

Computing distances in two pants complexes.

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ABSTRACT: We address the lack of local finiteness in the pants graph of the 5-holed sphere and the 2-holed torus by computational means. We find an explicit finite-time algorithm for the constructing of all geodesic paths between any two vertices and, in particular, the computing of distances in either graph.

KEYWORDS: pants complex; geodesic; algorithm.

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§1. Introduction.

Let Σ be a borderless, connected and orientable surface of finite type. After Hatcher-Thurston [HT], to the surface Σ one may associate a simplicial graph $\mathcal{P}(\Sigma)$, the *pants graph*, whose vertex set $X(\Sigma)$ comprises of all pants decompositions of Σ and any two vertices are connected by an edge if and only if they differ by an elementary move; see §2.2 for an expanded definition. This graph is connected [HT], and one may define a path-metric d on $\mathcal{P}(\Sigma)$, or more precisely $X(\Sigma)$, by first assigning length 1 to each edge and then regarding the result as a length space. The pants graph of the 4-holed sphere and of the 1-holed torus are both isometric to the Farey graph with similarly defined path-metric. With the trivial exception of the 3-holed sphere, the pants graph of a hyperbolic surface is otherwise non-planar.

The pants graph, with its own geometry, is a fundamental object to study. Brock [Br1] revealed deep connections with hyperbolic 3-manifolds and proved the pants graph is the correct combinatorial model for the Weil-Petersson metric on Teichmüller space, for the two are quasi-isometric. The isometry group of (\mathcal{P}, d) is also correct in so far as the study of surface groups is concerned for Margalit [Mar] proved it is almost always isomorphic to the mapping class group; indeed, injections between pants graphs are all induced by surface inclusions [A2]. In addition, Masur-Schleimer [MasSch] proved the pants graph to be one-ended for closed surfaces of genus at least 3. With only a few exceptions, the pants graph is not hyperbolic in the sense of Gromov [MasMi2, BrFa, Beh, A1, APS2] or even strongly relatively hyperbolic [BrMas]. The extrinsic geometry of many natural subgraphs of the pants graph is studied in [APS1,2] and [S3].

The focus of this paper is a question of Brock's from [Br2].

Question: [Brock] *Does there exist an explicit algorithm which takes as input a surface Σ and any two pants decompositions $\mu, \nu \in X(\Sigma)$, and returns the distance $d(\mu, \nu)$ in $\mathcal{P}(\Sigma)$ between μ and ν ?*

(We regard μ and ν as already represented on the surface Σ in minimal position, though no procedure for choosing such representatives is offered here.)

It is natural to pose this question after reading Leasure's thesis [Lea], in which an affirmative answer to the analogous question for Harvey's *curve graph* [Har] is given for closed surfaces; see also [S1, S2]. While a complete solution for the pants graph remains out of reach for now, we shall in effect prove that one does follow from an affirmative solution to Conjecture 5 of [APS1]. Since this conjecture is shown therein to hold for the 5-holed sphere and for the 2-holed torus, a result recalled as Proposition 13 in §5.1, we find an algorithm for the computing of distances in the corresponding pants graphs.

Theorem 1 *There exists an explicit finite-time algorithm which takes as input a surface Σ , either the 5-holed sphere or the 2-holed torus, and two pants decompositions $\mu, \nu \in X(\Sigma)$, and returns all the geodesic paths connecting μ and ν . In particular, we may read off the distance $d(\mu, \nu)$ between μ and ν .*

The algorithm is admittedly impractical, with astronomical running time even for low distances. The proof of Theorem 1 factors through a second theorem, the analogue of Theorem 6 from [S1] restricted to the 5-holed sphere and the 2-holed torus. The more general statement, without restriction on surface type, would follow from an affirmative solution to Conjecture 5 of [APS1].

Theorem 2 *There exists a computable increasing function $F_0 : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds: For Σ the 5-holed sphere or the 2-holed torus, if (ν_0, \dots, ν_n) is a geodesic in the pants graph $\mathcal{P}(\Sigma)$ then $\iota(\nu_1, \nu_n) \leq F_0(\iota(\nu_0, \nu_n))$.*

Inducting along a given geodesic path, and by reversing this geodesic and repeating, one soon finds the following consequence of Theorem 2.

Theorem 3 *There exists a computable increasing function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds: For Σ the 5-holed sphere or the 2-holed torus, if (ν_0, \dots, ν_n) is a geodesic in the pants graph $\mathcal{P}(\Sigma)$ then, for each index j , both $\iota(\nu_0, \nu_j)$ and $\iota(\nu_j, \nu_n)$ are at most $F(\iota(\nu_0, \nu_n))$.*

The pants graph is nowhere locally finite. It is a noteworthy consequence of Theorem 3 that the pants graph of the 5-holed sphere and the pants graph of the 2-holed torus are both *finitely geodesic*, so that between any two vertices there exist only finitely many geodesic paths. That is, all the Gromov hyperbolic pants graphs are finitely geodesic. This reproves an observation of Lecuire's [Lec] using [APS1]:

Theorem 4 *The pants graph of the 5-holed sphere and the pants graph of the 2-holed torus are finitely geodesic.*

The same argument can also be used to reprove the more general observation of Lecuire’s [Lec], that Conjecture 5 of [APS1] implies the assertion that any pants graph is finitely geodesic; that the converse holds is elementary and follows by first supposing it fails and then applying iterates of an appropriately chosen Dehn twist. In other words, Conjecture 5 of [APS1] is equivalent to asserting the pants graphs are all finitely geodesic, and to asserting the existence of an explicit finite time algorithm for constructing all geodesic paths between any two pants decompositions in any pants graph. Each therefore implies the existence of a distance computing algorithm. Thus, Brock’s question is intimately related to the synthetic geometry of the pants graph and is very much part of a programme to push the geometric understanding of the pants graph from the coarse category to the metric category.

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§2. Background and definitions.

We supply all the background and terminology needed both to understand the statements of our main results, and to make sense of their proofs. Throughout,

we define a *loop* as the homeomorphic image of a standard circle.

§2.1. **Curves and pants decompositions.** A loop on Σ is said to be *trivial* if it bounds a disc and *peripheral* if a component of its complement contains a non-compact one-holed disc. A *curve* is by definition the free homotopy class of a non-trivial and non-peripheral loop. We denote the free homotopy class of a simple, essential and non-peripheral loop e by $[e]$.

