Renormalization group and perturbation theory about fixed points in two-dimensional field theory

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The behavior of the renormalization group is investigated in the neighborhood of the fixed points described by the "minimal" conformal theories M_p with $p \gg 1$. In the leading approximation in $1/p$ a field theory is constructed which corresponds to the renormalization-group trajectory connecting the fixed points M_p and M_{p-1}.

1. INTRODUCTION

The critical behavior of statistical systems is directly connected with fixed points of the renormalization group.[1,2] The field theory corresponding to a fixed point possesses conformal invariance on all scales substantially exceeding the ultraviolet cutoff.[3,4] A full analysis of the critical point should include, first, the construction of the conformally invariant field-theory solution corresponding to the fixed point itself, and, second, the calculation of the corresponding universality class, i.e., in essence, the description of the structure of the renormalization group in a certain neighborhood of this point.[2] In two-dimensional theory (which is discussed in the present article) the first of these problems can be solved in many cases.[5] Several infinite series of exact solutions of two-dimensional conformal field theory are now known,[5–10] and methods for constructing new solutions exist[7,11–13]; there is even hope of finding a complete classification of such solutions. In a number of cases it has proved possible to relate such solutions to the critical (or multicritical) points of two-dimensional models from statistical physics.[5,6,14–17] However, the question of the "physical" interpretation of conformally invariant solutions remains largely open.

If the solution corresponding to the fixed point itself is known exactly, the properties of the renormalization group in the neighborhood of this point can, in principle, be calculated using perturbation theory. However, this approximation is useful only if the renormalization group exhibits topologically nontrivial behavior (e.g., has other fixed points) within a sufficiently small neighborhood[1)] of the initial point. Such a situation occurs if the given conformal field theory contains spinless fields with anomalous dimensions d close to 2 (to $\mathscr{D}$, in a $\mathscr{D}$-dimensional theory), i.e., $d = \mathscr{D} - 2\varepsilon$, $|\varepsilon| \ll 1$. This well known circumstance serves as the basis for the famous ε-expansion method.[2] In our paper we consider the analogous approximation directly in the two-dimensional case.

In Sec. 2 we discuss the general properties of the renormalization group in two-dimensional field theory. In particular, it is shown that in a renormalizable theory that has n coupling constants $g = (g^1, g^2, \ldots, g^n)$ and satisfies the positivity condition,[6,18] the renormalization-group equations[1,19]

$$dg^i = \beta^i(g)\,dt \tag{1.1}$$

possess the following property. There exists a function $c(g)$ such that: a)

$$\frac{d}{dt}c = \beta^i(g)\frac{\partial}{\partial g^i}c(g) \leqslant 0, \tag{1.2}$$

where the stationary points of $c(g)$ coincide with the fixed points of the renormalization group, i.e., the conditions $\beta(g) = 0$ and $\partial c(g)/\partial g^i = 0$ are equivalent;

b) for each fixed point $g = g_*$ $[\beta(g_*) = 0]$ the value $c_* = c(g_*)$ coincides with the central charge of the Virasoro algebra in the corresponding conformal field theory.[20]

This "c-theorem" has an obvious consequence. If two fixed points g_{*0} and g_{*1} are linked by a renormalization-group trajectory, i.e., there exists a solution $g(t)$ of (1.1) satisfying the conditions $g(-\infty) = g_{*0}$ and $g(\infty) = g_{*1}$ (see the figure), then the values c_0 and c_1 of the "central charge" in the conformal theories g_{*0} and g_{*1} obey the inequality

$$c_0 > c_1. \tag{1.3}$$

This statement makes it possible to give a renormalization-group meaning to the "ordering" of the conformal field-theory solutions by the magnitude of the central charge c.

In Sec. 3 we use perturbation theory to find the power expansions of the β-functions [to order $(g - g_*)^2$] and of the function $c(g)$ [to order $(g - g_*)^3$] about the fixed point g_*, with the coefficients expressed in terms of the anomalous dimensions and structure constants of the operator algebra of the conformal theory g_*. It is found that, up to the indicated order, the relation

$$\beta^i(g) = -\frac{1}{12}G^{ij}(g)\frac{\partial c(g)}{\partial g^j} \tag{1.4}$$

is fulfilled, where $G^{ij}(g)$ is the symmetric positive-definite matrix that specifies the metric in the space of the coupling constants g.[2)]

The simplest known series of exact conformally invariant solutions are the "minimal models" M_p, $p = 3, 4, 5, \ldots$ (Refs. 5, 6, 15). The models M_p are related to degenerate representations of the Virasoro algebra; the corresponding values of the central charge are

$$c_p = 1 - 6/p(p+1). \tag{1.5}$$

In Sec. 4 we consider perturbation theory about these solutions in the case $p \gg 1$. An essential point is that in each of the

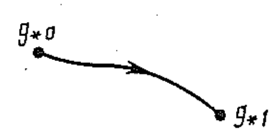

FIG. 1. Renormalization-group trajectory linking the fixed points g_{*0} and g_{*1}.

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models M_p there is a field $\varphi = \varphi_{(1,3)}$ with anomalous dimension $d_{(1,3)} = 2\Delta_{(1,3)} = 2 - 2\varepsilon$, where the quantity

$$\varepsilon = 2/(p+1) \tag{1.6}$$

can be regarded for $p \gg 1$ as a small parameter. In Sec. 4 we investigate the perturbation theory with the Euclidean action

$$\mathcal{H} = \mathcal{H}^{(p)} + \mathcal{H}_{int}^{(p)}, \quad \mathcal{H}_{int}^{(p)} = g \int \varphi_{(1,3)}^{(p)}(x)\, d^2x, \tag{1.7}$$

where $\mathcal{H}^{(p)}$ corresponds to the strictly conformal theory M_p that describes, obviously, the ultraviolet asymptotic form of the theory (1.7). The renormalization-group method makes it possible to sum the perturbation-theory series in the region $0 < g \lesssim \varepsilon$. It is found that for $g > 0$ the infrared asymptotic form of the theory (1.7) also possesses conformal invariance, and is described by the model M_{p-1}. In other words, we shall construct the field theory corresponding to the renormalization-group trajectory that links the two fixed points g_{*0} and g_{*1} (see the figure), where g_{*0} and g_{*1} correspond to the conformal theories M_p and M_{p-1}, respectively.

