Geodesic axes in the pants complex of the five-holed sphere.

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ABSTRACT: We study the synthetic geometry of the pants graph of the 5-holed sphere, establishing the existence of geodesics connecting any vertex or ideal point to any ideal point. We prove the existence of geodesic axes for sufficiently high powers of any pseudo-Anosov mapping class, and that large link hierarchies from Harvey's curve graph induce geodesic paths.

KEYWORDS: pants complex; Weil-Petersson metric; Farey graph.

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§1. Introduction.

Let Σ be a compact, connected and orientable surface, of genus $g(\Sigma)$ and $\#\partial\Sigma$ boundary components. After Hatcher-Thurston [HT], to the surface Σ one may associate a simplicial graph $\mathcal{P}(\Sigma)$, the *pants graph*, whose vertex set $X(\Sigma)$ comprises all pants decompositions of Σ and any two vertices are connected by an edge if and only if they differ by an elementary move; see §2.2 for an expanded definition. This graph is connected [HT], and one may define a path-metric d on $\mathcal{P}(\Sigma)$, or more precisely $X(\Sigma)$, by assigning length 1 to each edge and regarding the result as a length space. Brock [Br1,2] recently related distances in the pants graph with volumes of hyperbolic 3-manifolds and proved the pants graph is quasi-isometric to Weil-Petersson's metric on Teichmüller space.

The pants graph of the 5-holed sphere is hyperbolic in the sense of Gromov [BrFa, A1, Beh] and thus admits a canonical boundary at infinity, characterised topologically in [BrMas]. Classical theory dictates any pair of ideal points, or a vertex and an ideal point, is necessarily connected by a *quasi*-geodesic. However pants graphs are nowhere locally finite and so one cannot conclude there are infinite geodesics, for there are elementary examples of connected graphs quasi-isometric to $\mathbb{R}$, even admitting a hyperbolic isometry, in which there are no infinite geodesics whatsoever.

The central purpose of this work is to set about overcoming such obstacles.

Theorem 1 *Let Σ be the 5-holed sphere. Any geodesic ray in $\mathcal{P}(\Sigma)$ is eventually contained in any Farey graph from which it remains a bounded distance.*

The intersection of any geodesic path with any Farey graph is always path-connected, and any geodesic ray can only parallel at most one Farey graph. It will become apparent from our method of proof that any K–quasi-geodesic ray,

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for $K < \frac{3}{2}$, remaining a bounded distance from a Farey graph must intersect this Farey graph infinitely many times and at regular intervals. The *hierarchies* from [MasMi2] induce a class of uniform quasi-geodesics that are also eventually contained in any Farey graph they parallel [BrMas].

By $\overline{X}$ we denote the disjoint union $X_0 \sqcup \partial X$, namely the *bordification of* X, for any hyperbolic metric graph X with vertex set X_0. Let us also introduce the notation $\mathcal{G}(x, y)$ for the set of all pants decompositions belonging to geodesic paths connecting two points $x, y \in \overline{\mathcal{P}(\Sigma)}$. We note $\mathcal{G}(x, y) = \mathcal{G}(y, x)$ and $x \in \mathcal{G}(x, y)$ for all x and y. As yet we cannot be certain $\mathcal{G}(x, y)$ is non-empty whenever x is an ideal point.

When Σ is the 5-holed sphere, the set $\mathcal{G}(x, y)$ is locally finite.

Theorem 2 *Let Σ be the 5-holed sphere. Then, for any two points $x, y \in \overline{\mathcal{P}(\Sigma)}$ and any ball B in $\mathcal{P}(\Sigma)$, the set $B \cap \mathcal{G}(x, y)$ is finite.*

An alternative proof in the case $x, y \in X(\Sigma)$ are vertices is given in [Sh1].

A fairly immediate corollary of Theorem 2 is the full combinatorial analogue of the convexity properties enjoyed by the frontier in the corresponding completed Weil-Petersson metric as identified by Masur-Wolf-Farb [MasW] and Wolpert [Wo].

Theorem 3 *Let Σ be the 5-holed sphere. Then, the bordification of any Farey graph in $\mathcal{P}(\Sigma)$ is totally geodesic: For any Farey graph $\mathcal{F}$, any geodesic path connecting two points of $\overline{\mathcal{F}}$ is entirely contained in $\mathcal{F}$.*

Note there exist Gromov hyperbolic graphs containing a totally geodesic Farey graph whose boundary has convex hull properly containing the Farey graph; an example can be explicitly constructed by gluing either arm of a bi-infinite ladder to a bi-infinite geodesic path of a Farey graph. In the pants graph of a surface of complexity 3 not all Farey graphs are of totally geodesic bordification, in the sense there exist geodesic rays remaining a bounded distance from a Farey graph but not intersecting this Farey graph. For instance, in such pants graphs there exist Farey graphs intersecting a convex plane in an axis [APS2]. Indeed only those Farey graphs whose vertices all contain a pair of non-separating or outer curves can have totally geodesic bordification. In the pants graph of a surface of complexity at least 4 there are no Farey graphs with totally geodesic bordification whatsoever, given that each Farey graph intersects in a geodesic a plane ruled by geodesics parallel to this intersection.

We extend our notation $\mathcal{G}$ to consider pairs of subsets of the pants graph, defining $\mathcal{G}(A, B)$ to be the set of all pants decompositions lying on geodesic paths beginning in A and ending in B for all subsets $A, B \subseteq X(\Sigma)$. In other words, we define $\mathcal{G}(A, B)$ to be equal to the union of all $\mathcal{G}(x, y)$ such that $x \in A, y \in B$. Note, $\mathcal{G}(A, B) = \mathcal{G}(B, A)$ and $A \subseteq \mathcal{G}(A, B)$ for all A, B. The subset $\mathcal{G}(A, B)$ is mostly locally finite in the following sense.

Theorem 4 *Let Σ be the 5-holed sphere. Let B_0, B_1 and B_2 be balls of radius at most r in the pants graph $\mathcal{P}(\Sigma)$ such that $d(B_0, B_i) \geq 12(2r + k) + 7$ for $i \in \{1, 2\}$. Then, $B_0 \cap \mathcal{G}(B_1, B_2)$ is finite.*

Theorem 5 *Let Σ be the 5-holed sphere. Let x be a vertex and B a ball of radius at most r in the pants graph $\mathcal{P}(\Sigma)$ such that $d(x, B) \geq 25r + 12k + 7$. Then, $B(x, r) \cap \mathcal{G}(\{x\}, B)$ is finite.*

We do not offer any explicit bounds on the cardinality of the intersections. The constant k depends only on the choice of hyperbolicity constant, and is such that any two geodesic paths connecting two balls of radius r are contained in the other's $2r + k$-regular neighbourhood.

Implicit in the proof of Theorem 4 and relevant to the study of the action of pseudo-Anosov mapping classes carried out in §7 is the following.

Theorem 6 *Let Σ be the 5-holed sphere. If B_0, B_1 and B_2 are balls in the pants graph such that no Farey graph intersecting B_0 intersects B_1 or B_2, then $B_0 \cap \mathcal{G}(B_1, B_2)$ is finite.*

Theorems 4 and 5 together form the basis of a diagonal sequence argument.

Theorem 7 *Let Σ be the 5-holed sphere. Then, any pair of points from $\overline{\mathcal{P}(\Sigma)}$ is connected by a geodesic path.*

It follows the Gromov boundary is *visual*, so that the boundaries defined in terms of quasi-geodesic rays and in terms of geodesic rays are equal. Theorem 7 and Theorem 1 together imply the ideal boundaries of two distinct Farey subgraphs are disjoint and, being closed subsets, therefore have positive nearest point distance in any of the Gromov boundary metrics.

Corollary 8 *Let Σ be the 5-holed sphere or the 2-holed torus. For $\mathcal{F}_1$ and $\mathcal{F}_2$ distinct Farey subgraphs of $\mathcal{P}(\Sigma)$, the ideal boundaries $\partial\mathcal{F}_1$ and $\partial\mathcal{F}_2$ are disjoint.*

An argument of Delzant's [De], adapted from the setting of locally finite hyperbolic graphs, is applicable and finds geodesic axes. This is a combinatorial analogue of Theorem 1.1 from [DaWe], regarding the existence of unique geodesics axes in the Weil-Petersson metric invariant under the action of pseudo-Anosov mapping classes, as stated for the 5-holed sphere.