Given any two curves α and β , their *intersection number* $\iota(\alpha, \beta)$ is defined equal to $\min\{|\alpha \cap \beta| : a \in \alpha, b \in \beta\}$. We say two curves are *disjoint* if they have zero intersection number, and otherwise say they *intersect essentially*. A *pants decomposition* is then by definition a collection of pairwise distinct and pairwise disjoint curves, maximal subject to inclusion; for the 5-holed sphere and the 2-holed torus only, a pants decomposition contains exactly two curves. We add that our definition differs only slightly from the more usual definition of a pants decomposition, for ours provides the torus with a pants decomposition. The meaning of “pair of pants decompositions” remains regrettably ambiguous; we read this as “(pair of)(pants decompositions)”, in reference to two pants decompositions considered as a pair. The intersection number of two finite sets of curves is defined additively.

§2.2. **Pants graph.** The pants graph $\mathcal{P}(\Sigma)$ of a surface Σ is defined by taking as vertices all the pants decompositions of Σ , and declaring two pants decompositions μ and ν to be connected by an edge if and only if they are related by an *elementary move*, so that $\mu \cap \nu$ is a corank 1 *multicurve* and the remaining two curves together either fill a 4-holed sphere and intersect twice or fill a 1-holed torus and intersect once; consider Figure 1 below.

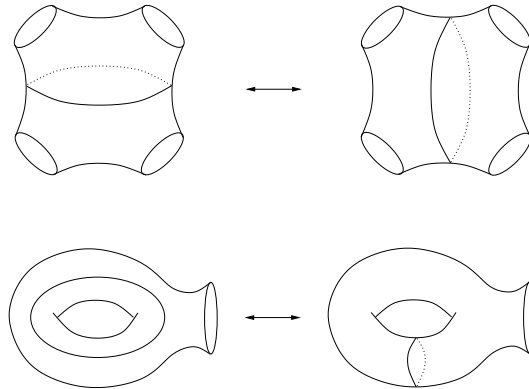


Figure 1: The two types of elementary move defining the pants graph.

It is noted by Hatcher-Thurston [HT] that the pants graph is connected. We may thus endow the pants graph with the standard combinatorial path-metric d , assigning length 1 to each edge. Notice that between the 5-holed sphere and the 2-holed torus the corresponding pants graphs are not isometric, for among other reasons in only that of the 5-holed sphere does every vertex belong to a circuit of length 5 not contained in a single Farey graph. Indeed, there are no such circuits in the pants graph of the 2-holed torus whatsoever (see Lemma 5.6 of [Luo]).

There is however a natural distance non-increasing bijection from the pants graph of the 2-holed torus to the pants graph of the 5-holed sphere, whose inverse is 2-Lipschitz, found by regarding the 2-holed torus as a branched double cover of the disc with four cone points in its interior and each of order 2. In particular, the two graphs are thus quasi-isometric. Combining this with Brock's Theorem 1.1 of [Br1] recovers a natural quasi-isometry from the Weil-Petersson metric on the Teichmüller space of the 2-holed torus to that of the 5-holed sphere.

Proposition 5 *There exists a 1-Lipschitz map ϕ from the pants graph of the 2-holed torus to the pants graph of the 5-holed sphere, with a 2-Lipschitz inverse ϕ^{-1} , such that $\iota(\phi(\mu), \phi(\nu)) \leq 2\iota(\mu, \nu)$ and $\iota(\phi^{-1}(\mu), \phi^{-1}(\nu)) \leq 2\iota(\mu, \nu)$ for all pants decompositions μ and ν on the respective surfaces. Furthermore, both ϕ and ϕ^{-1} map pants decompositions containing a common curve to pants decompositions containing a common curve.*

An elementary proof of the above is provided in §3. An alternate argument, providing us instead with a quasi-isometry of non-explicit parameters, factors through the Weil-Petersson metric and Brock's Theorem 1.1 [Br1].

When Σ is either the 5-holed sphere or the 2-holed torus, the metric d has an explicit linear upper bound in terms of the geometric intersection number.

Lemma 6 *Suppose Σ is either the 5-holed sphere or the 2-holed torus. For any two pants decompositions μ and ν of Σ , we have*

$$d(\mu, \nu) \leq 2\iota(\mu, \nu) + 8.$$

If one considers only the 5-holed sphere, the linear upper bound offered above improves to $\frac{1}{2}\iota(\alpha, \beta) + 8$. There exist generic upper-bounds on distance involving non-computed constants, by appealing to the distance estimate of Theorem 6.12 of [MasMi2], as interpreted for the pants graph, and then the remark following Lemma 2.1 of [MasMi1] and Section 4 therein. The attraction of the bound in Lemma 6 is that it is explicit.

An elementary proof of Lemma 6 proceeds very much like the usual proof that the curve graph of a surface is connected, following an argument of Lickorish [Li], though somewhat complicated by the need to consider many curves

and thus difficult to extend to other surfaces. Since it seems an explicit bound on distance in terms of geometric intersection numbers has yet to be provided in the literature, a full proof of Lemma 6 is offered here in §4.

§2.3. **Arcs graph.** Some of our arguments will be phrased in terms of arcs and arcs pairs, rather than curves and pants decompositions.

Let \mathbb{S}_5^2 denote the 2-sphere with 5 distinguished points, referred to as *marked points*. An *arc* is the relative homotopy class of a compact interval on \mathbb{S}_5^2 connecting precisely two marked points. For an arc α we denote the set of such marked points by $\partial\alpha$. The *arcs graph* \mathcal{A} is formed by taking as vertices all pairs of distinct and disjoint arcs, called *arc pairs*, connecting four distinct marked points of \mathbb{S}_5^2 ; two vertices are then connected by an edge if they are related by an *elementary move*, so that one is obtained from the other by replacing an arc with a second arc sharing only one end and otherwise disjoint from the original arc pair.

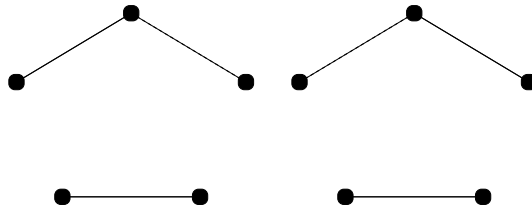


Figure 2: Two arcs pairs related by an elementary move.

The intersection number of a collection of arcs and curves with a second collection of arcs and curves is defined additively, with only intersection away from the marked points contributing; thus a pair of adjacent arcs pairs has zero intersection number. Note that unlike a pants decomposition, every arcs pair has zero intersection number with infinitely many arcs pairs. We assign length 1 to each edge of \mathcal{A} , and endow \mathcal{A} with the induced path-metric $d_{\mathcal{A}}$. For Σ the 5-holed sphere, the natural collapsing map Φ from curves to arcs induces the canonical isometry $\Phi : \mathcal{P}(\Sigma) \rightarrow \mathcal{A}$ satisfying $4\iota(\Phi(\mu), \Phi(\nu)) \leq \iota(\mu, \nu)$ for any two pants decompositions $\mu, \nu \in X(\Sigma)$.