2. THE RENORMALIZATION GROUP IN TWO-DIMENSIONAL FIELD THEORY

A field theory can be regarded as a set of correlation functions

$$\langle A_1(x_1) A_2(x_2) \ldots A_N(x_N) \rangle, \tag{2.1}$$

where the local fields $A_a(x)$ are elements of an infinite-dimensional vector space $\mathcal{A}$ and form a closed (associative and commutative) algebra under operator expansions.[3,5,22,23] Below, we consider a two-dimensional Euclidean field theory, so that $x = (x^1, x^2) \in \mathbf{R}^2$.

The spatial symmetries of a homogeneous and isotropic theory are guaranteed by the presence of a symmetric stress tensor $T_{\mu\nu}(x) \in \mathcal{A}$, satisfying the continuity equation

$$\partial_\mu T_{\mu\nu} = 0. \tag{2.2}$$

The expressions

$$i\hat{P}_\mu A(x) = \oint dy^\lambda \epsilon_{\lambda\nu} T_\mu{}^\nu(y) A(x),$$
$$i\hat{S} A(x) = \oint dy^\lambda \epsilon_{\lambda\nu} \epsilon_{\mu\rho} (y-x)^\rho T^{\mu\nu}(y) A(x) \tag{2.3}$$

(where A is an arbitrary element of $\mathcal{A}$) serve to define the operators $\hat{P}$ and $\hat{S}$ acting in $\mathcal{A}$. Here, $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$, $\varepsilon_{12} = 1$, and the integrals are taken over a small contour surrounding the point x; in view of (2.2), these integrals do not depend on the shape of the contour. The operator $\hat{P}_\mu$ coincides with the derivative

$$i\hat{P}_\mu A(x) = \partial_\mu A(x), \tag{2.4}$$

and the spectrum of the operator $\hat{S}$ in $\mathcal{A}$ consists of integers (for Bose fields) and half-odd-integers (for Fermi fields), so that

$$\mathcal{A} = \bigoplus_{s=0,\pm 1/2,\pm 1,\ldots} \mathcal{A}^{(s)}, \quad \hat{S}\mathcal{A}^{(s)} = s\mathcal{A}^{(s)}. \tag{2.5}$$

In the Lagrangian formulation of the theory the correlators (2.1) are defined as averages of the form

$$\int [D\varphi] A_1(x_1) \ldots A_N(x_N) e^{-\mathcal{H}[\varphi]}, \tag{2.6}$$

where φ is a certain set of "fundamental" fields (the nature of which is not important here), the symbols $A_a(x)$ denote any local "composite" fields, and $\mathcal{H}[\varphi]$ is the Euclidean action, which is an integral of the local density:

$$\mathcal{H}[\varphi] = \int H(\varphi(x), \partial_\mu\varphi(x), \ldots)\, d^2x. \tag{2.7}$$

We assume that in the definition of $\mathcal{H}$ we have included a constant (i.e., independent of φ) term that ensures the normalization of the distribution function, and so in (2.6) we have not written the factor Z^{-1}. In the given formalism the components of the stress tensor $T_{\mu\nu}$ describe the variations of the density $H(x) \equiv H(\varphi(x), \partial_\mu\varphi(x), \ldots)$ under infinitesimal coordinate transformations $x_\mu \to x_\mu + \delta x_\mu$. In particular, under the scale transformation

$$x_\mu \to (1 + {}^1\!/_2 dt) x_\mu \tag{2.8}$$

we have

$$H(x) \to H(x) - dt\,\Theta(x), \tag{2.9}$$

where we have introduced the notation $\Theta(x) = -T_{\mu\mu}(x)$. As a rule, the correctdefinition of the functional integrals (2.6) in the nonperturbative region encounters certain difficulties. Nevertheless, we shall make use of certain general properties of the expression (2.6).

We shall asume that there is an n-parameter family of field theories, i.e., the correlation functions (2.1) depend on the "coupling constants" $g^i = (g^1, g^2, \ldots, g^n)$. Then there exist spinless[3)] fields $\Phi_i \in \mathcal{A}^{(0)}$ and linear operators $\hat{B}_i : \mathcal{A}^{(s)} \to \mathcal{A}^{(s)}$ such that

$$\frac{\partial}{\partial g^i} \langle A_1(x_1) \ldots A_N(x_N) \rangle = \sum_{a=1}^{N} \langle \hat{B}_{i,a} A_1(x_1) \ldots A_N(x_N) \rangle + \int d^2y \langle A_1(x_1) \ldots A_N(x_N) \Phi_i(y) \rangle, \tag{2.10}$$

where the operator $\hat{B}_{i,a}$ acts on the field $A_a(x_a)$. According to (2.6),

$$\Phi_i(x) = \frac{\partial}{\partial g^i} H(x), \quad \hat{B}_i A(x) = \frac{\partial}{\partial g^i} A(x). \tag{2.11}$$

The subspace $\Phi \subset \mathcal{A}^{(0)}$ spanned by the fields Φ_i is associated with the tangent space $T_g Q$, where Q is the "interaction space" with coordinates g^i.

We shall assume that the field theories under consideration are renormalizable, i.e., for all $g \in Q$ the field Θ lies in Φ. We can then write

$$\Theta(x) = \sum_{i=1}^{n} \beta^i(g) \Phi_i(x), \tag{2.12}$$

where the coefficients β^i are called β-functions.

Since the field Θ can be represented in the divergence form $\Theta(x) = -\partial_\mu (x^\nu T_{\mu\nu})$, the relation

$$\sum_{a=1}^{N} \left\langle \left(\frac{1}{2} x_a{}^\mu \frac{\partial}{\partial x_a{}^\mu} + \hat{D}_a \right) A_1(x_1) \ldots A_N(x_N) \right\rangle = \int d^2y \langle A_1(x_1) \ldots A_N(x_N) \Theta(y) \rangle, \tag{2.13}$$

is valid, where the operator $\hat{D} : \mathcal{A}^{(s)} \to \mathcal{A}^{(s)}$ describes variations of the fields $A \in \mathcal{A}$ under the infinitesimal scale transformations (2.8):

$$A(0)\to A(0)+dt\hat{D}A(0). \qquad (2.14)$$

We note that the integral in the right-hand side of (2.13) can diverge as $y\to x_a$. In this case the operator $\hat{D}$ depends on the cutoff parameter R_0 in such a way as to cancel this divergence in (2.13), since by (2.1) we understand the "renormalized" correlation functions, independent of R_0. This applies in equal measure to the relations (2.10) and the operators $\hat{B}_i$.

