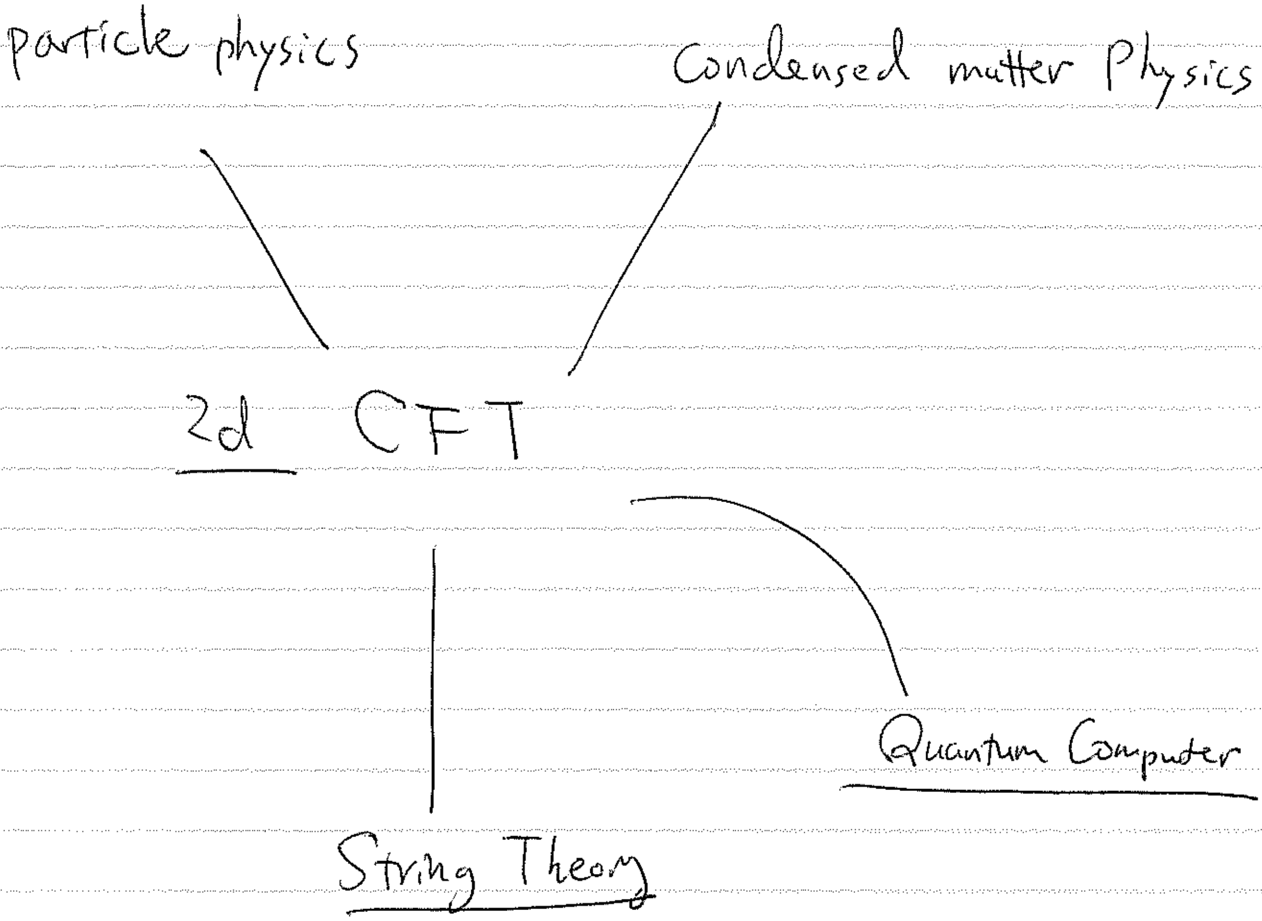
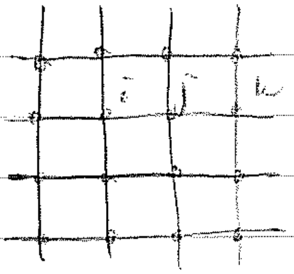

particle physics
Condensed matter Physics
2d CFT
Quantum Computer
String Theory

Ising Model

Lattice



(e.g. 2d square lattice)