§2.4. **Parallel intervals.** For any compact subset A of Σ , we say that a compact interval h is *properly embedded* in (Σ, A) if $h \cap A = \partial h$ and there are no discs, or *bigons*, bordered by h and a smooth interval in A . Two compact intervals h_0 and h_1 in (Σ, A) are *parallel* if there exists a continuous map $H : h_0 \times [0, 1] \rightarrow \Sigma$ such that $H^{-1}(A) = (h_0 \cap A) \times [0, 1]$ and $H(h_0 \times \{i\}) = h_i$ for $i \in \{0, 1\}$. Two intervals h_0 and h_1 in (Σ, A) are *homotopically disjoint* if there exists a continuous map $H : h_0 \times [0, 1] \rightarrow \Sigma$ such that $H(h_0 \times \{0\}) = h_0$, $H^{-1}(A) = (h_0 \cap A) \times [0, 1]$

and $H(h_0 \times \{1\}) \cap h_1$ is empty.

§3. Proof of Proposition 5.

In this section we let Σ denote the 2-holed torus. Let h denote any hyperelliptic involution of Σ ; by Kerckhoff's solution [Ke] to the Nielsen realisation problem, there exists a hyperbolic metric on Σ for which h is isotopic to an isometry. Replacing h by an isotopic homeomorphism if need be, we assume h is a self-isometry so that it stabilises every simple closed geodesic. Every pants decomposition of Σ is uniquely represented by a pair of simple closed geodesics in this metric.

To the surface Σ we introduce the equivalence relation $x \sim hx$, and denote by $\pi : \Sigma \rightarrow \Sigma / \sim$ the quotient map. The space $\pi(\Sigma)$ is conformally a disc with four singularities of order 2. Let Σ' denote the 5-holed sphere, regarded as the subsurface of $\pi(\Sigma)$ obtained by removing each singular point and the boundary of $\pi(\Sigma)$.

Since h stabilises every simple closed geodesic, the π -image of any simple closed geodesic is a simple closed loop or an interval and the π -image of two disjoint simple closed geodesics is a pair of disjoint simple closed loops or intervals. Thus π induces a well-defined map $\phi : X(\Sigma) \rightarrow X(\Sigma')$ on pants decompositions, defined by taking the isotopy class on Σ' (without multiplicity) of each essential component in the boundary of a small regular neighbourhood of $\pi(\mu)$ for each pair of simple closed geodesics μ on Σ . Even if a given pair of simple closed geodesics μ passes through the π -preimage of any singularity, since pants decompositions are maximal collections of disjoint curves the image $\phi(\mu)$ is uniquely determined. The inverse ϕ^{-1} of ϕ is defined by taking $\phi^{-1}(\mu)$ to be the pants decomposition represented among the essential components of $\pi^{-1}(\mu)$, for each pants decomposition μ represented on Σ' in minimal position. That ϕ and ϕ^{-1} both at most double intersection numbers is apparent from their construction, and in fact both respectively double the intersection number of some pair of pants decompositions. Also apparent from their construction, both ϕ and ϕ^{-1} map pants decompositions containing a common curve to pants decompositions containing a common curve.

Two pants decompositions of intersection number 4 are distance 2 apart in both the pants graph of the 5-holed sphere and the pants graph of the 2-holed torus. Thus, we already know that ϕ and ϕ^{-1} are both 2-Lipschitz. With regard to ϕ^{-1} one can do no better: there exist adjacent pants decompositions in the pants graph of the 5-holed sphere mapped by ϕ^{-1} to a pair of pants decompositions of intersection number 4 and thus of distance 2.

Lemma 7 *The map ϕ is 1-Lipschitz.*

Proof: Since adjacent pants decompositions on the 2-holed torus containing distinct non-separating curves intersect exactly once, their ϕ -images intersect

twice and thus are adjacent in the pants graph of the 5-holed sphere. The issue therefore rests with pairs of adjacent pants decompositions that intersect twice. Note exactly one of two such pants decompositions contains a separating curve.

We begin with the general remark that the mapping class group $\text{Map}(\Sigma)$ of the 2-holed torus acts naturally on the vertex set $X(\Sigma')$ of the pants graph of the 5-holed sphere, by defining $f\nu = \phi f \phi^{-1}(\nu)$ for each mapping class $f \in \text{Map}(\Sigma)$ and each pants decomposition $\nu \in X(\Sigma')$. We as yet make no claims about the nature of this action, other than it respects the intersection of two pants decompositions over a single curve, equivalently that it permutes all the Farey subgraphs, and thus descends to an action on Harvey's curve graph $\mathcal{G}(\Sigma')$ of the 5-holed sphere by automorphisms. The automorphism group of $\mathcal{G}(\Sigma')$ is naturally a homomorphic image of the mapping class group $\text{Map}(\Sigma')$ (see Theorem 2.12 of [Ko]). In particular, the action of $\text{Map}(\Sigma)$ on $X(\Sigma')$ necessarily preserves the intersection number of two pants decompositions. Thus, for any mapping class $f \in \text{Map}(\Sigma)$ of the 2-holed torus and any two pants decompositions $\mu, \nu \in X(\Sigma')$ of the 5-holed sphere, we have:

$$\begin{aligned} \iota(\phi(f\mu), \phi(f\nu)) &= \iota(\phi(f\phi^{-1}\phi(\mu)), \phi(f\phi^{-1}\phi(\nu))) \\ &= \iota(\phi f \phi^{-1}(\phi(\mu)), \phi f \phi^{-1}(\phi(\nu))) \\ &= \iota(f\phi(\mu), f\phi(\nu)) \\ &= \iota(\phi(\mu), \phi(\nu)). \end{aligned}$$

The action of $\text{Map}(\Sigma)$ on the set $X(\Sigma)$ is cofinite, indeed its orbit space has cardinality 2. To finally verify that ϕ is 1-Lipschitz, it now suffices to check that ϕ maps at least one pair of adjacent pants decompositions with intersection number 2 to an adjacent pair of pants decompositions. (\diamond)

This completes a proof of Proposition 5.

§4. Proof of Lemma 6.

The purpose of this section is to prove Lemma 6. In fact, we need only prove Lemma 10, supplying an explicit upper bound for distances in the arcs graph: an explicit upper bound for distances in the pants graph of the 5-holed sphere and of the 2-holed torus is soon deduced.

The following provides us with two useful criteria for recognising small distances in the arcs graph in terms of geometric intersection numbers.