Theorem 9 *Let Σ be the 5-holed sphere. Then, for any pseudo-Anosov mapping class $\phi \in \mathrm{Map}(\Sigma)$ there exists a positive integer N such that ϕ^N leaves invariant a geodesic axis in $\mathcal{P}(\Sigma)$.*

We do not rule out the possibility that the integer N offered above might depend on the conjugacy class of ϕ. If we combine Theorem 9 with Corollary 3 of [APS1] we find that all the non-elliptic mapping classes must leave invariant a geodesic axis, when first raised to sufficiently high powers.

Corollary 10 *Let Σ be the 5-holed sphere. Then, for any mapping class $\phi \in \mathrm{Map}(\Sigma)$ such that no positive power of ϕ fixes a pants decomposition there exists a positive integer N such that ϕ^N leaves invariant a geodesic axis in $\mathcal{P}(\Sigma)$.*

The action of the mapping class group on the pants graph of the 5-holed sphere is not "weakly properly discontinuous (WPD)" in the sense of Bestvina-Fujiwara [BeF], and thus not "acylindrical" in the sense of Bowditch [Bo1] or of [Sh2] either, since Dehn twists fix pointwise unbounded subsets of the pants graph and commute with many partial pseudo-Anosov mapping classes, which act as hyperbolic isometries. Though unlike the action on the curve graph, the action can nevertheless be described geometrically by restricting the definition of WPD to consider only hyperbolic isometries with virtually cyclic centralisers.

A further application of our work is to provide the first non-trivial family of geodesic paths in the pants graph induced by Masur-Minsky's *hierarchies* [MasMi2] from Harvey's curve graph $\mathcal{C}$, and in the pants graph of the 5-holed sphere geodesics are often described thus. To this end the class of subgraphs known to have totally geodesic bordification is expanded with the following statement.

Theorem 11 *Let Σ be the 5-holed sphere, and let $\mathcal{F}$ and $\mathcal{F}'$ be a pair of Farey subgraphs of the pants graph $\mathcal{P}(\Sigma)$ that intersect. Then, the union $\mathcal{F} \cup \mathcal{F}'$ has totally geodesic bordification.*

We point out that the union of three successively intersecting Farey graphs is not always convex, for there exist 5-circuits in the pants graph of the 5-holed sphere not contained in a single Farey graph. That said, Theorem 11 and a version of Theorem 1 for sufficiently long geodesic paths (Theorem 21 in §3) do combine to yield the following.

Theorem 12 *Let Σ be the 5-holed sphere. There exists a constant h depending only on the choice of hyperbolicity constant such that the following holds: Suppose $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is a finite sequence of Farey subgraphs such that $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$*

intersect in a single point x_i for which $d(x_i, x_{i+1}) \geq 20hn+15$ for each i. Then, the union $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ has totally geodesic bordification.

There exist arbitrarily long hierarchies in the curve graph of the 5-holed sphere with arbitrarily large link distances at each vertex. Indeed these can be explicitly constructed by considering arbitrarily high powers of partial pseudo-Anosov mapping classes and appealing to both parts of Lemma 6.2 from [MasMi2] inductively. Theorem 12 implies hierarchies typically induce geodesic paths in the pants graph of the 5-holed sphere.

Corollary 13 *Let Σ be the 5-holed sphere. Then, the edge set of a hierarchy in $\mathcal{C}(\Sigma)$ with sufficiently large link distances at each vertex corresponds to a geodesic path in $\mathcal{P}(\Sigma)$. A family of such hierarchies exists.*

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§2. Background and definitions.

We supply all the background and terminology needed both to understand the statements of our main results, and to make sense of their proofs. The expert is encouraged to read from §2.5 before jumping to §3. A *loop* will be a closed 1-submanifold, the homeomorphic image of a standard circle.

§2.1. **Curves and pants decompositions.** A loop on Σ is said to be *trivial* if it bounds a disc and *peripheral* if it bounds an annulus whose other boundary component belongs to $\partial\Sigma$. A *curve* is by definition the free homotopy class of a non-trivial and non-peripheral loop. It will at times be convenient to speak of a set containing a single curve as also being a curve.

Given any two curves α and β, their *intersection number* $\iota(\alpha, \beta)$ is defined equal to $\min\{|\mathrm{a} \cap \mathrm{b}| : \mathrm{a} \in \alpha, \mathrm{b} \in \beta\}$. We say two curves are *disjoint* if they have zero intersection number, and otherwise say they *intersect essentially.* The *curve graph* $\mathcal{C}(\Sigma)$ after Harvey [Ha] is connected and is the graph defined by taking as vertices all curves on Σ and declaring a pair of distinct curves spans an edge if and only if the two curves are disjoint. We denote the set of vertices by $\mathcal{C}_0$ and the set of edges by $\mathcal{C}_1$.

A *pants decomposition* is by definition a collection of pairwise distinct and pairwise disjoint curves maximal subject to inclusion among all multicurves. The complement of any pants decomposition on Σ is the disjoint union of non-compact 3-spheres. A pants decomposition comprises $\xi(\Sigma) := \max\{0, 3g(\Sigma) + \#\partial\Sigma - 3\}$ curves, a quantity commonly referred to as the *complexity of* Σ, and as such the pants decompositions of the 5-holed sphere all have exactly two curves and naturally correspond with elements of $\mathcal{C}_1(\Sigma)$. Two pants decompositions have zero intersection number if and only if they are equal.

§2.2. **Pants graph.** The pants graph $\mathcal{P}(\Sigma)$ of a surface Σ is defined by taking as vertex set the set $X(\Sigma)$ of all the pants decompositions of Σ, and declaring two pants decompositions μ and ν to be connected by an edge if and only if they are related by an *elementary move*, so that the multicurve $\mu \cap \nu$ comprises $\xi(\Sigma) - 1$ curves and the remaining two curves together either fill a 4-holed sphere and intersect twice or fill a 1-holed torus and intersect once; consider Figure 1. In particular, in the pants graph of the 5-holed sphere two pants decompositions are connected by an edge if and only if they contain a common curve and the remaining two curves intersect twice.

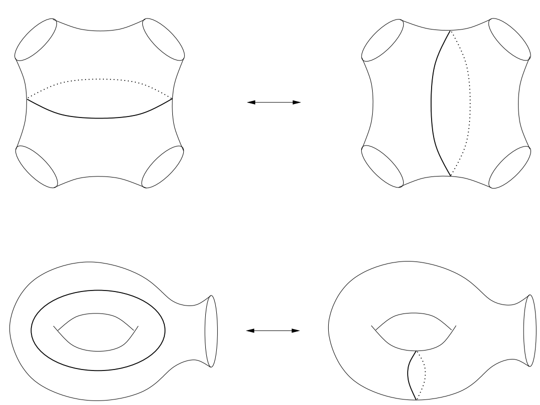

Figure 1: The two types of elementary move defining the pants graph.

It was known to Hatcher-Thurston [HT] that the pants graph is connected.

We may thus endow the pants graph with the standard combinatorial path-metric d, assigning length 1 to each edge. For two subsets $A, B \subseteq X(\Sigma)$ we define their nearest point distance $d(A, B)$ equal to $\min\{d(x, y) : x \in A,\ y \in B\}$. We say A *is*, or *remains*, *within distance* $D \geq 0$ *of* B if A is contained in the closed D-regular neighbourhood of B.

The mapping class group surjects to the isometry group of the pants graph via its natural action, and this surjection is almost always an isomorphism [Mar]. (See [A2] also.) In particular this surjection is an isomorphism in the case of the 5-holed sphere, where all pseudo-Anosov mapping classes, partial or otherwise, act as hyperbolic isometries.

§2.3. **Geodesics and quasi-geodesics.** A path in the pants graph is *geodesic* if it is of minimal length. A path π indexed by a subinterval of $\mathbb{Z}$ is said to be a K*–quasi-geodesic*, for a constant $K \geq 1$, if $|i - j| \leq K d(\pi^i, \pi^j)$ for any two indices i and j. In other words, a path is a K–quasi-geodesic if and only if it is $(K, 0)$–quasi-geodesic in the standard sense of [GH]. A path π is said to be a *quasi-geodesic* if there exists such a constant K.