Combining (2.10), (2.12), and (2.13), we can obtain the Callan-Symanzik equation

$$\left\{\sum_{a=1}^{N}\left(\frac{1}{2}x_a^\mu\frac{\partial}{\partial x_a^\mu}+\hat{\Gamma}_a(g)\right)-\sum_{i=1}^{n}\beta^i(g)\frac{\partial}{\partial g^i}\right\}\langle A_1(x_1)\dots A_N(x_N)\rangle=0, \qquad (2.15)$$

where the operator

$$\hat{\Gamma}=\hat{D}+\beta^i\hat{B}_i=\hat{D}+\beta^i\frac{\partial}{\partial g^i} \qquad (2.16)$$

is called the matrix of anomalous dimensions and does not contain any dependence on R_0. It follows from (2.15) that the field theories corresponding to two points $g(t_1)$ and $g(t_2)$ on the same integral curve of Eqs. (1.1) differ only by a scale transformation $x_\mu\to e^{t_1-t_2}x_\mu$. Compatibility of (2.9), (2.11), (2.12), and (2.14) requires that the subspace Φ be invariant for the operator $\hat{\Gamma}$, i.e.,

$$\hat{\Gamma}\Phi_i=\gamma_i^j(g)\Phi_j=\left(\delta_i^j-\frac{\partial\beta^j}{\partial g^i}\right)\Phi_j. \qquad (2.17)$$

This relation ensures the absence of renormalizations of the components of the stress tensor:

$$(\hat{\Gamma}-1)\Theta=(\hat{\Gamma}-1)T=(\hat{\Gamma}-1)\bar{T}=0, \qquad (2.18)$$

where we have introduced for later convenience the notation

$$T=T_{11}-T_{22}+2iT_{12}\in\mathcal{A}^{(2)},\quad \bar{T}=T_{11}-T_{22}-2iT_{12}\in\mathcal{A}^{(-2)}. \qquad (2.19)$$

Information on the global topological properties of the vector field $\beta^i(g)$ is of special interest, since it permits us to judge the phase structure of the field theory. Here we shall show that in a two-dimensional field theory satisfying the positivity condition[6,18] the renormalization-group flow described by Eq. (1.1) has a "dissipative" character, i.e., we shall derive the inequality (1.2). For this we consider the two-point functions

$$\langle T(z,\bar{z})T(0,0)\rangle=F(\tau)/z^4, \qquad (2.20a)$$

$$\langle T(z,\bar{z})\Theta(0,0)\rangle=H(\tau)/z^3\bar{z}, \qquad (2.20b)$$

$$\langle \Theta(z,\bar{z})\Theta(0,0)\rangle=G(\tau)/z^2\bar{z}^2. \qquad (2.20c)$$

The formulas (2.20) serve as definitions of the functions $F(\tau)$, $H(\tau)$, and $G(\tau)$ of the scalar argument $\tau=\log(z\bar{z})$, where

$$z=x_1+ix_2,\quad \bar{z}=x_1-ix_2. \qquad (2.21)$$

The equations (2.2), which in the notation (2.19) have the form

$$\partial_{\bar{z}}T=\partial_z\Theta,\quad \partial_{\bar{z}}\Theta=\partial_z\bar{T}, \qquad (2.22)$$

lead to the following relations for these functions:

$$\dot{F}=\dot{H}-3H,\quad \dot{H}-H=\dot{G}-2G, \qquad (2.23)$$

where the dot denotes the derivative with respect to τ. We introduce the quantity

$$c=2F+4H-6G. \qquad (2.24)$$

The equation

$$\dot{c}=-12G, \qquad (2.25)$$

which is a simple consequence of (2.23), shows that $c(\tau)$ is a monotonically decreasing function of τ, since $G(\tau)\geqslant 0$ by virtue of the positivity condition.[18] If we fix τ (say, set $\tau=0$), then the quantities F, H, and G will depend only on the coupling constants g; then, from (2.25) and the renormalization-group equations (2.15) and (2.18) we obtain

$$\beta^i\frac{\partial}{\partial g^i}c(g)=-12G_{ij}(g)\beta^i(g)\beta^j(g), \qquad (2.26)$$

where the symmetric matrix

$$G_{ij}(g)=G_{ij}(0,g),\quad G_{ij}(\tau,g)=(z\bar{z})^2\langle\Phi_i(z,\bar{z})\Phi_j(0,0)\rangle \qquad (2.27)$$

is positive-definite because of the positivity condition adopted. We note that $ds^2=G_{ij}(g)dg^i dg^j$ can be regarded as the metric in Q.

The renormalization-group fixed points g_{*l} are determined by the conditions

$$\beta^i(g_{*l})=0,\quad i=1,2,\dots,n, \qquad (2.28)$$

where we have introduced the index l labeling the solutions of Eqs. (2.28). For $g=g_{*l}$ we have $\Theta=0$, and conformal symmetry appears in the theory. Namely, with this condition it follows from (2.22) that

$$T=T(z),\quad \bar{T}=\bar{T}(\bar{z}). \qquad (2.29)$$

Therefore, in analogy with (2.3) we can introduce an infinite set of operators $\hat{L}_n, \hat{\bar{L}}_n$, $n=0,\pm1,\pm2$, acting in $\mathcal{A}$ in accordance with the formulas

$$\hat{L}_nA(z,\bar{z})=\oint d\zeta(\zeta-z)^{n+1}T(\zeta)A(z,\bar{z}),$$

$$\hat{\bar{L}}_nA(z,\bar{z})=\oint d\bar{\xi}(\bar{\xi}-\bar{z})^{n+1}\bar{T}(\bar{\xi})A(z,\bar{z}), \qquad (2.30)$$