Lemma 8 *Suppose that $\{\delta_1, \delta_2\}$ and $\{\delta'_1, \delta'_2\}$ are two arc pairs on \mathbb{S}_5^2 such that $\iota(\delta_i, \delta'_i) = 0$ for $i \in \{1, 2\}$, $\iota(\delta_1, \delta'_2) \leq 1$, and $\iota(\delta_2, \delta'_1) \leq 1$. Then, $d_{\mathcal{A}}(\{\delta_1, \delta_2\}, \{\delta'_1, \delta'_2\}) \leq 4$.*

Proof: If $\iota(\delta_i, \delta'_j) = 0$ for $i, j \in \{1, 2\}$ then the arc pairs $\{\delta_1, \delta_2\}$ and $\{\delta'_1, \delta'_2\}$ are equal, have one arc in common, or together form four sides of a pentagon

configuration. One can quickly verify that $d_{\mathcal{A}}(\{\delta_1, \delta_2\}, \{\delta'_1, \delta'_2\}) \leq 2$. In the case $\iota(\delta_1, \delta'_2) = 0$ and $\iota(\delta_2, \delta'_1) = 1$, if $\delta_1 \in \{\delta'_1, \delta'_2\}$ then a similar computation shows our distance is at most 2, otherwise our distance is at most 3.

In the remaining case, for which $\iota(\delta_1, \delta'_2) = \iota(\delta_2, \delta'_1) = 1$, in exactly two elementary moves we pass from $\{\delta_1, \delta_2\}$ to the arc pair $\{\delta''_1, \delta''_2\}$ such that $\delta''_1, \delta''_2, \delta'_1$ and δ'_2 together make up four sides of a pentagon configuration. We thus need only two further moves, one on δ''_1 and one on δ''_2 , to arrive at $\{\delta'_1, \delta'_2\}$. By the triangle inequality, it follows that $d_{\mathcal{A}}(\{\delta_1, \delta_2\}, \{\delta'_1, \delta'_2\})$ is at most 4 (indeed it is exactly 4). \diamond

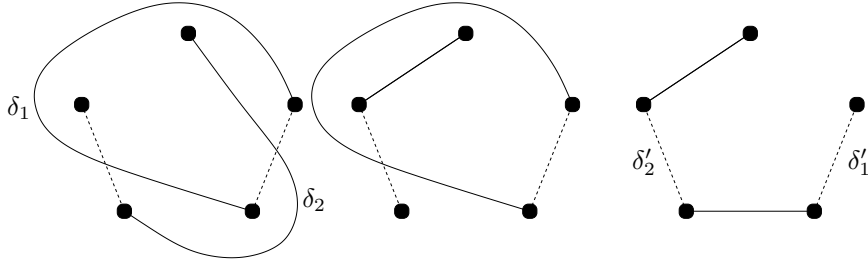


Figure 3: The first three vertices of a path.

A proof of Lemma 9 is similar and left as an exercise to the reader.

Lemma 9 *Suppose $\{\delta_1, \delta_2\}$ and $\{\delta'_1, \delta'_2\}$ are two arc pairs on \mathbb{S}_5^2 such that $\iota(\delta_1, \delta'_1) \leq 2$ and $\iota(\delta_2, \delta'_1) \leq 1$, and such that $\iota(\delta_{1\sigma}, \delta'_2) \leq 2$ and $\iota(\delta_{2\sigma}, \delta'_1) \leq 1$ for some permutation $\sigma \in \text{Sym}\{1, 2\}$. Then, $d_{\mathcal{A}}(\{\delta_1, \delta_2\}, \{\delta'_1, \delta'_2\}) \leq 8$.*

We add the reader need be in no doubt that in this setting an explicit bound on $d_{\mathcal{A}}(\{\delta_1, \delta_2\}, \{\delta'_1, \delta'_2\})$ does exist, taking a gratuitously large power of 10 if so desired. This is akin to many coarse arguments in hyperbolic geometry. That said, an exhaustive case-by-case argument does reveal that 8 is sufficient.

Let α and β be two arcs on \mathbb{S}_5^2 such that $\iota(\alpha, \beta) \geq 2$; we describe a standard procedure to obtain a new arc $\alpha\beta$, also depicted in Figure 4. This procedure is non-commutative, for $\alpha\beta$ will generally be distinct from $\beta\alpha$. First realise α and β in minimal position, and denote by b any subinterval of β connecting two points of $\alpha \cap \beta$ but otherwise disjoint from α . Let a_1 and a_2 be the two disjoint compact subintervals of α connecting either end of b to the ends of α . The homotopy class $\alpha\beta$ of $a_1 \cup b \cup a_2$ is an arc on \mathbb{S}_5^2 connecting the same two marked points as α and such that $\iota(\alpha, \alpha\beta) = 0$ and such that $\iota(\alpha\beta, \beta) \leq \iota(\alpha, \beta) - 1$ if b is not empty. While it is not formally correct to regard $\alpha\beta$ as anything other than a family of arcs determined by α and β , for our purposes we may regard

$\alpha\beta$ as a preferred arc.

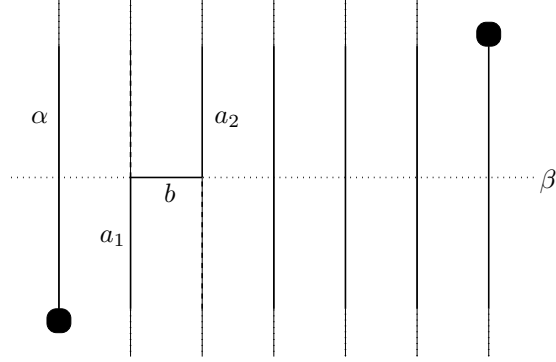


Figure 4: The surgery of one arc along another.

Consider an arc system $\{\alpha_1, \alpha_2\}$ and an arc β such that $\iota(\alpha_i, \beta) \geq 2$ for $i \in \{1, 2\}$. There exist disjoint arcs of the form $\alpha_1\beta$ and $\alpha_2\beta$ such that $\alpha_1\beta \neq \alpha_1$, $\iota(\alpha_1\beta, \alpha_2) \leq 1$, and $\iota(\alpha_2\beta, \alpha_1) \leq 1$. Moreover, since α_1 and α_2 are not incident on a common marked point so $\alpha_1\beta$ and $\alpha_2\beta$ are not incident on a common marked point. Thus, $\{\alpha_1\beta, \alpha_2\beta\}$ is an arcs pair, and we note that $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1\beta, \alpha_2\beta\}$ together satisfy the conditions of Lemma 8. Thus $d_{\mathcal{A}}(\{\alpha_1, \alpha_2\}, \{\alpha_1\beta, \alpha_2\beta\}) \leq 4$, and the proof of Lemma 8 even provides us with an explicit path.

