Homework 1 (MAT 1711, CFT I)

Consider the free field theory in 0-dimension with n real commuting variables $X_1, \ldots, X_n$ and action

$$S = \frac{1}{2} \sum_{i,j=1}^{n} X_i A_{ij} X_j .$$

Define $\overset{*}{*}\, \mathrm{e}^{\beta X_i} \overset{*}{*}$ by

$$\overset{*}{*}\, \mathrm{e}^{\beta X_i} \overset{*}{*} \ \overset{\mathrm{def}}{=}\ \mathrm{e}^{-\frac{1}{2}\beta^2 \langle X_i X_i \rangle}\, \mathrm{e}^{\beta X_i} .$$

(1) Going back to Gaussian integral, show that

$$\langle\ \overset{*}{*}\, \mathrm{e}^{\beta_1 X_{i_1}} \overset{*}{*} \ \cdots\ \overset{*}{*}\, \mathrm{e}^{\beta_s X_{i_s}} \overset{*}{*}\ \rangle = \prod_{1 \le a < b \le s} \mathrm{e}^{\beta_a \beta_b \langle X_{i_a} X_{i_b} \rangle} . \qquad (1)$$

(2) Show that $\overset{*}{*}\, \mathrm{e}^{\beta X_i} \overset{*}{*}$ agrees with $:\mathrm{e}^{\beta X_i}:$ where $:\mathcal{O}:$ is defined in the class. Show the operator product identity

$$:\mathrm{e}^{\beta_1 x_{i_1}}::\mathrm{e}^{\beta_2 x_{i_2}}:= \mathrm{e}^{\beta_1 \beta_2 \wick{\c X_{i_1} \c X_{i_2}}} :\mathrm{e}^{\beta_1 X_{i_1} + \beta_2 X_{i_2}}: . \qquad (2)$$

This reproduces (1) for the case $s = 2$. (Notation: $\langle X_i X_j \rangle = \wick{\c X_i \c X_j}$.)

For those who were not in the class, $:\mathcal{O}:$ are recursively defined by $:1:= 1$ and

$$\begin{aligned}
X_i &= :X_i:, \\
X_i X_j &= :X_i X_j: + \wick{\c X_i \c X_j}, \\
X_i X_j X_k &= :X_i X_j X_k: + :\wick{\c X_i \c X_j} X_k: + :X_i \wick{\c X_j \c X_k}: + :\wick{\c X_i X_j \c X_k}:, \\
X_i X_j X_k X_l &= :X_i X_j X_k X_l: \\
&\quad + :\wick{\c X_i \c X_j} X_k X_l: + :\wick{\c X_i X_j \c X_k} X_l: + :\wick{\c X_i X_j X_k \c X_l}: \\
&\quad + :X_i \wick{\c X_j \c X_k} X_l: + :X_i \wick{\c X_j X_k \c X_l}: + :X_i X_j \wick{\c X_k \c X_l}: \\
&\quad + \wick{\c X_i \c X_j} \wick{\c X_k \c X_l} + \wick{\c1 X_i \c2 X_j \c1 X_k \c2 X_l} + \wick{\c1 X_i \c2 X_j \c2 X_k \c1 X_l} \\
&\quad \cdots
\end{aligned}$$

It has the property that

$$\langle :X_{i_1} X_{i_2} \cdots X_{i_s}: \rangle = 0 \quad \text{if } s \ge 1.$$

It avoids self-contraction in operator product. For example,

$$\begin{aligned}
:X_i X_j::X_k X_l: \ &= \ \wick{\c1 X_i \c2 X_j \c1 X_k \c2 X_l} + \wick{\c1 X_i \c2 X_j \c2 X_k \c1 X_l} + + :\wick{\c X_i X_j \c X_k} X_l: + :\wick{\c X_i X_j X_k \c X_l}: \\
&\quad + :X_i \wick{\c X_j \c X_k} X_l: + :X_i \wick{\c X_j X_k \c X_l}: + :X_i X_j X_k X_l: .
\end{aligned}$$