with $\hat{S}=\hat{L}_0-\hat{\bar{L}}_0$ and $\hat{P}\equiv\hat{P}_1+i\hat{P}_0=\hat{L}_{-1}$. The operators $\hat{L}$ obey the commutation relations of the Virasoro algebra:

$$[\hat{L}_n,\hat{L}_m]=(n-m)\hat{L}_{n+m}+(c_l/12)(n^3-n)\delta_{n+m,0}, \qquad (2.31)$$

where c_l is a real number (the central charge) characterizing the given conformal field theory.[5] The two-point function (2.20a) with $g=g_{*l}$ can be expressed in terms of the constant c_l:

$$\langle T(z)T(0)\rangle=c_l/2z^4. \qquad (2.32)$$

Since for $g=g_{*l}$ the functions H and G in (2.20) vanish, we obtain from (2.24)

$$c(g_{*l})=c_l. \qquad (2.33)$$

There is some simplification of the formula (2.26) in the "one-charge" case $n=1$, when without loss of generality we can set $G_{11}=1$ (G_{11} is the only component of the matrix G_{ij}), since (for $n=1$) this can always be achieved by a suitable replacement $g \to \bar{g}(g)$. With this choice we obtain from (2.26)

$$\beta(g) = -\frac{1}{12}\frac{d}{dg}c(g). \tag{2.34}$$

Let g_{*0} and g_{*1} be fixed points, i.e., different roots of the equation $\beta(g) = 0$. Then the values c_0 and c_1 of the central charge in the corresponding conformal theories are connected by the relation

$$c_1 - c_0 = \int_{g_{*0}}^{g_{*1}} \beta(g)\,dg. \tag{2.35}$$

3. PERTURBATION THEORY NEAR A FIXED POINT

We shall concentrate attention on the vicinity of a certain fixed point $g_{*0} \in Q$. It is convenient to assume that the coordinate origin in Q coincides with g_{*0}, i.e., $g_{*0} = 0$.

The structure of the conformal field theory (that arises when $g = g_{*0} = 0$) is described in Ref. 5. We recall the principal facts. The space $\mathscr{A}$ for $g = 0$ contains a certain (possibly infinite) set of "primary" fields $\varphi_\alpha \in \mathscr{A}$, obeying the equations

$$\begin{gathered} L_n\varphi_\alpha = \bar{L}_n\varphi_\alpha = 0 \quad \text{for} \quad n > 0, \\ L_0\varphi_\alpha = \Delta_\alpha\varphi_\alpha, \quad \bar{L}_0\varphi_\alpha = \bar{\Delta}_\alpha\varphi_\alpha. \end{gathered} \tag{3.1}$$

Here the parameters $(\Delta_\alpha, \bar{\Delta}_\alpha)$ characterize the field φ_α and are called conformal dimensions. In reality, $s_\alpha = \Delta_\alpha - \bar{\Delta}_\alpha$ is the spin of the field φ_α, while $d_\alpha = \Delta_\alpha + \bar{\Delta}_\alpha$ coincides with the scaling dimension. The fields φ_α are mutually orthogonal [with respect to the metric (2.26)], i.e.,

$$\langle \varphi_\alpha(z, \bar{z})\varphi_\beta(0, 0)\rangle = \delta_{\alpha\beta}(z)^{-2\Delta_\alpha}(\bar{z})^{-2\bar{\Delta}_\alpha}. \tag{3.2}$$

The space $\mathscr{A}$ can be represented as the sum $\alpha[\varphi_\alpha]$, where each of the subspaces $[\varphi_\alpha]$ ("conformal classes") is spanned by all possible independent monomials of the form

$$\hat{L}_{-n1}\hat{L}_{-n2}\cdots\hat{L}_{-n_N}\hat{\bar{L}}_{-m1}\cdots\hat{\bar{L}}_{-m_M}\varphi_\alpha \tag{3.3}$$

with $n_i, m_j > 0$, and corresponds to an irreducible representation of the Virasoro algebra with central charge $c_0 = c(g_{*0})$. The fields (3.3) ("descendants" of the field φ_α) have dimensions

$$\left(\Delta_\alpha + \sum_{i=1}^{N} n_i, \ \bar{\Delta}_\alpha + \sum_{j=1}^{M} m_j\right). \tag{3.4}$$

The fields constituting the space $\mathscr{A}$ form a closed operator algebra. For example, the product $\varphi_\alpha(x)\varphi_\beta(0)$ can be represented in the form

$$\varphi_\alpha(z, \bar{z})\varphi_\beta(0, 0) = \sum_\gamma C_{\alpha\beta\gamma} z^{\Delta_\gamma - \Delta_\alpha - \Delta_\beta}\bar{z}^{\bar{\Delta}_\gamma - \bar{\Delta}_\alpha - \bar{\Delta}_\beta}[\varphi_\gamma(0, 0) + \ldots], \tag{3.5}$$

where in the square brackets in the right-hand side we have omitted the series in integer positive powers of z and $\bar{z}$ that describes the contribution of the "descendants" of the field φ_α; the numerical coefficients in this series are completely determined by the requirements of conformal invariance of the operator algebra (3.5) (Ref. 5). The coefficients $\mathbf{C}_{\alpha\beta\gamma}$ in (3.5) (the structure constants of the operator algebra) are symmetric in the indices α, β, γ and are connected with the three-point functions by the relations

$$\begin{aligned} &\langle \varphi_{\alpha_1}(z_1, \bar{z}_1)\varphi_{\alpha_2}(z_2, \bar{z}_2)\varphi_{\alpha_3}(z_3, \bar{z}_3)\rangle \\ &= C_{\alpha_1\alpha_2\alpha_3} z_{12}^{\lambda_3} z_{13}^{\lambda_2} z_{23}^{\lambda_1}\bar{z}_{12}^{\bar{\lambda}_3}\bar{z}_{13}^{\bar{\lambda}_2}\bar{z}_{23}^{\bar{\lambda}_1}, \end{aligned} \tag{3.6}$$

where $z_{12} = z_1 - z_2$, etc., $\lambda_1 = \Delta_{\alpha 1} - \Delta_{\alpha 2} - \Delta_{\alpha 3}$, etc.