We are now ready to construct a path in \mathcal{A} between two given arc pairs $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$. By applying the above procedure iteratively, first to $\{\alpha_1, \alpha_2\}$ relative to β_1 and then relative to β_2 , we produce a finite sequence of arc pairs $\{\alpha_1, \alpha_2\} = \{\alpha_1^0, \alpha_2^0\}, \{\alpha_1^1, \alpha_2^1\}, \dots, \{\alpha_1^n, \alpha_2^n\}$ such that consecutive arc pairs are of distance at most 4 and, reindexing if need be, such that $\iota(\alpha_1^n, \beta_1) \leq 2$ and $\iota(\alpha_2^n, \beta_1) \leq 1$, and such that $\iota(\alpha_{1\sigma}^n, \beta_2) \leq 2$ and $\iota(\alpha_{2\sigma}^n, \beta_2) \leq 1$ for some permutation $\sigma \in \text{Sym}\{1, 2\}$. According to Lemma 9, we have $d_{\mathcal{A}}(\{\alpha_1^n, \alpha_2^n\}, \{\beta_1, \beta_2\}) \leq 8$. We also note that

$$\iota(\{\alpha_1^{j-1}, \alpha_2^{j-1}\}, \{\beta_1, \beta_2\}) \leq \iota(\{\alpha_1^j, \alpha_2^j\}, \{\beta_1, \beta_2\}) - 2$$

for each $j \in \{1, \dots, n\}$.

The conclusion of this discussion is the following estimate.

Lemma 10 *Let μ and ν be any two vertices of \mathcal{A} . Then,*

$$d_{\mathcal{A}}(\mu, \nu) \leq 2\iota(\mu, \nu) + 8.$$

From this we soon deduce the following.

Corollary 11 *Suppose Σ is the 5-holed sphere. For $\mu, \nu \in X(\Sigma)$ any two pants decompositions of Σ , we have*

$$d(\mu, \nu) \leq \frac{1}{2}\iota(\mu, \nu) + 8.$$

Proof: The canonical isometry $\Phi : \mathcal{P}(\Sigma) \rightarrow \mathcal{A}$ satisfies $4\iota(\Phi(\mu), \Phi(\nu)) \leq \iota(\mu, \nu)$ for any two pants decompositions $\mu, \nu \in X(\Sigma)$. Thus: $d(\mu, \nu) = d(\Phi(\mu), \Phi(\nu)) \leq 2\iota(\Phi(\mu), \Phi(\nu)) + 8 \leq \frac{1}{2}\iota(\mu, \nu) + 8$. \diamond

Corollary 12 *Suppose Σ is the 2-holed torus. For $\mu, \nu \in X(\Sigma)$ any two pants decompositions of Σ , we have*

$$d(\mu, \nu) \leq 2\iota(\mu, \nu) + 8.$$

Proof: Let μ and ν be any two pants decompositions of Σ , and recall the biLip-schitz map ϕ from the pants graph of the 2-holed torus to that of the 5-holed sphere offered by Proposition 5. We have: $d(\mu, \nu) = d(\phi^{-1}\mu, \phi^{-1}\nu) \leq 2d(\mu, \nu) \leq \iota(\mu, \nu) + 8 \leq 2\iota(\mu, \nu) + 8$. \diamond

This completes a proof of Lemma 6. \diamond

§5. Proof of Theorem 2.

§5.1. Totally geodesic subgraphs.

We begin with a formulation of Theorem 1 from [APS1] for Σ the 5-holed sphere and the 2-holed torus. In effect, this will serve as a viable substitute for the algebraic tightness condition of Bowditch [Bo] whenever it was called upon in [S1, S2].

Proposition 13 *Let (ν_0, \dots, ν_n) be a geodesic in the pants graph $\mathcal{P}(\Sigma)$. Then, $\nu_0 \cap \nu_n \subseteq \nu_j$ for each index $j \in \{0, \dots, n\}$.*

Proof: The vertices of any Farey subgraph of the pants graph of the 5-holed sphere or the 2-holed torus all contain a common curve (Lemma 2 of [Mar]), and conversely any curve uniquely determines a Farey subgraph. Theorem 1 of [APS1] asserts that every Farey subgraph of the pants graph is totally geodesic.

That is, if P_1 and P_2 are two pants decompositions of either surface, every vertex on a geodesic path connecting P_1 and P_2 must contain the set of curves $P_1 \cap P_2$. \diamond

§5.2. Curve surgery.

Lemma 14 *Let γ and β be any two curves on Σ , and let $c \in \gamma, b \in \beta$ be simple representatives such that $|c \cap b| = \iota(\gamma, \beta)$. Let J be any compact non-empty subinterval of b . Suppose there exists a compact subinterval h of c such that $|h \cap J| = 3$. Then, $N(h \cup J)$ contains a simple loop representing a curve δ such that $\iota(\delta, \gamma) > 0$ and $\iota(\delta, \beta) > 0$.*

Proof: We construct a parameterised loop $p : [0, 1] \rightarrow \Sigma$ as follows. Let x_1, x_2 and x_3 denote the three points of $h \cap J$ where x_2 separates x_1 and x_3 along h , and let h_1 denote the subinterval of h connecting x_1 and x_2 and let h_2 the subinterval connecting x_2 and x_3 . The construction falls into three types (see Figure).

- (Type 1) If there exists $i \in \{1, 2\}$ such that h_i is incident to J from both sides we form p by traversing h_i and then the subinterval of J spanned by ∂h_i .
- (Type 2) Otherwise, if x_2 does not separate x_1 and x_3 along J we form p by traversing h from x_1 and then the subinterval of J spanned by x_3 and x_1 .
- (Type 3) If instead x_2 separates x_1 and x_3 along J we form p by traversing h_1 from x_1 to x_2 , J from x_2 to x_3 , h_2 from x_3 to x_2 , and then J from x_2 to x_1 .

We extend p to a periodic map $p : \mathbb{R} \rightarrow \Sigma$ such that $p(t+1) = p(t)$ for all $t \in \mathbb{R}$. Let $\pi : \mathbb{H}^2 \rightarrow \Sigma$ denote a covering map and \tilde{p} any lift of p . We denote $\tilde{p}(i)$ by z_i , the lift of b containing z_i by \tilde{b}_i , and the lift of c containing z_i by \tilde{c}_i for each integer $i \in \mathbb{Z}$. We note \tilde{b}_i and \tilde{c}_i are uniform quasi-geodesics, and that \tilde{b}_{i-1} and \tilde{b}_{i+1} are separated in \mathbb{H}^2 by \tilde{b}_i and that \tilde{c}_{i-1} and \tilde{c}_{i+1} are separated in \mathbb{H}^2 by \tilde{c}_i for every $i \in \mathbb{Z}$ regardless of the type defining p . In particular we note $z_0 \neq z_1$, so $p(\mathbb{R})$ is not homotopically trivial, and that $\tilde{p}(\mathbb{R})$ intersects every \tilde{b}_i and every \tilde{c}_i in a single component.