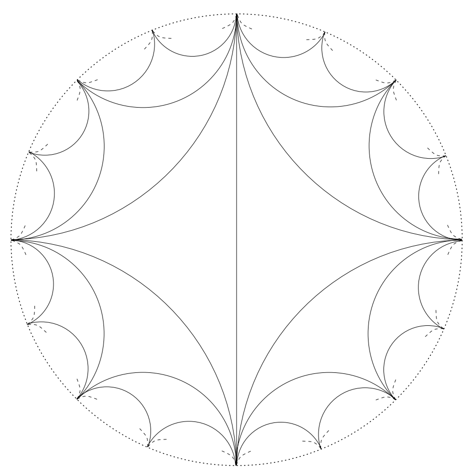

Figure 2: The Farey graph can be represented on a disc.

§2.4. **Farey graphs.** There are numerous ways to build a Farey graph $\mathcal{F}$, any two producing isomorphic graphs. We can start with the rational projective line $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, identifying 0 with $\frac{0}{1}$ and ∞ with $\frac{1}{0}$, and take this to be the vertex set of $\mathcal{F}$. Then, two projective rational numbers $\frac{p}{q}, \frac{r}{s} \in \widehat{\mathbb{Q}}$, where p and q are coprime and r and s are coprime, are deemed to span an edge, or 1-simplex, if and only if $|ps - rq| = 1$. The result is a connected graph in which every edge separates. The graph $\mathcal{F}$ can be represented on a disc; see Figure 2. We say a graph *is a Farey graph* if it is isomorphic to $\mathcal{F}$.

It is well-known that the pants graph of the 4-holed sphere, and the pants graph of the 1-holed torus, are Farey graphs [HT, Se, BoE, Bo2]. It follows the subgraph spanned by all pants decompositions containing any given curve on the 5-holed sphere is isomorphic to the Farey graph; the converse, that every Farey subgraph is determined thus, is proven in [Mar]. That is, Farey graphs in the pants graph of the 5-holed sphere naturally correspond with curves. The Farey subgraph corresponding to the curve α is denoted $\mathcal{F}_\alpha$. Two distinct Farey subgraphs are disjoint or intersect in a single vertex.

§2.5. **Subsurface projections.** Let Σ be the 5-holed sphere. For a curve α, let Y denote the complexity 1 subsurface of Σ disjoint from α. For a second curve β distinct and possibly disjoint from α, we introduce the finite set $\pi_\alpha(\beta)$ of all pants decompositions containing α and disjoint from at least one component of $\beta \cap Y$. We refer to such a component as an *α-footprint of β*, and if such a component is not a curve we call it a *wave*. We define $\pi_\alpha(\alpha)$ to be equal to the empty set. If β and β' are disjoint curves both distinct from a third curve α then $d(z, z') \leq 1$ for all $z \in \pi_\alpha(\beta)$ and $z' \in \pi_\alpha(\beta')$.

Given a pants decomposition $x = \{\beta_1, \beta_2\}$ we define $\pi_\alpha(x)$ to be equal to the union $\pi_\alpha(\beta_1) \cup \pi_\alpha(\beta_2)$, a set of diameter at most 1. Note that for any curve α and any pants decomposition x, the set $\pi_\alpha(x)$ is contained in the Farey graph $\mathcal{F}_\alpha$. Moreover, if x contains α and y is a second pants decomposition adjacent to x but not containing α, then $\pi_\alpha(y) = \{x\}$. We may regard each subsurface projection as a map from the disjoint union $\mathcal{C}_0(\Sigma) \sqcup \mathcal{C}_1(\Sigma)$ to the power set of $X(\Sigma)$. The maps π_α are reminiscent of the subsurface projections defined by Masur-Minsky [MasMi2], and are examples of those discussed in [APS1].

We recall a restricted version of Theorem 2 from [APS1].

Proposition 14 (APS1) *Let Σ be the 5-holed sphere and α a curve. Let $(x_0, \ldots, x_n)$ be a path in the pants graph $\mathcal{P}(\Sigma)$. For each index $i \leq n-1$ and for each $z_i \in \pi_\alpha(x_i)$, there exists $z_{i+1} \in \pi_\alpha(x_{i+1})$ such that $d(z_i, z_{i+1}) \leq 1$.*

We note that subsurface projections to any Farey graph are fairly close to nearest point projections in the following sense.

Lemma 15 *Let x be any pants decomposition and α a curve both on the 5-holed sphere. If $z \in \pi_\alpha(x)$ and $z' \in \mathcal{F}_\alpha$ is any vertex nearest to x, then $d(z, z') \leq d(x, z')$.*

Proof: Let π be any geodesic path connecting x to z' and oriented thus. Appealing to Proposition 14, we can project π to a path in $\mathcal{F}_\alpha$ connecting z to z' and of length no greater than $d(x, z')$. Thus, $d(z, z') \leq d(x, z')$. $\diamondsuit$

Corollary 16 *Let x be any pants decomposition and α any curve on the 5-holed sphere. For any $z \in \pi_\alpha(x)$, we have $d(x,z) \le 2d(x,\mathcal{F}_\alpha)$.*

Proof: According to Lemma 15, if $z' \in \mathcal{F}_\alpha$ is any vertex nearest to x then for any $z \in \pi_\alpha(x)$ we have $d(z,z') \le d(x,z')$. The triangle equality now yields $d(x,z) \le d(x,z') + d(z',z) = d(x,\mathcal{F}_\alpha) + d(z,z') \le d(x,\mathcal{F}_\alpha) + d(x,\mathcal{F}_\alpha) = 2d(x,\mathcal{F}_\alpha)$, as required. $\diamondsuit$

Remark: *The statements of both Lemma 15 and Corollary 16 are true for all subsurface projections to Farey graphs as defined in* [APS1]*, regardless of the surface type. A proof parallels the above.*

We also record the potentially useful observation that no pants decomposition can project to the pants decomposition $\{\alpha,\beta\}$ in both of the subsurface projections π_α and π_β other than $\{\alpha,\beta\}$ itself. That is:

Lemma 17 *Let Σ be the 5-holed sphere, and let $\{\alpha,\beta\}$ be a pants decomposition. Then, $\pi_\alpha^{-1}(\{\alpha,\beta\}) \cap \pi_\beta^{-1}(\{\alpha,\beta\}) = \{\{\alpha,\beta\}\}$.*

Proof: Let $x \in \pi_\alpha^{-1}(\{\alpha,\beta\}) \cap \pi_\beta^{-1}(\{\alpha,\beta\})$. Then, there exists an α-footprint b of x disjoint from β and there exists a β-footprint a of x disjoint from α. If x is distinct from $\{\alpha,\beta\}$ then neither a nor b can be a curve, and as such both are waves supported on the 3-holed sphere Y bordered by α and β. As a and b are not incident on a common boundary component of Y, they intersect essentially. Thus, x self-intersects and this is absurd. It follows x is equal to $\{\alpha,\beta\}$. $\diamondsuit$

§2.6. **Extrinsic geometry of Farey graphs.** In any pants graph every Farey graph is totally geodesic in the following sense, a version of Theorem 1 from [APS1] restricted to consider 5-holed sphere and implied by Proposition 14.

Proposition 18 (APS1) *Let Σ be the 5-holed sphere. Then, every Farey graph in $\mathcal{P}(\Sigma)$ is totally geodesic: Every finite geodesic beginning and ending in any given Farey graph is entirely contained in this Farey graph.*

§2.7. **Hierarchies.** The link of any vertex in the curve graph of the 5-holed sphere contains no edges. However, we can naturally identify any link as the vertex set of the pants graph of the 4-holed sphere and thus the vertex set of a Farey graph. We pullback the canonical path-metric of the Farey graph to the link. A *hierarchy* in the curve graph of the 5-holed sphere we then regard as obtained from a geodesic path by connecting each pair of vertices of distance 2 by a geodesic in the link of the intermediary vertex. In other words, we consider hierarchies as defined in [MasMi2] modulo their markings.