We now consider the vicinity of the fixed point $g = 0$, assuming that the functions $\beta^i(g)$ [satisfying the condition $\beta^i(0) = 0$] can be expanded in Taylor series in powers of g. Of course, the linear part of this expansion is completely determined by the spectrum of the anomalous dimensions of the spinless (see footnote 3) fields in the conformal theory $g = 0$. We denote $\Phi_i^0 = \Phi_i|_{g=0}$, $\Phi_i^0 \in \mathscr{A}^{(0)}_{g=0}$, where for $g \neq 0$ the fields Φ_i are defined by the expressions (2.11). It is convenient to choose the coordinate system in Q in such a way that the fields Φ_i^0 possess well defined dimensions $\Delta_i = \bar{\Delta}_i$ and are orthonormal, i.e., $G_{ij}(0) = \delta_{ij}$. Then it is clear from the expressions (3.2), (3.6) and the similar formulas for the correlation functions of the "descendants" that for $g = 0$ the equations (2.15) with $\beta^i = 0$ are fulfilled, and

$$\hat{\Gamma}(0)\Phi_i^0 = \gamma_i^j(0)\Phi_j^0, \quad \gamma_i^j(0) = \Delta_i\delta_i^j. \tag{3.7}$$

Taking into account the relation (2.17), we obtain the well known expressions

$$\beta^i(g) = \epsilon_i g^i + O(g^2), \tag{3.8}$$

where $\varepsilon_i = 1 - \Delta_i$.

The next terms of the expansion (3.8) can be calculated, in principle, by means of perturbation theory. Of course, in the general case the first few terms of this expansion do not permit us to make a judgement on the global topological properties of the renormalization group. We shall consider, however, the case when the dimensions Δ_i of the fields Φ_i^0 are close to unity, i.e.,

$$1 - \Delta_i = \epsilon_i, \quad |\epsilon_i| \sim \epsilon \ll 1. \tag{3.9}$$

In this case it may be expected that the nonlinear terms in (3.8) become comparable to the linear part when $g^i \sim \varepsilon$. Thus, the renormalization group possesses nontrivial behavior (i.e., can have other fixed points) in the region $g_i \lesssim \varepsilon$ in which perturbation theory is applicable. We shall calculate the next term of the expansion (3.8) in the case (3.9), assuming that the Φ_i^0 are primary fields.[4)]

According to (2.10) we have

$$\begin{aligned} &\frac{\partial}{\partial g^k}\langle \Phi_i(x)\Phi_j(0)\rangle|_{g=0} \\ &= \langle (B_k^0\Phi_i^0)(x)\Phi_j^0(0)\rangle + \langle \Phi_i^0(x)(\hat{B}_k^0\Phi_j^0)(0)\rangle \\ &\quad + \int d^2y\langle \Phi_i^0(x)\Phi_j^0(0)\Phi_k^0(y)\rangle, \end{aligned} \tag{3.10}$$

where $\hat{B}_k^0 = \hat{B}_k|_{g=0}$. The integral in the right-hand side can be represented in the form

$$(x^2)^{1-\Delta_i-\Delta_j-\Delta_k}C_{ijk}I_{ij}^k + \langle (\hat{b}_k\Phi_i^0)(x)\Phi_j^0(0)\rangle + \langle \Phi_i^0(x)(\hat{b}_k\Phi_j^0)(0) \tag{3.11}$$

where the $\mathbf{C}_{ijk}$ are the structure constants,

$$I_{ij}^k = 2\pi\frac{\Gamma(\Delta_i - \Delta_j - \Delta_k + 1)\Gamma(\Delta_j - \Delta_i - \Delta_k + 1)\Gamma(2\Delta_k - 1)}{\Gamma(2 - 2\Delta_k)\Gamma(\Delta_i + \Delta_k - \Delta_j)\Gamma(\Delta_j + \Delta_k - \Delta_i)}$$

$$= \frac{4\pi\epsilon_k}{(\epsilon_i+\epsilon_k-\epsilon_j)(\epsilon_j+\epsilon_k-\epsilon_i)}(1+O(\epsilon^3)),$$

and the terms with the operator $\hat{b}_k$ appear in the case of divergence of the integral; the form of the operator $\hat{b}_k$ depends on the method of cutoff, and is not important for us. It is convenient to choose the operators $\hat{B}^0_k$ in such a way that the expression (3.10) vanishes for $x^2 = 1$; this corresponds to a special choice of coordinate system in Q:

$$G_{ij}(g)=\delta_{ij}+O(g^2), \tag{3.12}$$

where G_{ij} is the "metric" (2.27). We then obtain

$$\frac{\partial}{\partial g^k}\langle\Phi_i(x)\Phi_j(0)\rangle|_{g=0}$$
$$=(x^2)^{-\Delta_i-\Delta_j}C_{ijk}\{I_{ik}{}^j[(x^2)^{1-\Delta_k}-(x^2)^{\Delta_i-\Delta_j}]+(i\leftrightarrow j)\}, \tag{3.13}$$

where

$$I_{ik}{}^j=\frac{1}{2}(I_{ij}{}^k+I_{jk}{}^i-I_{ik}{}^j)=\frac{2\pi}{\epsilon_i+\epsilon_k-\epsilon_j}(1+O(\epsilon^2)). \tag{3.14}$$

Comparing (3.13) with (2.15), we have,

$$\gamma_i{}^j(g)=\Delta_i\delta_i{}^j+C_{ik}{}^jg^k+O(g^2), \tag{3.15}$$

where

$$C_{ik}{}^j=(\epsilon_i+\epsilon_k-\epsilon_j)I_{ik}{}^jC_{ijk}=2\pi C_{ijk}+O(\epsilon^2). \tag{3.16}$$

Thus, according to (2.17),

$$\beta^i(g)=\epsilon_ig^i-{}^1/_2C_{jk}{}^ig^jg^k+O(g^3). \tag{3.17}$$

Because of the symmetry of the coefficients $C_{ijk}=C_{jk}{}^i$, which holds for $\varepsilon \ll 1$, the vector $\beta^i(g)$ in the approximation (3.17) can be written as the gradient of the function

$$-{}^1/_{12}c(g)=-{}^1/_{12}c_0+{}^1/_2\epsilon_ig^ig^i-{}^1/_6C_{ijk}g^ig^jg^k+O(g^4). \tag{3.18}$$

By direct calculation we can convince ourselves that (3.17) coincides with the expansion of the function $c(g)$ defined in Sec. 2. Returning to an arbitrary coordinate system in Q, we obtain (to the accuracy indicated here) the relation (1.4).