There exists a component \tilde{e} of the boundary of a small regular neighbourhood of $\tilde{p}(\mathbb{R})$ such that $\pi(\tilde{e})$ is a simple loop and regardless of the type defining p . We denote this loop by e . Since \tilde{e} is homotopic to $\tilde{p}(\mathbb{R})$ we note e is not a trivial loop. Moreover, \tilde{e} intersects each \tilde{b}_i and each \tilde{c}_i in a single point. It follows e is also not peripheral, defining a curve δ whose intersection number with γ and β is positive. \diamond

§5.3. Realising curves.

From now on Σ is assumed to have a smooth structure, (ν_0, \dots, ν_n) is a geodesic in the pants graph with $n \geq 2$, c_n is the component of $v_n \in \nu_n$ containing an arbitrary compact subinterval J of $v_n \setminus v_0$ for $v_0 \in \nu_0$, and the multiloops $v_i \in \nu_i$, each i , together satisfy the following conditions:

- (1) the two components of v_i are disjoint, simple and smooth for each i ;
- (2) v_i and v_j have only transverse intersection away from any common loop for $i + 1 \leq j$;
- (3) $v_i \cap v_{i+1}$ contains a single loop for each $i \leq n - 1$;
- (4) $|v_i \cap v_{i+1}| = \iota(\nu_i, \nu_{i+1})$ for each $i \leq n - 1$;
- (5) $|v_i \cap v_n| = \iota(\nu_i, \nu_n)$ for each $i \leq n - 2$, and
- (6) $v_i \cup v_j \cup v_n$ has no points of valence 6 for $i + 1 \leq j \leq n - 1$.

We could for instance endow Σ with a complete hyperbolic metric and represent each ν_i uniquely by a collection of simple closed geodesics to give (1)–(5). Perturbing these geodesics if need be, in addition we then have (6).

§5.4. Minor intersection.

Lemma 15 *Suppose $|c_1 \cap J| \geq 9$ for some component c_1 of v_1 . Then, v_2 and J have non-empty and only transverse intersection.*

Proof: If $c_1 \subset v_2$ then v_2 already intersects J transversely, so suppose c_1 is not contained in v_2 . There then exists a subinterval h of c_1 such that $|h \cap J| = 3$ and both $v_0 \cap h$ and $v_2 \cap h$ are empty. According to Lemma 14 there exists a loop $e \subset N(h \cup J)$ representing a curve δ such that $\iota(\delta, \nu_0) = 0$ and $\iota(\delta, [c_1]) > 0$. It follows δ is contained in ν_0 but not contained in ν_1 . If $v_2 \cap J$ is empty then $e \cap v_2 = \emptyset$ and $\iota(\delta, \nu_2) = 0$, as such $\delta \in \nu_2$. It follows $\delta \in \nu_0 \cap \nu_2$ and yet $\delta \notin \nu_1$. This contradicts Proposition 13, and we conclude $v_2 \cap J$ must be non-empty. Since $|v_1 \cap J| \geq 3$ and $\iota(\nu_1, \nu_2) \leq 2$, we conclude the intersection of v_2 with J is transverse. \diamond

Corollary 16 *Suppose $|v_1 \cap J| \geq 17$. Then, v_2 and J have non-empty and only transverse intersection.*

Proof: By the pigeonhole principle there exists a component c_1 of v_1 such that $|c_1 \cap J| \geq 9$. The claim now follows from Lemma 15. \diamond

§5.5. Nested intersection.

Lemma 17 *For $m \geq 2$, suppose there exists a continuous map $H : [0, 1] \times [0, 1] \rightarrow \Sigma$ such that:*

- (1) $H([0, 1] \times \{0, \frac{1}{2}, 1\}) \subseteq J$;
- (2) $H(\{0, 1\} \times [0, 1]) \subseteq v_1$;
- (3) $H(\{\frac{1}{2}\} \times [0, 1]) \subseteq v_m$;
- (4) $H(\{\frac{1}{2}\} \times [0, 1])$ is disjoint from v_0 , and
- (5) $H(\{\frac{1}{2}\} \times [0, 1])$ does not intersect v_m transversely.

Let $J_0 \subseteq J$ be any compact interval containing $H(\{\frac{1}{2}\} \times \{0, \frac{1}{2}, 1\})$. Then, v_{m+1} and J_0 have non-empty and only transverse intersection.

Proof: We view the proof of Lemma 15 as a template for a proof here.

The image $H(\{\frac{1}{2}\} \times [0, 1])$ is necessarily an embedded interval, for otherwise v_0 contains at least two distinct parallel components and this is absurd. Let h denote the interval $H(\{\frac{1}{2}\} \times [0, 1])$ and c_m the component of v_m containing h . (See Figure 6.)

We apply Lemma 14, with $\gamma = [c_m], \beta = [c_n], c = c_m, b = c_n$ and to h along J_0 , to find a simple loop $e_m \subset N(h \cup J_0)$ representing a curve $\delta = [e_m]$ such that $\iota(\delta, [c_m]) > 0$ and as such $\delta \notin \nu_m$. Since $v_0 \cap h$ and $v_0 \cap J_0$ are both empty so, having made a suitable choice for $N(h \cup J_0)$, we see $e_m \cap v_0 = \emptyset$ and conclude $\iota(\delta, \nu_0) = 0$. It follows $\delta \in \nu_0$.

Suppose for contradiction that $v_{m+1} \cap J_0$ is empty. Then, since h does not intersect v_{m+1} transversely, it follows v_{m+1} is disjoint from $h \cup J_0$ and so $\iota(\delta, \nu_{m+1}) = 0$. In particular, $\delta \in \nu_{m+1}$. We conclude that the geodesic path $(\nu_0, \dots, \nu_{m+1})$ violates the statement of Proposition 13, since $\delta \in \nu_0 \cap \nu_{m+1}$ while $\delta \notin \nu_m$. Thus, $v_{m+1} \cap J_0$ is not empty. Since $|v_m \cap J_0| \geq 3$ and $|v_m \pitchfork v_{m+1}| \leq 2$ we conclude v_{m+1} only intersects J_0 transversely. \diamond

§5.6. Parallel intervals.

Lemma 18 *Let s be a non-negative integer. There exists an explicit constant $K_0 = K_0(s)$ such that in any set of at least K_0 compact intervals properly embedded in (Σ, J) there are at least s parallel intervals.*

Proof: A set of pairwise homotopically disjoint non-parallel intervals properly embedded in (Σ, J) has cardinality at most 7. By the pigeonhole principle, we can define $K_0(s) := 7s - 6$. \diamond

We may apply Lemma 18 piecewise to find the following.