§3. **Proof of Theorem 1.**

We begin by noting geodesic paths contract under projections to Farey graphs.

Lemma 19 *Let x_0, x_1, x_2, x_3 be a path in the pants graph of the 5-holed sphere such that no x_i contains the curve α. Then, for any $z_0 \in \pi_\alpha(x_0)$ there exists $j \in \{1, 2, 3\}$ and $z_j \in \pi_\alpha(x_j)$ such that $d(z_0, z_j) \leq j - 1$.*

Proof: The proof is separated into two parts.

Alternating path. We first suppose our path is *alternating*, so that $x_i \cap x_{i+1} \cap x_{i+2}$ is empty for both $i \in \{0, 1\}$. Let β_0 denote a curve from x_0 such that $z_0 \in \pi_\alpha(\beta_0)$. If $\beta_0 \in x_1$ then we immediately take $j = 1$ and equate z_1 with z_0, so let us suppose $\beta_0 \notin x_1$. We denote by β_i the curve from $x_{i-1} \cap x_i$ for $i \in \{1, 2, 3\}$, each being distinct from α and thus having non-empty projection set $\pi_\alpha(\beta_i)$. Let $z_i \in \pi_\alpha(\beta_i)$ for each index i. As our path is alternating the four curves β_i in fact form four sides of a "pentagon configuration" on the surface:

- $\iota(\beta_i, \beta_{i+1}) = 0$ for $i \in \{0, 1, 2\}$;
- $\iota(\beta_i, \beta_{i+2}) = 2$ for $i \in \{0, 1\}$ and,
- $\iota(\beta_0, \beta_3) = 2$.

There exists a unique curve δ extending this to a full pentagon configuration:

- $\iota(\delta, \beta_i) = 0$ for $i \in \{0, 3\}$, and
- $\iota(\delta, \beta_i) = 2$ for $i \in \{1, 2\}$.

If the curve δ is equal to α then β_0 is distinct and disjoint from both α and β_1. In which case $\pi_\alpha(\beta_1) = \pi_\alpha(\beta_0) = \{z_0\}$ and so $z_0 = z_1$. We may thus assume δ is distinct from α, denoting by w any element of the now non-empty set $\pi_\alpha(\delta)$. Consecutive curves in the cyclic sequence $\beta_0, \beta_1, \beta_2, \beta_3, \delta$ are disjoint so the projections z_0, z_1, z_2, z_3, w form a 5-circuit in $\mathcal{F}_\alpha$. However 5-circuits have diameter at most 2 and it follows $d(z_0, z_3) \leq 2$, as required.

Non-alternating path. If our path is not alternating, there exists $i \in \{0, 1\}$ such that $x_i \cap x_{i+1} \cap x_{i+2}$ is not empty. Let $\beta_0 \in x_0$ be such that $z_0 \in \pi_\alpha(\beta_0)$. There exists $\delta \in x_i \cap x_{i+1} \cap x_{i+2}$ and $\beta_3 \in \pi_\alpha(x_3)$ such that $\iota(\beta_0, \delta) = 0$ and $\iota(\delta, \beta_3) = 0$.

For all $z \in \pi_\alpha(\delta)$ and for all $z_3 \in \pi_\alpha(\beta_3)$, we have $d(z_0, z) \leq 1$ and $d(z, z_3) \leq 1$. The triangle inequality yields $d(z_0, z_3) \leq d(z_0, z) + d(z, z_3) \leq 1 + 1 = 2$. $\diamondsuit$

Applying Lemma 19 piecewise, we deduce the following.

Corollary 20 *Let x and y be any two pants decompositions of the 5-holed sphere connected by a geodesic path disjoint from a Farey graph $\mathcal{F}_\alpha$. Then, for any $z \in \pi_\alpha(x)$ there exists $w \in \pi_\alpha(y)$ such that $3d(z, w) \leq 2d(x, y) + 2$.*

Proof: Let π be any geodesic path connecting x to y. According to Lemma 19, there exists a maximal sequence of pairwise distinct vertices $x_0, \ldots, x_k$ of π, indexed so that $x_0 = x$, x_j separates x_{j-1} and x_{j+1} along π, $d(x_{j-1}, x_j) \leq 3$ and $d(x_k, y) \leq 2$, for which there exist projections $z_j \in \pi_\alpha(x_j)$ with $z_0 = z$ such that $d(z_{j-1}, z_j) \leq d(x_{j-1}, x_j) - 1$ for each index j. Thus, $3d(z_{j-1}, z_j) \leq 2d(x_{j-1}, x_j)$ for each index j. In the event $x_k \neq y$, there exists $w \in \pi_\alpha(y)$ such that $d(z_k, w) \leq 2$, and the difference $3d(z_k, w) - 2d(x_k, y)$ is always at most 2. Appealing to the triangle inequality for d we find:

$$\begin{aligned}
3d(z, w) &\leq 3\sum_{j=1}^{k} d(z_{j-1}, z_j) + 3d(z_k, w) \\
&\leq 2\sum_{j=1}^{k} d(x_{j-1}, x_j) + 3d(z_k, w) \\
&= 2d(x, x_k) + 3d(z_k, w) \\
&= 2(d(x, y) - d(x_k, y)) + 3d(z_k, w) \\
&= 2d(x, y) + (3d(z_k, w) - 2d(x_k, y)) \\
&\leq 2d(x, y) + 2.
\end{aligned}$$

To be more succinct, $3d(z, w) \leq 2d(x, y) + 2$ as claimed. $\diamondsuit$

Proof: (of Theorem 1.) Suppose otherwise. Then, there exists a geodesic ray π remaining within a finite distance D of some Farey graph $\mathcal{F}_\alpha$ but not eventually contained in $\mathcal{F}_\alpha$. We note that π does not contain any finite length segments beginning and ending in $\mathcal{F}_\alpha$ but not entirely contained in $\mathcal{F}_\alpha$, for by Proposition 18 the Farey graph $\mathcal{F}_\alpha$ is a totally geodesic subgraph of the pants graph. By restricting if need be, we may thus assume π is disjoint from $\mathcal{F}_\alpha$.

For any two vertices x and y of π, the segment connecting x to y is disjoint from $\mathcal{F}_\alpha$ and so, by Corollary 20, for any $z \in \pi_\alpha(x)$ there exists $w \in \pi_\alpha(y)$ such that $3d(z, w) \leq 2d(x, y) + 2$. Combining this with Corollary 16 and the triangle inequality for d we have:

$$
\begin{aligned}
3d(x,y) &\leq 3d(x,z)+3d(z,w)+3d(w,y) \\
&\leq 6d(x,\mathcal{F}_\alpha)+3d(z,w)+6d(\mathcal{F}_\alpha,y) \\
&\leq 6D+3d(z,w)+6D \\
&= 3d(z,w)+12D \\
&\leq (2d(x,y)+2)+12D \\
&= 2d(x,y)+12D+2.
\end{aligned}
$$

To be more succinct, $d(x,y) \leq 12D+2$ for all vertices x and y of π. In particular, π is therefore a bounded subset and this is a contradiction. It follows that π is eventually contained in $\mathcal{F}_\alpha$ as claimed. $\diamondsuit$

The full implication of the above argument is the following.

Theorem 21 *Let Σ be the 5-holed sphere. Then, for $D \in \mathbb{N}$, any geodesic path in $\mathcal{P}(\Sigma)$ of length at least $12D+3$ remaining distance at most D from a Farey graph intersects this Farey graph.*

A similar argument treats a spectrum of quasi-geodesic rays.

Theorem 22 *Let Σ be the 5-holed sphere and $K \in [1, \frac{3}{2})$. Then, any K–quasi-geodesic ray in $\mathcal{P}(\Sigma)$ remaining a bounded distance from a Farey graph intersects this Farey graph.*

§4. Proof of Theorem 2.

We begin with the following statements regarding convergence to laminations in the Hausdorff topology, the proofs of which are inspired by an argument of Luo's presented in [MasMi1] itself inspired by an argument of Kobayashi's [K].