4. THE RENORMALIZATION GROUP IN THE VICINITY OF A FIXED POINT M_p WITH $p \gg 1$

We shall carry out a preliminary investigation of the vicinities of the fixed points described by the minimal models of conformal field theory,[5,6,15] and concentrate attention on the minimal models of the the "main series" M_p, $p = 3, 4, 5, \ldots$, which satisfy the positivity condition.[6] The value of the central charge c_p for the model M_p is given by the formula (1.5). In Ref. 6 it is shown that the models M_p exhaust all the conformally invariant field-theory solutions with $c < 1$ that satisfy the positivity condition. The space $\mathscr{A}$ of the model M_p contains $p(p-1)/2$ spinless primary fields $\varphi_{(n,m)}$, labelled by the two integers $n = 1, 2, \ldots, p-1$ and $m = 1, 2, \ldots, p$; $\varphi_{(n,m)} = \varphi(p-n, p+1-m)$, and the field $\varphi_{(p-1,p)} = \varphi_{(1,1)} = I$ coincides with the unit operator. The dimensions of the fields $\varphi_{(n,m)}$ are equal to

$$\Delta_{(n,m)}=\bar{\Delta}_{(n,m)}=[((p+1)n-pm)^2-1]/4p(p+1). \tag{4.1}$$

The space $\mathscr{A} = \bigoplus_{n,m}[\varphi_{(n,m)}]$ forms a closed algebra, the structure of which is described in Ref. 5 and, in more detail, in Ref. 15. For the following it is important that the subspace $\mathscr{A}_1 = \bigoplus_n [\varphi_{(1,n)}] \subset \mathscr{A}$ is a subalgebra.

We shall consider a fixed point M_p with $p \gg 1$. We note first of all that there are two series of primary fields $\varphi_{(n,n+2)}$ and $\varphi_{(n+2,n)}$ whose dimensions for $n \ll p$ are close to unity. Therefore, we may expect that the renormalization group will display nontrivial behavior in a small (of size $\sim 1/p$) neighborhood of the point M_p. In order to investigate this neighborhood, it is necessary to calculate the β-functions from the formula (3.17), using the fields[5] $\varphi_{(n,n+2)}$ and $\varphi_{(n+2,n)}$ as the Φ_i. Since $p \gg 1$ the dimension of the space Q in this case turns out to be very large (in fact, in the leading approximation it must be regarded as infinite). In its general form this problem will be considered in another paper. Here we shall investigate only the field theory that arises when the model M_p is perturbed by the operator $\varphi_{(1,3)}$ with dimension

$$\Delta_{(1,3)}=1-\epsilon,\quad \epsilon=2/(p+1). \tag{4.2}$$

Since the field $\varphi_{(1,3)}$ is the only field in the subalgebra $\mathscr{A}_1$ that has dimension close to unity, the corresponding renormalization group can be constructed as a single-charge renormalization group.[6]

Let $H^p(x, g)$ be the density of the action of this perturbed theory [$H^{(p)}(x, 0)$ describes the fixed point M_p]:

$$\Phi(x,g)=\frac{\partial}{\partial g}H^{(p)}(x,g),\qquad \Phi(x,0)=\varphi_{(1,3)}. \tag{4.3}$$

We shall assume that the "coordinate" g in Q is chosen so that

$$G(g)=\langle\Phi(x,g)\Phi(0,g)\rangle|_{x^2=1}=1. \tag{4.4}$$

The field $\Theta = -T_{\mu\mu}$ can be written in the form

$$\Theta=\beta(g)\Phi. \tag{4.5}$$

According to (3.17), the first terms of the expansion of the function $\beta(g)$ are

$$\beta(g)=\epsilon g-{}^1/_2(2\pi C)g^2+O(g^3), \tag{4.6}$$

where the structure constant $C = C_{(1,3)(1,3)(1,3)}$ can be obtained from the general formulas[15]

$$C(\epsilon)=\frac{4}{\sqrt{3}}\frac{(1-2\epsilon)^2}{(1-\epsilon)(1-3\epsilon/2)}\left[\frac{\Gamma(1-\epsilon/2)}{\Gamma(1+\epsilon/2)}\right]^{3/2}\frac{\Gamma^2(1+\epsilon)}{\Gamma^2(1-\epsilon)}$$
$$\times\left[\frac{\Gamma(1+3\epsilon/2)}{\Gamma(1-3\epsilon/2)}\right]^{1/2}\frac{\Gamma(1-2\epsilon)}{\Gamma(1+2\epsilon)}=\frac{4}{\sqrt{3}}\left(1-\frac{3\epsilon}{2}+O(\epsilon^2)\right). \tag{4.7}$$

Thus,

$$\beta(g)=\epsilon g-\frac{4\pi}{\sqrt{3}}\left(1-\frac{3\epsilon}{2}\right)g^2-\frac{4(2\pi)^2}{3}g^3+\ldots, \tag{4.8}$$

where we have written also the next term of the expansion of the β-function, retaining only terms of order ε^3 for $g \sim \varepsilon$. It can be seen from (4.8) that there is a fixed point

$$2\pi g_{*1}=(\sqrt{3}/2)\epsilon(1+\epsilon/2+O(\epsilon^2)). \tag{4.9}$$

Taking into account the posiitivity condition, which cannot be violated in the perturbed theory, and the statement of Sec. 2 concerning the decay of the c-function, we can predict that the fixed point (4.9) corresponds to a particular one of the models M_q with $q < p$. Calculation of the central charge at the point g_{*1} using formula (2.35) gives

$$c(g_{*1})=c_p-{}^3/_2\varepsilon^2-{}^3/_2\varepsilon^3+\dots \quad (4.10)$$

which, to the necessary accuracy, agrees with the expression

$$c(g_{*1})=1-6/p(p-1)=c_{p-1}. \quad (4.11)$$

Thus, the fixed point g_{*1} is described by the minimal model M_{p-1}.