At each site i, there is a variable (spin) $\sigma_i = 1$ or -1

$$\mathcal{H} = \underbrace{- J \underset{\substack{\uparrow \\ \text{nearest neighbor}}}{\sum_{\langle i,j \rangle}} \sigma_i \sigma_j}_{\text{exchange interaction}} \underbrace{- \mu_B H \sum_i \sigma_i}_{\text{Zeeman energy}}$$

↑ ↑

$J>0$ tries to align spins — tries to align spins in the same direction as H.

Things to compute !

Partition function $Z(T,H) = \sum_{\{\sigma_i\}} e^{-\mathcal{H}/kT}$ — V = # sites

$$= e^{-\frac{V}{kT} F(T,H)} \qquad (F(T,H) \text{: free energy})$$

magnetization $M_i = \langle \mu_0 \sigma_i \rangle = \dfrac{\sum_{\{\sigma_i\}} \mu_B \sigma_i \, e^{\mathcal{H}/kT}}{Z}$

correlation functions

$$\langle \delta\sigma_i \, \delta\sigma_j \rangle = \langle (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle) \rangle$$

0-d lattice

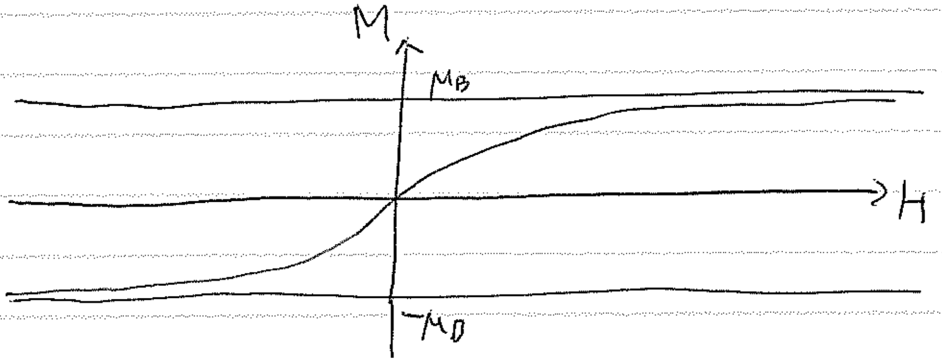
. $\sigma = \pm 1$ $\quad \mathcal{H} = -\mu_B H \sigma$

partition fcn $Z = \sum_{\sigma=1,-1} e^{-\mathcal{H}/kT} = e^{\mu_B H/kT} + e^{-\mu_B H/kT}$

$$= 2\cosh\left(\frac{\mu_B H}{kT}\right)$$

magnetization $M = \langle \mu_B \sigma \rangle = \sum_{\sigma=1,-1} \mu_B \sigma e^{-\mathcal{H}/kT} / Z$

$$= \frac{\mu_B e^{\mu_B H/kT} - \mu_B e^{-\mu_B H/kT}}{e^{\mu_B H/kT} + e^{-\mu_B H/kT}} = \mu_B \tanh\left(\frac{\mu_B H}{kT}\right)$$



paramagnetism

magnetic susceptibility

$$\chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} = \frac{\mu_B^2}{kT} \sim \frac{1}{T}$$