Lemma 23 *If $(x_i)_i$ is a bounded sequence of pants decompositions on any surface converging in the Hausdorff topology to a lamination λ, then λ does not contain an irrational leaf.*

Proof: Let w be any vertex. Passing to a subsequence of $(x_i)_i$ if need be, since $(x_i)_i$ is bounded we may assume that $d(w,x_0)=d(w,x_i)$ for each i. Let π_i be any geodesic path connecting w to x_i, whose jth vertex we denote by π_i^j. We regard each π_i as an element of a finite product of lamination spaces, endowed with the product Hausdorff topology. The latter is again compact so, passing to further subsequence if need be, the sequence $(\pi_i)_i$ converges to a sequence of laminations $w = \lambda_0, \lambda_1, \ldots, \lambda_k = \lambda$.

We note that if λ is to contain an irrational leaf, then λ_{k-1} must also contain this same leaf for otherwise $\iota(\pi_i^{k-1}, \pi_i^k)$ diverges with i and yet π_i^{k-1} and π_i^k have uniformly bounded intersection number, being adjacent vertices of the pants graph. As λ_0 is a pants decomposition, and as such does not contain an irrational leaf, so there exists an index j such that λ_j contains an irrational leaf but λ_{j-1} does not. In which case, $\iota(\pi_i^{j-1}, \pi_i^j)$ diverges and this again is absurd. Thus, λ does not contain an irrational leaf as claimed. $\diamondsuit$

Lemma 24 *Let Σ be the 5-holed sphere. If $(x_i)_i$ is a sequence of pants decompositions contained in a ball B in $\mathcal{P}(\Sigma)$ and converging in the Hausdorff topology to a lamination λ containing a closed leaf α, then $B \cap \mathcal{F}_\alpha$ is not empty.*

Proof: We note if λ contains α as an isolated leaf then x_i also contains α for all i sufficiently large. It follows each such x_i is a vertex of $\mathcal{F}_\alpha$ and we conclude $B \cap \mathcal{F}_\alpha$ is not empty. It suffices to assume λ contains α as a non-isolated leaf.

Let w denote a centre of B. Passing to a subsequence of $(x_i)_i$ if need be, we may assume that $d(w, x_0) = d(w, x_i)$ for each i. Let π_i be any geodesic path connecting w to x_i. We regard each π_i as an element of a finite product of lamination spaces, endowed with the product Hausdorff topology. The latter is again compact so, passing to further subsequence if need be, the sequence $(\pi_i)_i$ converges to a sequence of laminations $w = \lambda_0, \lambda_1, \ldots, \lambda_k = \lambda$. As adjacent pants decompositions have bounded intersection number we observe λ_{k-1} contains α, either as an isolated leaf or as a non-isolated leaf. However, λ_0 is a pants decomposition. Iterating this observation, there exists some index j such that λ_j contains α as an isolated leaf. For sufficiently large i, the pants decomposition π_j^i therefore contains α and so is a vertex of the Farey graph $\mathcal{F}_\alpha$ determined by α. We conclude that B intersects $\mathcal{F}_\alpha$ as claimed. $\diamondsuit$

Remark: *If $\mathcal{P}_\alpha$ denotes the subgraph of any pants graph $\mathcal{P}$ spanned by all pants decompositions containing a curve α, and B is a ball in $\mathcal{P}$, under the hypotheses of Lemma 24 a parallel argument yields B intersects $\mathcal{P}_\alpha$. A similar argument yields the general fact that a sequence of pants decompositions $(x_i)_i$ is unbounded if and only if $(\iota(x_i, z))_i$ is also unbounded for every pants decomposition z.*

We use Theorem 1 to complete a proof of Theorem 2.

Proof: (of Theorem 2.) Suppose for contradiction the statement is false. Then, there exist two points $x, y \in \overline{\mathcal{P}(\Sigma)}$ and a ball B such that $B \cap \mathcal{G}(x, y)$ is infinite. Let $(x_i)_i$ be any infinite sequence of distinct vertices from $B \cap \mathcal{G}(x, y)$. We regard each x_i as a geodesic lamination. Since the Hausdorff topology on the set of all geodesic laminations is compact, passing to subsequence if need be we may assume $(x_i)_i$ converges to a geodesic lamination λ. The classification of geodesic laminations (Lemma 4.4 of [CBl]) yields just three possibilities:

- λ is a pants decomposition;
- λ contains an irrational leaf, or
- λ contains a non-isolated closed leaf.

The first possibility, that λ is a pants decomposition, is impossible since it implies the sequence $(x_i)_i$ is eventually constant. The second possibility is eliminated by Lemma 23. This leaves only the third possibility, that λ contains a non-isolated closed leaf. We denote this leaf by α, also regarded as a curve. Appealing to Lemma 24, the set B intersects the Farey graph $\mathcal{F}_\alpha$.

While not every vertex of B may be extended to a geodesic path connecting x to y, by the definition of $\mathcal{G}(x,y)$ each vertex x_i may be extended thus to a geodesic path π_i indexed so that $\pi_i^0 = x_i$. We also regard $(\pi_i)_i$ as an element of a product of lamination spaces, compact in the Tychonoff product topology. On passing to a further subsequence if need be, $(\pi_i)_i$ converges to a sequence of laminations $(\mu_j)_j$ indexed so that $\mu_0 = \lambda$. Adjacent pants decompositions have bounded intersection number, so we observe that μ_{-1} and μ_1 also contain α. We iterate this observation to find minimal $s_- \in -\mathbb{N} \sqcup \{-\infty\}$ and maximal $s_+ \in \mathbb{N} \sqcup \{\infty\}$ such that μ_j contains α as a non-isolated leaf for each $s_- < j < s_+$. If both s_- and s_+ are finite, then for sufficiently large i the geodesic path π_i contains a segment beginning and ending in the Farey graph $\mathcal{F}_\alpha$, namely at $\pi_i^{s_-}$ and at $\pi_i^{s_+}$ respectively, but not entirely contained in $\mathcal{F}_\alpha$ and this violates Proposition 18. Thus at least one of s_- and s_+ is infinite. We assume, reindexing if need be, that s_+ is infinite and so y is an ideal point.

The hyperbolicity of $\mathcal{P}(\Sigma)$ implies the existence of a constant k depending only on the chosen hyperbolicity constant and such that any two geodesic paths connecting two common points of $\overline{\mathcal{P}(\Sigma)}$ remain within distance k of each other. In particular, the geodesic paths π_i all remain within distance k of each other. The ball $B(\pi_0^j, 10k)$ contains the sequence $(\pi_i^j)_i$ and so, by Lemma 24, has non-empty intersection with the Farey graph $\mathcal{F}_\alpha$ for each $s_- < j < s_+$. Thus, the positive segment π_0^+ of π_0, and each π_i^+ therefore, is a geodesic ray remaining a distance uniformly bounded, independent of i, from the Farey graph $\mathcal{F}_\alpha$. According to Theorem 1, each π_i^+ is eventually contained in $\mathcal{F}_\alpha$ for all i. If π_0 is bi-infinite, so that s_- is also not finite, a parallel argument proves π_i^- is eventually contained in $\mathcal{F}_\alpha$, for all i.

In all cases we conclude that π_i contains a finite geodesic segment beginning and ending in $\mathcal{F}_\alpha$ for all i. However, for i sufficiently large, x_i intersects α and so no such geodesic segment can be entirely contained in $\mathcal{F}_\alpha$. This is contrary to the statement of Proposition 18. $\diamondsuit$

Remark: *In the above, by Theorem 21 we only really need* $12(\operatorname{diam}(B)+k)+3$ *consecutive Hausdorff limits with* k *the universal constant appearing in the state-*

ment of Theorems 4 and 5. Theorem 2 is proven without the axiom of choice.

We soon deduce Theorem 3 as a corollary of Theorem 2.