Corollary 19 *Let s be a non-negative integer. There exists an explicit constant $K = K(s)$ such that in any set Δ of at least K pairwise homotopically*

disjoint compact intervals each intersecting J three times and whose boundaries are contained in J there are at least s intervals parallel in (Σ, J) .

Proof: A fairly crude constant $K(s)$ can be obtained as follows. Suppose Δ contains at least $K_0(K_0(s))$ such intervals. Each of these intervals is expressed as the union of two intervals properly embedded in (Σ, J) . By Lemma 18, there exist $K_0(s)$ intervals in Δ containing a properly embedded subinterval parallel to a common interval. Among these intervals, Lemma 18 implies there exist s parallel intervals. We thus set $K(s) = K_0(K_0(s))$. \diamond

§5.7. Large intersection.

Lemma 20 *There exists a computable function $G : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds. For m a non-negative integer, if $|v_1 \cap J| \geq G(m)$ there exist nested compact subintervals $J_1 \supseteq \cdots \supseteq J_m$ of J such that:*

- (1) $\partial J_i \subset v_i$ for each $i \in \{1, \dots, m\}$, and
- (2) v_{m+1} and J have non-empty and only transverse intersection.

Proof: We define $G(0) := 0$ and note $G(1) := 17$ satisfies our assertion for $m = 1$, by Corollary 16. We regard the latter as the base case for an induction on m .

Suppose inductively we have found $G(m)$. If $|v_1 \cap J| \geq K(10m)G(m)$ we can partition J into the union of $K(10m)$ compact subintervals J_1^i each intersecting v_1 at least $G(m)$ times. The inductive hypothesis further implies the existence of nested compact intervals $J_1^i \supseteq \cdots \supseteq J_m^i$ such that $\partial J_j^i \subset v_j$ and $v_{m+1} \cap J_m^i \neq \emptyset$ for each index i and j .

Let Δ denote the set of all compact subintervals h of v_1 such that $|h \cap J| = 3$, $\partial h \subset J$ and $\partial h \cap J_1^i \neq \emptyset$ for some index i . We apply Corollary 19 to this collection Δ of at least $K(10m)$ intervals to find at least $10m$ parallel intervals. Among these parallel intervals two are related by a homotopy $H : [0, 1] \times [0, 1] \rightarrow \Sigma$ such that in addition $H([0, 1] \times \{0\}) = J_1^i$ for some index i . As $|v_j \cap v_{j+1}| \leq 2$ for each $j \in \{0, \dots, m\}$ and $J_1^i \supseteq \cdots \supseteq J_m^i$, there exists a homotopy H so that furthermore $I_j := H(\{\frac{1}{2} \pm \frac{m+1-j}{2m}\} \times [0, 1]) \subseteq v_j$ and the images I_j are disjoint from v_0 and have no transverse intersection with v_{j+1} for each $j \in \{1, \dots, m+1\}$. The homotopy H is “swept out” by $J_1^i \supseteq \cdots \supseteq J_m^i$, so to speak.

Let \widehat{J}_j denote the intersection of all subintervals of J containing $I_j \cap J$ for each $j \in \{1, \dots, m+1\}$. Note that $\widehat{J}_1 \supseteq \cdots \supseteq \widehat{J}_{m+1}$ and that $\partial \widehat{J}_j \subset v_j$ for each $j \in \{1, \dots, m+1\}$. We also note that I_{m+1} is a now subinterval of v_{m+1} intersecting \widehat{J}_{m+1} at least three times. Appealing to Lemma 17, we deduce v_{m+2} and \widehat{J}_{m+1} have non-empty and only transverse intersection.

We define $G(m+1)$ equal to $K(10m)G(m)$, completing the induction. \diamond

§5.8. **Intersection bounds.**

A proof of Theorem 2 can be completed as follows. We define a function $F_0 : \mathbb{N} \rightarrow \mathbb{N}$ by

$$F_0(n) := nG(2n + 7) + n \quad (1)$$

for all $n \geq 1$ and $F_0(0) := 0$.

Suppose for contradiction

$$\iota(\nu_1, \nu_n) > F_0(\iota(\nu_0, \nu_n)). \quad (2)$$

The intersection number $\iota(\nu_1 \setminus \nu_0, \nu_n)$ can be expressed as the sum of $|(v_1 \setminus v_0) \cap J_i|$ over each component J_i of $v_n \setminus v_0$, of which there are exactly $\iota(\nu_0, \nu_n)$, and so there exists a compact subinterval J of $v_n \setminus v_0$ such that

$$\iota(\nu_0, \nu_n)|(v_1 \setminus v_0) \cap J| \geq \iota(\nu_1 \setminus \nu_0, \nu_n). \quad (3)$$

As $\iota(\nu_1, \nu_n) = \iota(\nu_1 \setminus \nu_0, \nu_n) + \iota(\nu_0 \cap \nu_1, \nu_n)$ and $\iota(\nu_0, \nu_n) \geq \iota(\nu_0 \cap \nu_1, \nu_n)$, adding $\iota(\nu_0 \cap \nu_1, \nu_n)$ to both sides of inequality (3) we find

$$\iota(\nu_0, \nu_n)|v_1 \cap J| + \iota(\nu_0, \nu_n) \geq \iota(\nu_1, \nu_n). \quad (4)$$

Combining inequalities (2) and (4), we find

$$\iota(\nu_0, \nu_n)|v_1 \cap J| + \iota(\nu_0, \nu_n) > F_0(\iota(\nu_0, \nu_n)). \quad (5)$$

We note $G(n)$ is increasing in n . Appealing to Lemma 6 and the definition of F_0 in (1), from inequality (5) we find

$$|v_1 \cap J| > G(2\iota(\nu_0, \nu_n) + 7) \geq G(n - 1).$$

However, condition (2) of Lemma 20 now implies v_n has non-empty transverse self-intersection and this is absurd. With this we complete the contradiction, and a proof of Theorem 2.

§6. **Proof of Theorem 3.**

By Theorem 2 we have $\iota(\nu_m, \nu_n) \leq F_0(\iota(\nu_{m-1}, \nu_n))$ for each index $m \in \{1, \dots, n-2\}$. As $k \leq F_0(k)$ for all $k \in \mathbb{N}$, we have

$$F_0(\iota(\nu_{m-1}, \nu_n)) \leq F_0(F_0(\iota(\nu_{m-2}, \nu_n))) \leq \dots \leq F_0^m(\iota(\nu_0, \nu_n)).$$

If we define $F(n) = F_0^{2n+7}(n)$ for all $n \in \mathbb{N}$, then by Lemma 6 we have

$$\iota(\nu_m, \nu_n) \leq F(\iota(\nu_0, \nu_n))$$

for all indices m . By reversing the geodesic and repeating this argument we find

$$\iota(\nu_0, \nu_m) \leq F(\iota(\nu_0, \nu_n))$$

for all indices m . This completes the proof of Theorem 3.