Curie's law

$d > 0$ $\boxed{J=0}$: decoupled system of V sites

$$Z = \sum_{\{\sigma_i\}} e^{-\sum_i \left(-\mu_0 H \sigma_i / kT\right)} = \sum_{\{\sigma_i\}} \prod_i e^{\mu_0 H \sigma_i / kT}$$

$$= \prod_i \sum_{\sigma_i} e^{\mu_0 H \sigma_i / kT} = \prod_i 2\cosh\left(\frac{\mu_0 H}{kT}\right)$$

$$= \left(2\cosh\frac{\mu_0 H}{kT}\right)^V$$

$$\langle M_i \rangle = \langle \mu_0 \sigma_i \rangle = \mu_0 \tanh\left(\frac{\mu_0 H}{kT}\right) \cdot 1$$ paramagnetic

$$\langle \delta\sigma_i \, \delta\sigma_j \rangle = \langle (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle) \rangle$$

$$= \left(\langle \sigma_i \rangle - \langle \sigma_i \rangle \cdot 1\right)\left(\langle \sigma_j \rangle - \langle \sigma_j \rangle \cdot 1\right)$$

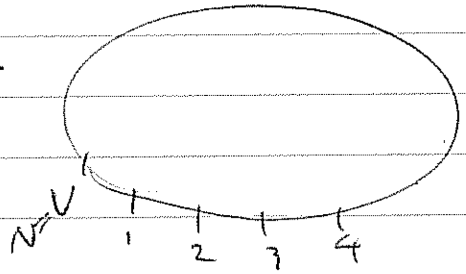
$$= 0$$

No correlation between different sites

(Of course! — No interaction)

$J \neq 0$

1-d case



$$\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - \mu_B H\sum_{i=1}^{N}\sigma_i \qquad \sigma_{N+1} \equiv \sigma_1$$

Want to compute Z, $\langle\sigma_i\rangle$, $\langle\sigma_i\sigma_j\rangle$, ---

(interested in the limit $N \to \infty$ (Thermodynamic limit)

$$Z = \sum_{\sigma_1,\cdots,\sigma_N} e^{-\mathcal{H}/kT} = \sum_{\sigma_1,\cdots,\sigma_N} e^{\overset{K}{\boxed{\frac{J}{kT}}}\sum_i \sigma_i\sigma_{i+1} + \overset{h}{\overset{\parallel}{\boxed{\frac{\mu_B H}{kT}}}}\sum_i \sigma_i}$$

$$= \sum_{\sigma_1\cdots\sigma_N}\prod_i e^{K\sigma_i\sigma_{i+1} + h\left(\frac{\sigma_i+\sigma_{i+1}}{2}\right)}$$

$$= \sum_{\sigma_1\cdots\sigma_N} t_{\sigma_1,\sigma_N}\, t_{\sigma_{N-1},\sigma_{N-2}}\cdots t_{\sigma_2,\sigma_1}$$

$$t_{\sigma,\sigma'} = e^{K\sigma\sigma' + h\left(\frac{\sigma'+\sigma}{2}\right)} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$

$$= \mathrm{Tr}(\overbrace{t\cdot t\cdots t}^{N}) = \mathrm{Tr}(t^N)$$

$$\det(\lambda - t) = (\lambda - e^{K+h})(\lambda - e^{K-h}) - e^{-2K}$$

$$= \lambda^2 - \underbrace{(e^{K+h} + e^{K-h})}_{2e^K\cosh(h)}\lambda + \underbrace{e^{2K} - e^{-2K}}_{2\sinh(\cancel{2K})}$$

$$\lambda = e^{K}\cosh(h) \pm \sqrt{e^{2K}\cosh^2(h) - 2\sinh(2K)}$$

$$= e^{K}\left[\cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4K}}\right] =: \lambda_\pm$$

$$Z = \mathrm{Tr}(t^N) = \lambda_+^N + \lambda_-^N$$

Free energy per site $Z = e^{-NF(T,H)/kT}$

$$F(T,H) = -\frac{kT}{N}\log\left(\lambda_+^N + \lambda_-^N\right)$$

$$= -kT\log\lambda_+ - \cancel{\frac{kT}{N}\log\left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right)} \xrightarrow{N\to\infty} 0$$

$$\therefore \lim_{N\to\infty} F(T,H) = \cancel{\log} - kT\log\lambda_+$$

$$= -J - kT\log\left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right)$$

