, K. Fujiwara, *Bounded cohomology of subgroups of the mapping class groups* : Geom. Topol. **6** (2002) 69–89.
- [3] B. H. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex* : J. reine angew. Math. **598** (2006) 105–129.
- [4] B. H. Bowditch, *Tight geodesics in the curve complex* : Invent. Math. **171** (2008) 281–300.

- [5] J. F. Brock, R. D. Canary, Y. N. Minsky, *The classification of Kleinian surface groups, II: The Ending Lamination Conjecture* : arXiv:math/0412006v1.
- [6] A. J. Casson, S. A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston* : London Math. Soc., Student Texts **9** (1982).
- [7] B. Farb, D. Margalit, *A primer on mapping class groups* : to appear, P. U. P. (December 2010).
- [8] A. Fathi, F. Laudenbach, V. Poénaru, *Travaux de Thurston sur les surfaces (seconde édition)* : Asterisque **66-67**, Société mathématique de France (1991).
- [9] C. D. Feustel, *Homotopic arcs are isotopic* : Proc. Amer. Math. Soc. **17** No. 4 (1966) 891–896.
- [10] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov* : Birkhäuser (1990).
- [11] M. Gromov, *Hyperbolic groups* : Essays in group theory (ed. S. M. Gersten), MSRI Publications **8**, Springer-Verlag (1987).
- [12] J. L. Harer, *The virtual cohomological dimension of the mapping class groups of an orientable surface* : Invent. Math. **84** (1986) 157–176.
- [13] W. J. Harvey, *Boundary structure of the modular group* : in “Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference” (ed. I. Kra, B. Maskit), Ann. Math. Stud. No. 97, P. U. P. (1981) 245-251.
- [14] J. P. Hempel, *3-manifolds as viewed from the curve complex* : Topology **40** (2001) 631–657.
- [15] N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces* : I. M. R. N. (1997) 651–666.
- [16] J. Jost, *Compact Riemann surfaces* : Springer Universitext, third edition (2006).
- [17] R. P. Kent IV : private communication (December 2004).
- [18] M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori* : Topology App. **95** (1999) 85–111.
- [19] J. P. Leasure, *Geodesics in the complex of curves of a surface* : PhD thesis University of Texas (2002) available on the Texas at Austin Digital Archive from January 2006, URL <http://hdl.handle.net/2152/1700>.
- [20] H. I. Levine, *Homotopic curves on surfaces* : Proc. Amer. Math. Soc. **14** No. 6 (1963) 986–990.
- [21] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds* : Ann. Math. (2) **76** No. 3 (1962) 531–540.

- [22] F. Luo, *Automorphisms of the complex of curves* : Topology **39** (2000) 283–298.
- [23] H. A. Masur, Y. N. Minsky, *Geometry of the complex of curves I: Hyperbolicity* : Invent. Math. **138** (1999) 103–149.
- [24] H. A. Masur, Y. N. Minsky, *Geometry of the complex of curves II: Hierarchical structure* : Geom. Funct. Anal. **10** (2000) 902–974.
- [25] Y. N. Minsky, *The classification of Kleinian surface groups, I: Models and bounds* : arXiv:math/0302208v3.
- [26] I. Richards, *On the classification of non-compact surfaces* : Trans. Amer. Math. **106** (1963) 259–269.
- [27] H. Seifert, W. Threlfall, *Lehrbuch der Topologie* : Teubner, Leipzig (1934).
- [28] K. J. Shackleton, *An acylindricity theorem for the mapping class group* : to appear in New York J. Math.