Proof: (of Theorem 3.) Suppose otherwise, for contradiction. Then, there exists a geodesic path π beginning and ending in $\overline{\mathcal{F}}$ but not entirely contained in $\mathcal{F}$. Let B be any ball centred on a vertex of $\mathcal{F}$ such that $B \cap \pi$ is not entirely contained in $\mathcal{F}$. If τ is a non-trivial Dehn twist pointwise fixing $\overline{\mathcal{F}}$, as such supported on the curve common to every vertex of $\mathcal{F}$, then τ also leaves invariant B and fixes fix the ends of π. However, every vertex of B not contained in $\mathcal{F}$ has an infinite τ-orbit. It follows $(\tau^i\pi)_i$ is an infinite sequence of geodesic paths connecting the same two points as π yet together of infinite intersection with B. This is a contradiction, violating Theorem 2. $\diamondsuit$

§5. **Proof of Theorems 4, 5 and 6.**

Proof: (of Theorem 4.) Suppose for contradiction the statement is false. Then, there exist three such metric balls B_0, B_1 and B_2 of radius at most r such that $B_0 \cap \mathcal{G}(B_1, B_2)$ is infinite. An argument parallel to that of §4 produces an infinite sequence of geodesic paths $(\pi_i)_i$, all of the same finite length, such that the vertices π_i^0 are all contained in B_0 and such that $(\pi_i)_i$ converges in the product topology to a sequence of geodesic laminations $(\mu_j)_j$ where μ_j is the Hausdorff limit of $(\pi_i^j)_i$ and μ_0 contains a curve α as a non-isolated leaf. We again conclude, by Lemma 24, that B_0 intersects the Farey graph $\mathcal{F}_\alpha$.

If there exist $s_- < 0 < s_+$ such that μ_{s_-} and μ_{s_+} contain α as an isolated leaf then for all sufficiently large i the geodesic path π_i contains a finite segment beginning and ending in the Farey graph $\mathcal{F}_\alpha$ but not entirely contained in $\mathcal{F}_\alpha$. This is absurd, violating Proposition 18, and so again we see, reindexing if need be, that μ_j contains α as a non-isolated leaf for all $j \geq 0$. Appealing to Lemma 24 once more, we conclude B_2 also intersects the Farey graph $\mathcal{F}_\alpha$. (To this point a parallel argument deduces Theorem 6.)

The hyperbolicity of $\mathcal{P}(\Sigma)$ implies the existence of a constant k, depending only on the chosen hyperbolicity constant, such that any two geodesic paths connecting one ball of radius at most r to another ball of radius at most r remain within distance $2r + k$ of each other. In particular, every geodesic path connecting B_0 to B_2 remains within distance $2r + k$ of any geodesic path in $\mathcal{F}_\alpha$ also connecting B_0 to B_2 and so, for all i, the geodesic path π_i remains within distance $2r + k$ of the Farey graph $\mathcal{F}_\alpha$. The paths π_i^+ each have length at least $12(2r + k) + 3$ and, when i is sufficiently large, each vertex of π_i^+ has positive intersection number with α and so cannot belong to $\mathcal{F}_\alpha$. This is contrary to Theorem 21 which assures us any geodesic path of length at least $12(2r+k)+3$ and remaining within distance $2r + k$ of $\mathcal{F}_\alpha$ must intersect $\mathcal{F}_\alpha$. This completes the proof of Theorem 4. $\diamondsuit$

Proof: (of Theorem 5.) The proof of Theorem 5 is parallel to the above, where $B(x,r)$ replaces B_0, $\{x\}$ replaces B_1 and B replaces B_2. As x is a pants decomposition, we deduce the existence of s_- such that μ_{s_-} contains the curve α as an isolated leaf and such that for all $j > s_-$ the lamination μ_j contains α as a non-isolated leaf. In particular, $B(x,r)$ intersects $\mathcal{F}_\alpha$. Lemma 24 implies B also intersects the Farey graph $\mathcal{F}_\alpha$. We thus have a family of geodesic paths π_i, for all i sufficiently large, of length at least $12(2r+k)+3$ remaining within distance $2r+k$ of $\mathcal{F}_\alpha$ but disjoint from $\mathcal{F}_\alpha$. However, this violates the statement of Theorem 21 and we have a contradiction. $\diamondsuit$

§6. **Proof of Theorem 7.**

We first show how to connect two ideal points x and y of $\mathcal{P}(\Sigma)$. Let $(w_i)_i$ be any quasi-geodesic connecting x to y. We denote by π_i any geodesic path connecting w_{-i} to w_i for each $i \in \mathbb{N}$. For each $r \geq 0$, we define $R(r)$ so that $i \geq R$ implies $d(w_0, w_{\pm i}) \geq 20(2r+k)+3$ say.

By Theorem 4, the intersection $B(w_0,r) \cap \mathcal{G}(B(w_{-i},r), B(w_i,r))$ is finite for all $i \geq R$. It follows from hyperbolicity that the intersection $B(w_0,r) \cap \bigcup_{i \geq R} \pi_i$ is also finite for all r sufficiently large. On passing to a subsequence of $(\pi_i)_i$ if need be, the intersection $\pi_i \cap B(w_0,r)$ is therefore equal to $\pi_{i+1} \cap B(w_0,r)$ for all $i \geq R$ and all r sufficiently large. This is the basis of a diagonal sequence argument extracting a sequence of geodesic paths converging on arbitrarily large balls centred on w_0 to a geodesic path connecting x to y.

If instead only y is an ideal point of $\mathcal{P}$, we consider a quasi-geodesic ray $(w_i)_i$ connecting the pants decomposition x to y and indexed so that $w_0 = x$. We denote by π_i any geodesic path connecting x to w_i for each $i \in \mathbb{N}$. By Theorem 5, the intersection $B(x,r) \cap \mathcal{G}(\{x\}, B(w_i,r))$ is finite for all $i \geq R$ and all r sufficiently large. It follows the intersection $B(x,r) \cap \bigcup_{i \geq R} \pi_i$ is also finite for all r sufficiently large. On passing to a subsequence of $(\pi_i)_i$ if need be, the intersection $\pi_i \cap B(w_0,r)$ is therefore equal to $\pi_{i+1} \cap B(w_0,r)$ for all $i \geq R$ and all r sufficiently large. This again is the basis of a diagonal sequence argument extracting a sequence of geodesic paths converging on arbitrarily large concentric balls centred on x to a geodesic ray connecting x to y. $\diamondsuit$

§7. **Proof of Corollary 8.**

Between the pants graph of the 5-holed sphere and the pants graph of the 2-holed torus there exist quasi-isometries mapping Farey subgraphs to Farey subgraphs. In particular there exist homeomorphisms between their Gromov boundaries carrying the boundary of any Farey subgraph to the boundary of a Farey subgraph. It therefore suffices to prove Corollary 8 for the 5-holed sphere.

We suppose for contradiction $\partial\mathcal{F}_1 \cap \partial\mathcal{F}_2 \neq \emptyset$. Let $z \in \partial\mathcal{F}_1 \cap \partial\mathcal{F}_2$ be any common ideal point and let $x \in X(\Sigma)$ be any vertex of $\mathcal{F}_1$. By Theorem 7 there exists a geodesic ray π connecting x to z. The ray π is entirely contained in

$\mathcal{F}_1$ by Theorem 3, and remains within a bounded distance of $\mathcal{F}_2$ and thus is eventually contained in $\mathcal{F}_2$ according to Theorem 1. In particular, $\mathcal{F}_1 \cap \mathcal{F}_2$ has cardinality at least 2 and it follows that in fact $\mathcal{F}_1 = \mathcal{F}_2$. However, this is a contradiction and we complete the proof of Corollary 8. $\diamondsuit$

§8. **Proof of Theorem 9.**

We provide an alternative proof of the existence of geodesic paths connecting the two ideal points fixed by a given pseudo-Anosov mapping class, recovering Theorem 7 in this case but without the need to precisely control constants. Thereafter we adapt an argument of Delzant's [De], already adapted by Bowditch [Bo1] to the setting of Harvey's curve graph to find "tight" geodesic axes.