$$M = \mu_B\langle\sigma_i\rangle = \frac{\mu_B}{N}\left\langle\sum_i\sigma_i\right\rangle = \frac{\frac{1}{N}\sum_{\sigma_1\cdots\sigma_N}\left(\sum_i\mu_B\sigma\right)e^{-\mathcal{H}/kT}}{\sum_{\sigma_1\cdots\sigma_N} e^{-\mathcal{H}/kT}}$$

$$= \frac{1}{N}\frac{\overset{kT}{\sum_{\sigma_1\cdots\sigma_N}}\frac{\partial}{\partial H}e^{-\mathcal{H}/kT}}{\sum_{\sigma_1\cdots\sigma_N}e^{-\mathcal{H}/kT}} = \frac{kT}{N}\frac{\partial}{\partial H}\log Z(T,H) = -\frac{kT}{N}\frac{\partial}{\partial H}F(T,H)$$

$$= \cdots = \mu_B\frac{\sinh(h)}{\sqrt{\sinh^2(h) + e^{-4K}}}$$ paramagnetism.

$$M = \mu_B \frac{\sinh\left(\frac{\mu_0 H}{kT}\right)}{\sqrt{\sinh^2\left(\frac{\mu_0 H}{kT}\right) + e^{-4J/kT}}}$$

Paramagnetism

$$\chi = \frac{\partial M}{\partial H}\bigg|_{H=0} = \frac{\mu_B^2}{kT} e^{2J/kT}$$

~~Curie's law (little modified)~~

$$M \bigg|_{T \searrow 0} = \mu_B \frac{\sinh\left(\frac{\mu_B H}{kT}\right)}{\sqrt{\sinh^2\left(\frac{\mu_B H}{kT}\right)}} = \begin{cases} \mu_B & H \searrow 0 \\ -\mu_B & H \nearrow 0. \end{cases}$$

ferromagnetic at zero temperature

$$\langle \sigma_i \sigma_j \rangle = \mathrm{tr}\left(\sigma_3 t^{i-j} \sigma_3 t^{N-(i-j)}\right)/Z \qquad (i>j)$$

$$\overset{h=0}{=} \cdots = (\tanh(K))^{i-j} = e^{-\frac{|i-j|}{\xi}}$$

exponential decay.

$$\xi = \frac{1}{\log(\tanh(K))} > 0 \quad \left(\longrightarrow \infty \quad \text{as } T \searrow 0 \quad \left(\begin{array}{l} \cancel{K} \to \infty \\ \tanh K \to 1 \end{array} \right) \right)$$

correlation length.

2d Ising Model — "solved" (Onsager 1940's)

$\exists\, T_c > 0$ $\quad \left(T_c = \frac{2}{\log(\sqrt{2}+1)} \cdot \frac{J}{k} \right.$ for square lattice $\left.\right)$

s.t.

$\underline{T > T_c}$: paramagnetic

2nd order phase transition

$$M = \begin{cases} > 0 & \text{for } H > 0 \\ = 0 & \phantom{\text{for }} H = 0 \\ < 0 & \phantom{\text{for }} H < 0 \end{cases} \sim \chi_m H$$

- $$\chi = \frac{\partial M}{\partial H}\bigg|_{H=0} \sim (T - T_c)^{-\gamma} \quad \text{as } T \gtrsim T_c$$
- $$\langle \sigma_i \sigma_j \rangle \Big|_{H=0} \sim e^{-\frac{|i-j|}{\xi}}, \quad \xi \sim |T - T_c|^{-\nu}$$

$\underline{T < T_c}$: ferromagnetic

$$M \Big|_{\substack{H \searrow 0 \\ H \nearrow 0}} = \pm M_* \sim \pm (T_c - T)^{\beta}$$

$\underline{T = T_c}$: $\langle \sigma_i \sigma_j \rangle \Big|_{H=0} \sim \frac{1}{|i-j|^{\eta}}$

γ, ν, β, η ⋯ <u>critical exponents</u>

$$\boxed{\gamma = \frac{7}{4}, \quad \beta = \frac{1}{8}, \quad \nu = 1, \quad \eta = \frac{1}{4}}$$