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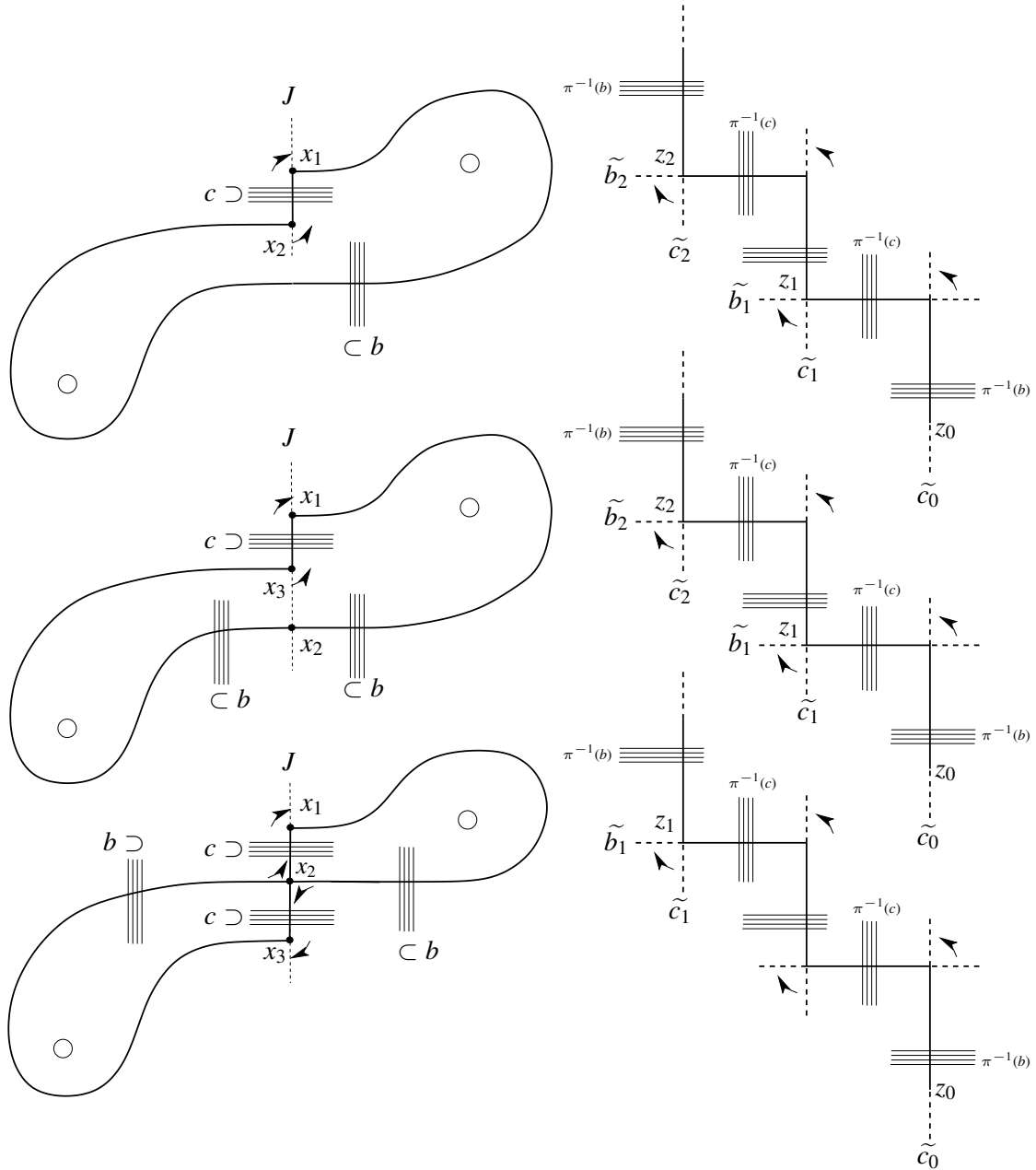


Figure 1: The three types of loop defining p (left) and compact pieces of their lifts (right) marching “left-right-left” in the hyperbolic plane.

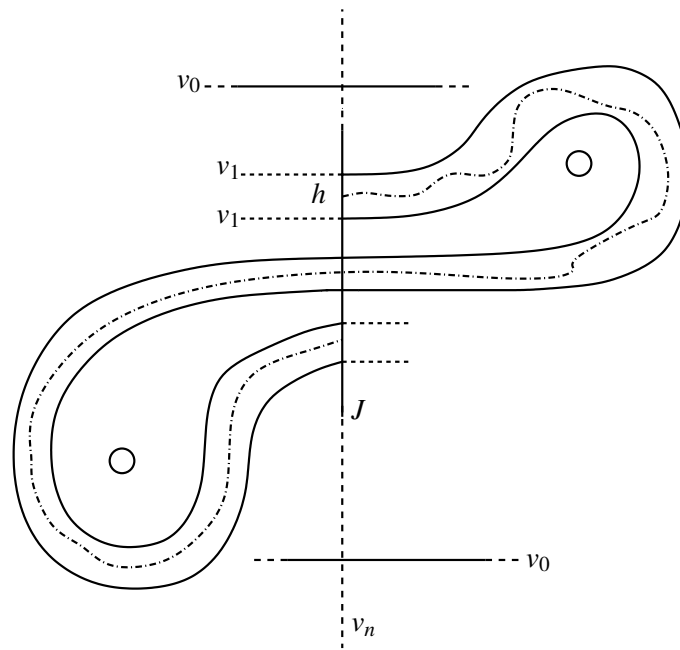


Figure 2: The subarc h of v_m is necessarily disjoint from the multiloop v_0 .