The slope of the β-function at the fixed point determines the anomalous dimension of the field $\Phi(x, g_{*1})$ in the conformal theory g_{*1}. From (4.8) we obtain

$$\left.\frac{d\beta}{dg}\right|_{g=g_{*1}}=-\varepsilon-\varepsilon^2+\dots \quad (4.12)$$

Therefore, to the indicated accuracy, this anomalous dimension is equal to

$$\Delta=1+\varepsilon+\varepsilon^2+\dots=1+2/(p-1). \quad (4.13)$$

The latter expression coincides with the dimension $\Delta_{(3,1)}$ in the model M_{p-1}. Consequently,

$$\Phi(x,0)=\varphi^{(p)}_{(1,3)}(x), \quad \Phi(x,g_{*1})=\varphi^{(p-1)}_{(3,1)}(x), \quad (4.14)$$

where the index (p) $[(p-1)]$ denotes that the corresponding field belongs to the model M_p (M_{p-1}).

Thus, the field theory constructed corresponds to the renormalization-group trajectory linking the fixed points M_p and M_{p-1} (therefore, we shall call it the theory $M_{p,p-1}$), and realizes a kind of "interpolation" between these two conformal theories: In the ultraviolet region this theory approximates to M_p, while the infrared asymptotic forms of this theory are described by the model M_{p-1}. We note that, in view of (4.14), the theory $M_{p,p-1}$ can be regarded as the model M_{p-1} perturbed by the field $\varphi_{(3,1)}$.

It is interesting to investigate the renormalizations of the fields $\varphi_{(n,m)}$ in the perturbed theory $M_{p,p-1}$ and to establish the corrspondence between these fields in the "asymptotic" theories M_p and M_{p-1}. For this it is necessary to calculate the anomalous-dimension matrix $\hat{\Gamma}(g)$. In the calculations it should be borne in mind that in the leading order in ε, as usual, only operators with close dimensions are effectively mixed. The information that we need on the structure of the operator algebra of the model M_p is contained in the symbolic formula

$$\varphi_{(n,m)}\varphi_{(1,3)} =C^{(n,m)}_{(n,m)}[\varphi_{(n,m)}]+C^{(n,m+2)}_{(n,m)}[\varphi_{(n,m+2)}]+C^{(n,m-2)}_{(n,m)}[\varphi_{(n,m-2)}], \quad (4.15)$$

where the square brackets denote the contribution of the corresponding field and of its "descendants"; the dependence on the coordinates, which is expressible by the standard power factors, has been omitted in (4.15). The structure constants $C^{(\beta)}_{(\alpha)}=C^{(\alpha)}_{(\beta)}=C_{(\alpha)(\beta)(1,3)}$ in (4.15) are equal to

$$C^{(n,m)}_{(n,m)}=\frac{(n-m)^2(m+1)}{2\sqrt{3}(m-1)}+O(\varepsilon) \quad (n\leqslant m),$$

$$C^{(n,n)}_{(n,n)}=\frac{n^2-1}{8\sqrt{3}}\varepsilon^2+O(\varepsilon^2),$$

$$C^{(n,m+2)}_{(n,m)}=\left(\frac{m+2}{3m}\right)^{1/2}+O(\varepsilon) \quad (n\leqslant m), \quad (4.16)$$

$$C^{(n,n+1)}_{(n,n-1)}=\frac{(n^2-1)^{1/2}}{\sqrt{3}n}+O(\varepsilon).$$

It is simplest of all to investigate the fields $\Phi_{(n,n)}(x,g)$: $\Phi_{(n,n)}(x,0)=\varphi^{(p)}_{(n,n)}(x)$ with $n\ll p$, which, for $g=0$, have the dimensions

$$\Delta_{(n,n)}=\frac{n^2-1}{4p(p+1)}=\frac{n^2-1}{16}\varepsilon^2\left(1+\frac{\varepsilon}{2}+O(\varepsilon^2)\right). \quad (4.17)$$

According to (4.15), the field $\Phi_{(n,n)}$ does not mix with other fields, so that

$$\hat{\Gamma}(g)\Phi_{(n,n)}=\gamma_{(n,n)}(g)\Phi_{(n,n)}, \quad (4.18)$$

where

$$\gamma_{(n,n)}(g)=\Delta_{(n,n)}+2\pi C^{(n,n)}_{(n,n)}g+\dots$$
$$=\varepsilon^2\frac{n^2-1}{16}\left(1+\frac{\varepsilon}{2}+\frac{4\pi}{\sqrt{3}}g+\dots\right.; \quad (4.19)$$

we have assumed that the field $\Phi_{(n,n)}$ is normalized by the condition $\langle\Phi_{(n,n)}(x)\Phi_{(n,n)}(0)\rangle|_{x^2=1}=1$. Thus, for $g=g_{*1}$ (4.9) the anomalous dimension of the field $\Phi_{(n,n)}$ is equal to

$$\gamma_{(n,n)}(g_{*1})=\varepsilon^2\frac{n^2-1}{16}\left(1+\frac{3\varepsilon}{2}+\dots\right)=\frac{n^2-1}{4p(p-1)}+O(\varepsilon^4), \quad (4.20)$$

and, consequently, $\Phi_{(n,n)}(x,g_{*1})=\varphi^{(p-1)}_{(n,n)}(x)$.