§7. Proof of Theorem 4 and Theorem 1.

For any two pants decompositions μ and ν of Σ , either the 5-holed sphere or the 2-holed torus, denote by $T(\mu, \nu)$ the set of pants decompositions P that contain the (possibly empty) set of curves $\mu \cap \nu$ and are such that $\iota(\mu, P) + \iota(P, \nu) \leq F(\iota(\mu, \nu))$, where F is the function offered by Theorem 2. Then, the set $T(\mu, \nu)$ is computable and has cardinality bounded above explicitly in terms of $\iota(\mu, \nu)$. To see this, take smooth representatives $u \in \mu$ and $v \in \nu$ such that $|u \pitchfork v| = \iota(\mu, \nu)$ and every intersection away from any common loop is transverse. We regard $u \cup v$ as a grid on Σ and mark on this grid, away from the finite set of points $u \pitchfork v$, at most $F(\iota(\mu, \nu))$ points and then attempt to connect these up to form a pants decomposition containing the multicurve $\mu \cap \nu$ and without introducing any additional intersection with either u or v . This establishes the statement of Theorem 4.

According to Theorem 2, the set $T(\mu, \nu)$ contains the vertices of every geodesic path in $\mathcal{P}(\Sigma)$ connecting μ to ν . In particular, every shortest path from μ to ν built from pants decompositions contained in $T(\mu, \nu)$ is a geodesic path. If we read off the length of any such path, we succeed in computing the distance $d(\mu, \nu)$ as required. This completes a proof of Theorem 1. \diamond

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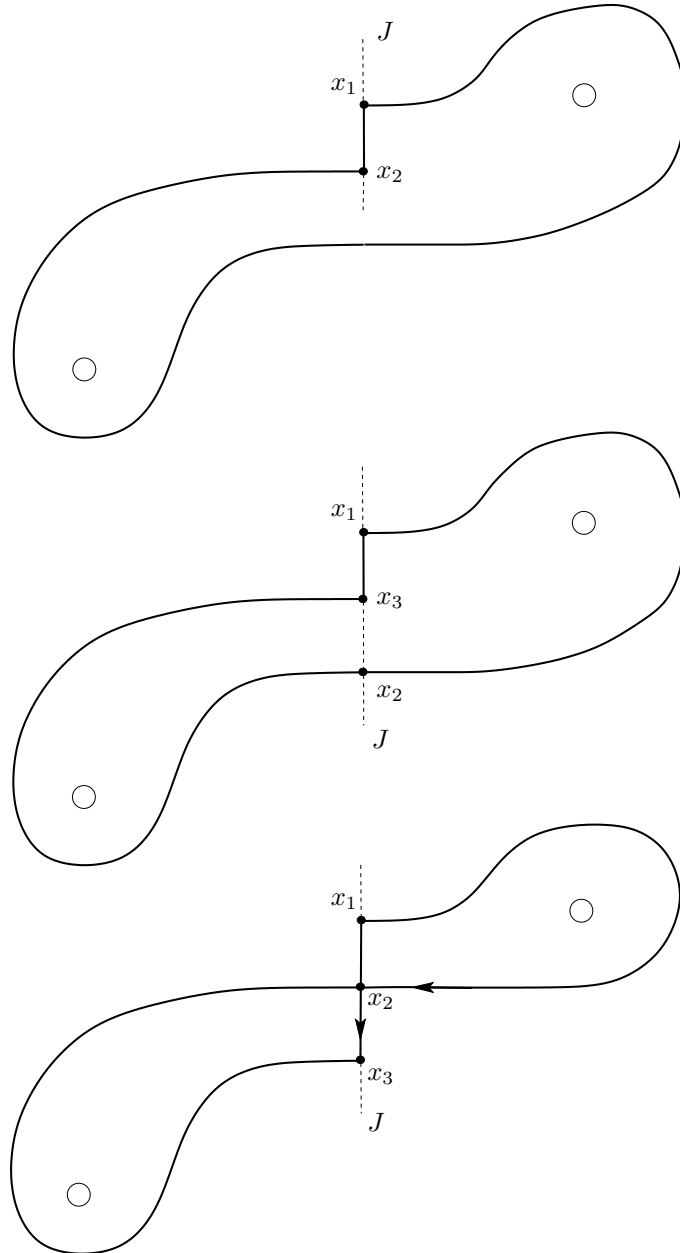


Figure 5: The three types of loop defining p .

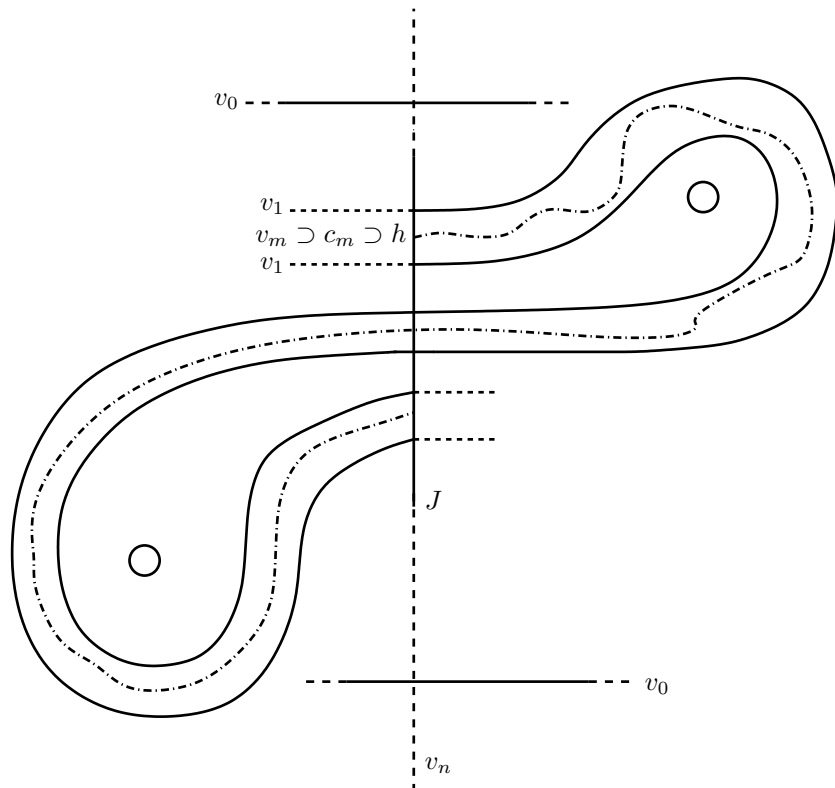


Figure 6: The subinterval h can be resolved along J to form a curve δ .