Lemma 25 *Let Σ be the 5-holed sphere. For any pseudo-Anosov mapping class $\phi \in \mathrm{Map}(\Sigma)$, there exists a bi-infinite geodesic path connecting the two ideal points fixed by ϕ.*

Proof: Let $(w_i)_i$ be any quasi-geodesic path connecting the two ideal points fixed by ϕ, and denote by π_i any geodesic path connecting w_{-i} to w_i for each $i \in \mathbb{N}$. For all $r \geq 0$ there exists $R(r) \geq 0$ such that every Farey graph intersecting the ball $B(w_0, r)$ is disjoint from both $B(w_{-i}, r)$ and $B(w_i, r)$ for all $i \geq R$. (In fact, R need only be such that the distance between w_0 and both $w_{\pm i}$ in Harvey's curve graph is at least $2r + 2$. As ϕ is pseudo-Anosov, such an R exists. See Proposition 7.6 of [MasMi2].) By Theorem 6, the intersection $B(w_0, r) \cap \mathcal{G}(B(w_{-i}, r), B(w_i, r))$ is finite. It follows the intersection $B(w_0, r) \cap \bigcup_{i \geq R} \pi_i$ is also finite for all r sufficiently large. On passing to a subsequence of $(\pi_i)_i$ if need be, the intersection $\pi_i \cap B(w_0, r)$ is therefore equal to $\pi_{i+1} \cap B(w_0, r)$ for all $i \geq R$ and all r sufficiently large. This is the basis of a diagonal sequence argument extracting a sequence of geodesic paths converging on arbitrarily large concentric balls centred on w_0 to a geodesic path connecting the fixed points of ϕ. $\diamondsuit$

Let us denote by x and y the two ideal points of $\mathcal{P}(\Sigma)$ fixed by a pseudo-Anosov mapping class ϕ. The set $\mathcal{L}(x, y)$ of all directed geodesic paths connecting x to y is non-empty by Lemma 25. This set is invariant under the action of ϕ, and by Theorem 2 the quotient of its edge set by $\langle \phi \rangle$ is finite. We distinguish these edge orbits by labeling them using only finitely many integers.

A geodesic path π connecting x to y and directed thus is said to be *lexicographically least* if every finite length subpath $\pi_0 \subset \pi$ is lexicographically least among all directed geodesic paths in $\mathcal{L}(x, y)$ connecting the two ends of π_0. To see that lexicographically least geodesics exist, consider any directed geodesic $(x_i)_i \in \mathcal{L}(x, y)$ and denote by $\pi_i \in \mathcal{L}(x, y)$ a geodesic lexicographically least from x_{-i} to x_i for each integer $i \in \mathbb{N}$. The local finiteness of $\mathcal{G}(x, y)$ offered by

Theorem 2 is the basis of a diagonal subsequence argument extracting a subsequence of $(\pi_i)_i$ converging on arbitrarily large concentric balls to a geodesic $\pi' \in \mathcal{L}(x,y)$. This limit is necessarily lexicographically least, for there otherwise exists i such that the geodesic π_i is not lexicographically least from x_{-i} to x_i.

The set of all lexicographically least geodesics connecting x to y is finite, for otherwise the local finiteness of $\mathcal{G}(x,y)$ given by Theorem 2 ensures there exist two lexicographically least geodesics $\pi_1, \pi_2 \in \mathcal{L}(x,y)$ connecting two vertices between which π_1 and π_2 are distinct. However, between any two vertices of $\mathcal{G}(x,y)$ there is only at most one lexicographically least geodesic segment.

The infinite cyclic group $\langle \phi \rangle$ permutes the finitely many lexicographically least geodesics among themselves. There thus exists a positive integer N such that ϕ^N stabilises some lexicographically least geodesic, an axis for ϕ^N. $\diamondsuit$

§9. **Proof of Theorem 11.**

We suppose for contradiction there exists a pair of distinct such Farey subgraphs $\mathcal{F}$ and $\mathcal{F}'$ sharing a single common vertex O and a finite geodesic path π beginning and ending in $\mathcal{F} \cup \mathcal{F}'$ but otherwise disjoint from the union. In view of Proposition 18, and reversing π if need be, we see that π necessarily begins in $\mathcal{F}$ and ends in $\mathcal{F}'$, is oriented thus, and does not pass through O. We can readily verify that the length $l(\pi)$ of π must be at least 3, as circuits in this pants graph of length at most 4 are only contained in a single Farey subgraph. We denote by α and α' the two curves constituting O, where α is common to every vertex of $\mathcal{F}$ making α' common to every vertex of $\mathcal{F}'$. As such, $\mathcal{F} = \mathcal{F}_\alpha$ and $\mathcal{F}' = \mathcal{F}_{\alpha'}$.

We note that $\pi_\alpha(\mathcal{F}') = \{O\}$ and so there exists a first vertex $x \in \pi$ such that $O \in \pi_\alpha(x)$. As $\pi_\alpha^{-1}(O) \cap \mathcal{F} = \{O\}$ and $x \neq O$ so $x \notin \mathcal{F}$ and there exists a vertex $x^- \in \pi$ immediately preceding x, that is x^- separates x from $\mathcal{F}$ along π or $x^- \in \mathcal{F}$. The choice of x implies $O \notin \pi_\alpha(x^-)$. In particular, every α-footprint of x^- must intersect α' essentially. It follows each α'-footprint of x^- is disjoint from a common wave on the complement of α' disjoint from α. Thus, O is within distance 1 of every vertex of $\pi_{\alpha'}(x^-)$ and therefore at least one vertex of $\pi_{\alpha'}(x)$.

This provides the basis for a projection of π to the union $\mathcal{F} \cup \mathcal{F}'$ completed in two parts: First, we use π_α to project x to O and then Proposition 14 to proceed from there along π towards $\mathcal{F}$. Second, we use $\pi_{\alpha'}$ to project x to a vertex within distance 1 of O before using Proposition 14 to project the remaining segment of π to $\mathcal{F}'$. We depict this projection in Figure 3.

The concatenation of the two paths, with one first inverted, is a path in $\mathcal{F} \cup \mathcal{F}'$ connecting the two ends of π. While the vertex x is projected to two vertices at most distance 1 apart, contributing at most an extra unit of length to the projection, the two vertices of π adjacent to the ends of π project to the ends of π, recovering two units of length. That is, any resulting projection is a path of length at most $l(\pi) + 1 - 2$, or $l(\pi) - 1$, connecting the ends of π. However, π is a geodesic path and so we have a contradiction. It follows such a union of two Farey graphs is always totally geodesic.

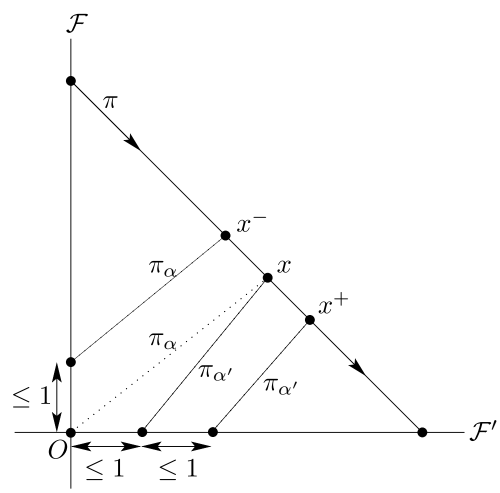

Figure 3: A piecewise projection.

Suppose π is instead a geodesic ray or a bi-infinite geodesic beginning and ending in the bordification of the union $\mathcal{F} \cup \mathcal{F}'$ of two distinct but intersecting Farey graphs. The hyperbolicity of this pants graph implies π remains a uniformly bounded distance from $\mathcal{F} \cup \mathcal{F}'$ and, by Theorem 7 applied to either Farey graph, is eventually contained in $\mathcal{F} \cup \mathcal{F}'$ in either direction. It follows that π can be expressed as the concatenation of one or two geodesic rays contained in $\mathcal{F} \cup \mathcal{F}'$ and a finite geodesic path. However, we already know that $\mathcal{F} \cup \mathcal{F}'$ is totally geodesic and so π must be entirely contained in $\mathcal{F} \cup \mathcal{F}'$. $\diamondsuit$

§10. **Proof of Theorem 12.**

The connected subgraph $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ is the union of n totally geodesic subgraphs of $\mathcal{P}(\Sigma)$. According to Lemma 4.1 of [Bo3], for instance, there exists a constant h depending only on the choice of hyperbolicity constant such that $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ is hn-quasiconvex. In particular, if π is any finite geodesic path beginning and ending in $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ then we may express π as a concatenation of geodesic paths π_j such that $\pi_j \subset N_{hn}(\mathcal{F}_j)$ for each j. If π_j does not contain either end of π, then the assumption on the distance $d(x_j, x_{j+1})$ implies the length of π_j is at least $12hn + 3$. As such, by Theorem 21 we know that each π_j intersects $\mathcal{F}_j$ in at least one vertex. We may thus also express π as a finite concatenation of geodesic paths π'_j such that π'_j begins in $\mathcal{F}_j$ and ends in $\mathcal{F}_{j+1}$. Theorem 11 implies $\pi'_j \subset \mathcal{F}_j \cup \mathcal{F}_{j+1}$ for each j, and so $\pi \subset \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ as required.