Next, we consider the fields $\Phi_{(n,n+1)}$ and $\Phi_{(n,n-1)}$ that go over, for $g=0$, into $\varphi^{(p)}_{(n,n+1)}$ and $\varphi^{(p)}_{(n,n-1)}$ and are normalized by the condition

$$\langle\Phi_{(\alpha)}(x)\Phi_{(\beta)}(0)\rangle|_{x^2=1}=\delta_{\alpha\beta}. \quad (4.21)$$

For $g=0$ these fields have the dimensions

$$\Delta_{(n,n+1)}=\frac{1}{4}-\frac{2n+1}{8}\varepsilon+O(\varepsilon^2),$$

$$\Delta_{(n,n-1)}=\frac{1}{4}+\frac{2n-1}{8}\varepsilon+O(\varepsilon^2).$$

It follows from (4.15) that in the leading approximation it is necessary to take into account the mixing of the fields $\Phi_{(n,n-1)}$ and $\Phi_{(n,n+1)}$, so that for each n we have a 2×2 matrix of anomalous dimensions:

$$\gamma^{(\beta)}_{(\alpha)}(g)=\begin{pmatrix}\Delta_{(n,n+1)} & 0\\ 0 & \Delta_{(n,n-1)}\end{pmatrix}$$
$$+\frac{2\pi g}{\sqrt{3}}\begin{pmatrix}\frac{n+2}{2n} & \frac{(n^2-1)^{1/2}}{n}\\ \frac{(n^2-1)^{1/2}}{n} & \frac{n-2}{2n}\end{pmatrix}+\dots \quad (4.22)$$

It is not difficult to check that for $g=g_{*1}$ (4.9) this matrix has eigenvalues

$$\bar{\Delta}_1=\frac{1}{4}+\frac{2n+1}{8}\varepsilon=\Delta^{(p-1)}_{(n+1,n)}+O(\varepsilon^2), \quad (4.23)$$

$$\bar{\Delta}_2=\frac{1}{4}-\frac{2n-1}{8}\varepsilon=\Delta^{(p-1)}_{(n-1,n)}+O(\varepsilon^2).$$

Thus, for $g=g_{*1}$ the fields $\Phi_{(n,n+1)}$ and $\Phi_{(n,n-1)}$ are linear combinations of $\varphi^{(p-1)}_{(n+1,n)}$ and $\varphi^{(p-1)}_{(n-1,n)}$.

Finally, we consider the fields $\Phi_{(n,n+2)}$ and $\Phi_{(n,n-2)}$ corresponding to $\varphi_{(n,n+2)}$ and $\varphi_{(n,n-2)}$ in M_p. In this case mixing of the fields $\Phi_{(n,n+2)}$ and $\Phi_{(n,n-2)}$ with the field $\tilde{\Phi}_{(n,n)}(x,g)$: $\tilde{\Phi}_{(n,n)}(x,0)=(2\Delta_{(n,n)})^{-1}\partial_z\partial_{\bar{z}}\varphi^{(p)}_{(n,n)}(x)$ (4.24) turns out to be important. We normalize each triple of fields $\Phi_{(n,n+2)}$, $\tilde{\Phi}_{(n,n)}$, and $\Phi_{(n,n-2)}$ by the condition (4.21) (this is why the factor $[2\Delta_{(n,n)}]^{-1}$ was introduced in (4.24)]. Then the corresponding matrix of anomalous dimensions is

$$\begin{pmatrix} \Delta_{(n,n+2)} & 0 & 0 \\ 0 & 1+\Delta_{(n,n)} & 0 \\ 0 & 0 & \Delta_{(n,n-2)} \end{pmatrix} + \frac{4\pi g}{\sqrt{3}} \left| \begin{matrix} \frac{n+1}{n+3} & \frac{n-1}{n+1}\left(\frac{n+2}{n}\right)^{1/2} & 0 \\ \frac{n-1}{n+1}\left(\frac{n+2}{n}\right)^{1/2} & \frac{4}{n^2-1} & \frac{n+1}{n-1}\left(\frac{n-2}{n}\right)^{1/2} \\ 0 & \frac{n+1}{n-1}\left(\frac{n-2}{n}\right)^{1/2} & \frac{n-3}{n-1} \end{matrix} \right|, \tag{4.25}$$

where

$$\Delta_{(n,n+2)}=1-\frac{n+1}{2}\epsilon+O(\epsilon^2), \quad \Delta_{(n,n-2)}=1+\frac{n-1}{2}\epsilon+O(\epsilon^2). \tag{4.26}$$

The spectrum of the anomalous dimensions of these models for $g=g_{*1}$ is described by the eigenvalues of the matrix (4.25), i.e.,

$$\bar{\Delta}_+=1+\frac{n+1}{2}\epsilon+O(\epsilon^2), \quad \bar{\Delta}_-=1-\frac{n-1}{2}\epsilon+O(\epsilon^2),$$

$$\bar{\Delta}_0=1+O(\epsilon^2). \tag{4.27}$$

consequently, for $g=g_{*1}$ this triple of fields is expanded in $\varphi^{(p-1)}_{(n+2,n)}$, $\varphi^{(p-1)}_{(n-2,n)}$, and $\partial_z\partial_{\bar{z}}\varphi^{(p-1)}_{(n,n)}$. We note that for all $0\leqslant g\leqslant g_{*1}$ the relation

$$\frac{2g}{n+1}\left(\frac{n+2}{n}\right)^{1/2}\Phi_{(n,n+2)}+\left(1-\frac{4g}{n^2-1}\right)\tilde{\Phi}_{(n,n)} + \frac{2g}{n-1}\left(\frac{n-2}{n}\right)^{1/2}\Phi_{(n,n-2)}=\frac{8}{\epsilon^2(n^2-1)}\partial_z\partial_{\bar{z}}\Phi_{(n,n)} \tag{4.28}$$

is valid. The renormalizations of the other fields $\Phi_{(n,m)}$ in the model $M_{p,p-1}$ can be investigated in an analogous manner. This, and also a more complete investigation of the vicinity of the fixed point M_p, lie outside the scope of the present paper.

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1) The metric characteristics of the "interaction space"[2] that are necessary in order to give statements of this kind a precise meaning are considered in Sec. 2.

2) A formula analogous to (1.4) was obtained earlier[21] in the two-loop approximation of perturbation theory in the general two-dimensional σ-model.

3) Here we consider only homogeneous and isotropic theories.

4) Generally speaking, the condition (3.9) can also be satisfied by fields of the form $\partial_z\partial_{\bar{z}}\varphi_\alpha$, where the φ_α are primary fields with dimensions $\Delta_\alpha \ll 1$. This more general case is not needed here, and will be considered in another paper.

5) In fact, in order to ensure renormalizability it is necessary to include here the "secondary" fields $\partial_z\partial_{\bar{z}}\varphi_{(n,n)}$ as well.

6) The operator $\varphi_{(1,3)}$ as a perturbation of the model M_p also has other specific properties. For example, such a perturbed theory possesses an infinite set of commuting integrals of motion (a proof of this statement in the framework of perturbation theory will be given in another paper).

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