Gaussian integrals

For an $n \times n$ symmetric matrix A_{ij} with positive eigenvalues

$$\int_{\mathbf{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j\right) = \sqrt{\frac{(2\pi)^n}{\det A}}. \tag{1}$$

$$\langle x_i x_j \rangle = \frac{\displaystyle\int_{\mathbf{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{e}^{-\frac{1}{2}x^T A x} x_i x_j}{\displaystyle\int_{\mathbf{R}^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{e}^{-\frac{1}{2}x^T A x}} = A^{-1}_{ij}.$$

For an $n \times n$ hermitian matrix A_{ij} with positive eigenvalues

$$\int_{\mathbf{C}^n} \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_n \exp\left(-\sum_{i,j=1}^{n} z_i^* A_{ij} z_j\right) = \frac{\pi^n}{\det A}. \tag{2}$$

$$\langle z_i z_j^* \rangle = \frac{\displaystyle\int_{\mathbf{C}^n} \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_n \, \mathrm{e}^{-z^\dagger A z} z_i z_j^*}{\displaystyle\int_{\mathbf{C}^n} \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_n \, \mathrm{e}^{-z^\dagger A z}} = A^{-1}_{ij}, \quad \langle z_i z_j \rangle = \langle z_i^* z_j^* \rangle = 0.$$

For an $n \times n$ antisymmetric matrix B_{ij},

$$\int \mathrm{d}\psi_1 \cdots \mathrm{d}\psi_n \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} \psi_i B_{ij} \psi_j\right) = \mathrm{Pf}(B) \ \left(= \pm\sqrt{\det B}\right). \tag{3}$$

$$\langle \psi_i \psi_j \rangle = \frac{\displaystyle\int \mathrm{d}\psi_1 \cdots \mathrm{d}\psi_n \, \mathrm{e}^{-\frac{1}{2}\psi^T B \psi} \psi_i \psi_j}{\displaystyle\int \mathrm{d}\psi_1 \cdots \mathrm{d}\psi_n \, \mathrm{e}^{-\frac{1}{2}\psi^T B \psi}} = B^{-1}_{ij}.$$

For an $n \times n$ matrix B_{ij},

$$\int \mathrm{d}\overline{\psi}_1 \mathrm{d}\psi_1 \cdots \mathrm{d}\overline{\psi}_n \mathrm{d}\psi_n \exp\left(-\sum_{i,j=1}^{n} \overline{\psi}_i B_{ij} \psi_j\right) = \det(B). \tag{4}$$

$$\langle \psi_i \overline{\psi}_j \rangle = \frac{\displaystyle\int \mathrm{d}\overline{\psi}_1 \mathrm{d}\psi_1 \cdots \mathrm{d}\overline{\psi}_n \mathrm{d}\psi_n \, \mathrm{e}^{-\psi^\dagger B \psi} \psi_i \overline{\psi}_j}{\displaystyle\int \mathrm{d}\overline{\psi}_1 \mathrm{d}\psi_1 \cdots \mathrm{d}\overline{\psi}_n \mathrm{d}\psi_n \, \mathrm{e}^{-\psi^\dagger B \psi}} = B^{-1}_{ij}, \quad \langle \psi_i \psi_j \rangle = \langle \overline{\psi}_i \overline{\psi}_j \rangle = 0.$$