That $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ is also of totally geodesic bordification can now be argued in much the same manner as the closing paragraph of §9. $\diamondsuit$

§11. **Proof of Corollary 13.**

All that remains is to show that the statement of Corollary 13 does not hold vacuously, making use of an argument due to Masur-Minsky [MasMi2] to inductively construct arbitrarily long hierarchies with arbitrarily large links. The constants M_1 and M_2 that appear here are recalled from their Lemma 6.2. The link of any vertex in $\mathcal{C}(\Sigma)$ is not connected but its vertex set naturally spans a graph isomorphic to the Farey graph, whose metric we pullback to the link.

The base case of the induction takes a geodesic path $\alpha_0, \alpha_1, \alpha_2$ in $\mathcal{C}(\Sigma)$ such that the distance between α_0 and α_2 in the link of α_1 is strictly greater than M_2. Suppose inductively we have constructed a geodesic path $\alpha_0, \ldots, \alpha_{n-1}$ in the curve graph such that the distance between the subsurface projections of α_0 and α_i to the link of α_{i-1} is strictly greater than M_2 for each index $i \geq 2$. Let α_n be any curve disjoint from α_{n-1} such that the distance between the subsurface projection of α_0 to the link of α_{n-1} and the curve α_n in the link of α_{n-1} is strictly greater than M_2. Lemma 6.2 of [MasMi2] implies that every geodesic path connecting α_0 to α_n must pass through α_{n-1}, and thus through each α_i by induction. That is, $\alpha_0, \ldots, \alpha_n$ is the only geodesic path in the curve graph connecting α_0 to α_n.

We may extend each such geodesic to a hierarchy by taking a geodesic in each vertex link. The converse given in Lemma 6.2 of [MasMi2] implies the lengths of the geodesics in the vertex links are approximated by distances between projections to links to uniform additive error M_1 and are thus arbitrarily large. Moreover each is respectively the distance between the pants decompositions $\{\alpha_{i-1}, \alpha_i\}$ and $\{\alpha_i, \alpha_{i+1}\}$ in the Farey subgraph indexed by α_i, and thus equal to their distance in the whole pants graph by Proposition 18. If we take each to be at least $20hn+15$ say, then Theorem 12 implies the path in the pants graph induced by such a hierarchy must be a geodesic. $\diamondsuit$

Addendum

In the pants graph of the 2-holed torus there exist infinite paths whose every vertex comprises two curves distinct from a non-separating curve α and intersecting α at most once. These paths are typically quasi-geodesics remaining within distance 2 of the Farey graph $\mathcal{F}_\alpha$ but do not contract by $\frac{1}{3}$ under π_α, as defined in [APS1]. Lemma 19 is false as stated for the 2-holed torus.

References

[A1] J. Aramayona, *The Weil-Petersson geometry of the five-times punctured sphere* : in "Spaces of Kleinian Groups", LMS Lecture Notes 329, C. U. P., editors Y. N. Minsky, M. Sakuma and C. M. Series, (2006) 219–231.

[A2] J. Aramayona, *Simplicial embeddings between pants graphs* : Geom. Dedi. **144** No. 1 (February 2010) 115–128.

[APS1] J. Aramayona, H. W. M. Parlier, K. J. Shackleton, *Totally geodesic subgraphs of the pants complex* : Math. Res. Lett. **15** No. 2-3 (2008) 309–320.

[APS2] J. Aramayona, H. W. M. Parlier, K. J. Shackleton, *Constructing convex planes in the pants complex* : Proc. Amer. Math. Soc. **137** (2009) 3523–3531.

[BeF] M. Bestvina, K. Fujiwara, *Bounded cohomology of subgroups of the mapping class groups* : Geom. Topol. **6** (2002) 69–89.

[Beh] J. A. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space* : Geom. Topol. **10** (2006) 1523–1578.

[Bo1] B. H. Bowditch, *Tight geodesics in the curve complex* : Invent. Math. **171** (2008) 281–300.

[Bo2] B. H. Bowditch, *Markoff triples and quasi-Fuchsian groups*, Proc. London Math. Soc. (3) 77:3 (1998) 697–736.

[Bo3] B. H. Bowditch, *Notes on Gromov's hyperbolicity criterion for path-metric spaces* : in "Group theory from a geometrical viewpoint" (ed. E.Ghys, A.Haefliger, A.Verjovsky) World Scientific (1991) 64–167.

[BoE] B. H. Bowditch, D. B. A. Epstein, *Natural triangulations associated to a surface*, Topology **27** (1988) 91–117.

[Br1] J. F. Brock, *Weil-Petersson translation distance and volumes of mapping tori* : Comm. Anal. Geom. **11** (2003) 987–999.

[Br2] J. F. Brock, *The Weil-Petersson metric and volumes of* 3*-dimensional hyperbolic convex cores* : J. Amer. Math. Soc. **16** No. 3 (2003) 495–535.

[BrFa] J. F. Brock, B. Farb, *Curvature and rank of Teichmüller space* : Amer. J. Math. **128** (2006) 1–22.

[BrMas] J. F. Brock, H. A. Masur, *Coarse and synthetic Weil-Petersson geometry: quasi-flats, geodesics, and relative hyperbolicity* : Geom. Topol. **12** No. 4 (2008) 2453–2495.

[CBl] A. J. Casson, S. A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston* : London Math. Soc., Student Texts **9** (1982).

[DaWe] G. D. Daskalopoulos, R. A. Wentworth, *Classification of Weil-Petersson isometries* : Amer. J. Math. **125** No. 4 (2003) 941–975.

[De] T. Delzant, *Sous-groupes distingués et quotients des groupes hyperboliques* : Duke Math. J. **83** (1996) 661–682.

[GH] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'aprés Mikhael Gromov* : Birkhäuser (1990).

[Ha] W. J. Harvey, *Boundary structure of the modular group* : in "Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference" (ed. I. Kra, B. Maskit), Ann. of Math. Studies No. 97, P. U. P. (1981) 245-251.

[HT] A. E. Hatcher, W. P. Thurston, *A presentation for the mapping class group of a closed orientable surface* : Topology **19** (1980) 221–237.

[K] T. Kobayashi, *Heights of simple loops and pseudo-Anosov homeomorphisms* : in "Braids (Santa Cruz, CA, 1986)" Contemp. Math. **78** (1988) 327–338.

[L] F. Luo, *Automorphisms of the complex of curves* : Topology **39** No. 2 (2000) 283–298.

[Mar] D. Margalit, *Automorphisms of the pants complex* : Duke Math. J. **121** No. 3 (2004) 457–479.

[MasMi1] H. A. Masur, Y. N. Minsky, *Geometry of the complex of curves I: Hyperbolicity* : Invent. Math. **138** No. 1 (1999) 103–149.

[MasMi2] H. A. Masur, Y. N. Minsky, *Geometry of the complex of curves II: Hierarchical structure* : Geom. Func. Anal. **10** No. 4 (2000) 902–974.

[MasW] H. A. Masur, M. Wolf, *The Weil-Petersson isometry group* : Geom. Dedi. **93** (2002) 177–190.

[O] T. Oetiker, H. Partl, I. Hyna, E. Schlegl, *The not so short introduction to LaTeX* 2_ϵ : available at CTAN:/tex-archive/info/lshort.

[Se] C. M. Series, *The modular surface and continued fractions* : J. London Math. Soc. **31** (1985) 69–80.

[Sh1] K. J. Shackleton, *Computing distances in two pants complexes* : preprint.

[Sh2] K. J. Shackleton, *An acylindricity theorem for the mapping class group* : preprint.

[Wo] S. A. Wolpert, *Geodesic length functions and the Nielsen problem*: J. Differential Geom. **25** No. 2 (1987) 275–296.

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