Sigma Model on S^1 and T-duality

$$X : \Sigma \longrightarrow S^1_{2\pi R} = \mathbb{R}/2\pi\mathbb{Z} \quad \text{(Circle of radius } R\text{)}$$

i.e. $X(t,\sigma)$ is a scalar field, but $X(t,\sigma) \equiv X(t,\sigma) + 2\pi R$

S = same as before

$\exists$ X-translation, t-translation, σ-translation symmetry

$\Rightarrow$ p = same as before $= \frac{1}{2\pi}\int_0^{2\pi} \partial_t X \, d\sigma$

H = same as before $= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2}(\partial_t X)^2 + \frac{1}{2}(\partial_\sigma X)^2 \right) d\sigma$

P = same as before $= \frac{1}{2\pi}\int_0^{2\pi} \partial_t X \partial_\sigma X \, d\sigma$

New Aspects

① (target space-) Momentum is quantized.

Zero mode sector $\mathcal{H}_0 = L^2(S^1_{2\pi R}, \mathbb{C}) \cdots e^{il x_0/R}$ $(l \in \mathbb{Z})$

$$\therefore \quad p = p_0 = -i\frac{\partial}{\partial x_0} = \frac{l}{R} \quad (l \in \mathbb{Z})$$

R

(2) $\exists$ Winding sectors

$X \equiv X + 2\pi R \rightsquigarrow$

$X(\sigma + 2\pi) = X(\sigma)$ only up to shift by $2\pi R \cdot \mathbb{Z}$

$X(\sigma + 2\pi) = X(\sigma) + 2\pi R m \iff m$: winding #.

The space of states have new sectors labeled by winding number $\underline{m \in \mathbb{Z}}$

- In the winding sector m, $X(t,\sigma)$ can be written as

$$X(t,\sigma) = \underset{\substack{\uparrow \\ S^1_{2\pi R}\text{-valued}}}{x_0(t)} + Rm\sigma + \sum_{n \neq 0} x_n(t) e^{in\sigma}$$

- "m" can also be regarded as a conserved charge (called topological charge) with conserved current J_W^μ

$$\begin{cases} J_W^t = \partial_\sigma X \\ J_W^\sigma = -\partial_t X \end{cases}$$

· Charge $W = \frac{1}{2\pi}\int_0^{2\pi} J_W^t \, d\sigma = \frac{1}{2\pi}\int_0^{2\pi} \partial_\sigma X \, d\sigma$

$$= \frac{1}{2\pi}\left(X(2\pi) - X(0)\right) = Rm$$

· Conservation equation $\partial_\mu J_W^\mu = \partial_t \partial_\sigma X - \partial_\sigma \partial_t X \equiv 0$

Identity !

Quantization : ~~as before~~

$$x_n,\ x_{-n} = x_n^* \rightsquigarrow a_n^i, a_n^{i\dagger} \rightsquigarrow \alpha_n, \alpha_{-n}, \tilde{\alpha}_n, \tilde{\alpha}_{-n}$$

$i = 1, 2.$ as before

denote

$$|l, m\rangle = |\tfrac{l}{R}\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots$$ in winding # m sector.

$$\alpha_n |l, m\rangle = \tilde{\alpha}_n |l, m\rangle = 0 \qquad n \geq 1$$

$$p\,|l, m\rangle = \frac{l}{R}|l, m\rangle$$

$$w\,|l, m\rangle = Rm\,|l, m\rangle$$

a general state : $\prod_{n=1}^{\infty} \alpha_{-n}^{N_n} \tilde{\alpha}_{-n}^{\tilde{N}_n} |l, m\rangle \rightarrow p = \frac{l}{R}$, $w = Rm$

$\therefore$ Space of States $\mathcal{H} = \bigoplus_{\substack{l \in \mathbb{Z} \\ m \in \mathbb{Z}}} \underbrace{\mathcal{H}_{(l,m)}}$

momentum l, winding # m } sector.

on $\mathcal{H}_{(l,m)}$:

$$H = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2}(\partial_t X)^2 + \frac{1}{2}(\partial_\sigma X)^2 \right) d\sigma$$

$$= \frac{1}{2}\left(\frac{l}{R}\right)^2 + \frac{1}{2}(Rm)^2 + \sum_{n=1}^{\infty} \left(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n \right) + \frac{-1}{12}$$

$$P = \frac{1}{2\pi}\int_0^{2\pi} \partial_t X \cdot \partial_\sigma X \; d\sigma$$

$$= l \cdot m + \sum_{n=1}^{\infty} \left(-\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n \right)$$

$$p = \frac{l}{R}$$

$$w = Rm$$

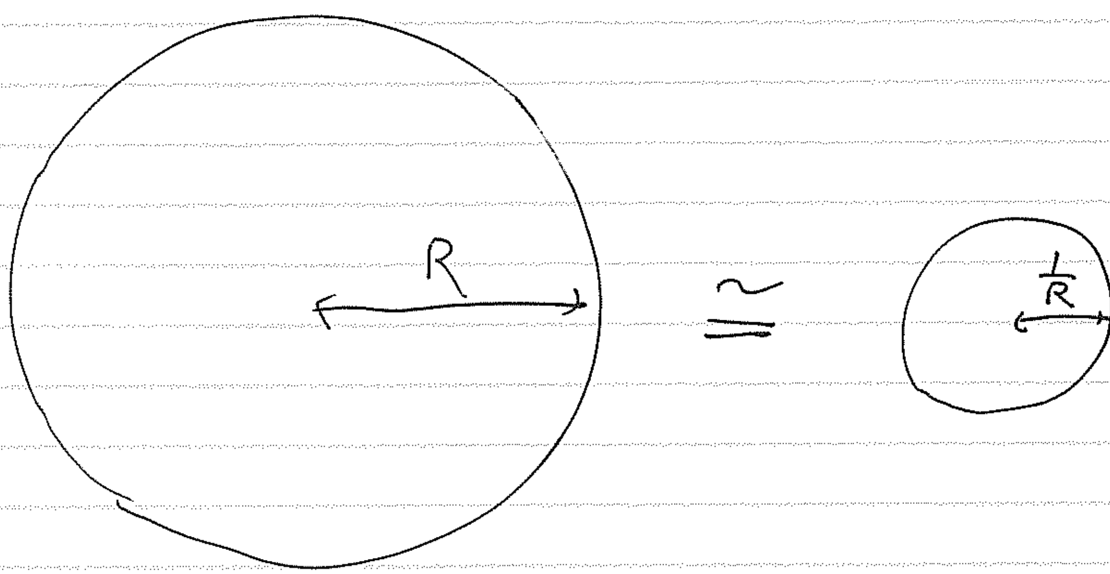
Notice! The spectrum is invariant under

$$R \longrightarrow \frac{1}{R}$$

provided l & m are exchanged!

<u>That is</u> Sigma model on $S^1_{2\pi R}$

$\underset{\text{equivalent}}{\cong}$ Sigma model on $S^1_{2\pi/R}$



larger circle $\longleftrightarrow$ smaller circle

This is called $\boxed{\text{T-duality}}$

Dictionary

$S^1_{2\pi R}$ $\longleftrightarrow$ $S^1_{2\pi R'}$ $\quad R' = \frac{1}{R}$

Momentum (Noether charge) $\longleftrightarrow$ Winding # (Topological charge)

$$p = \frac{\ell}{R} \quad \xleftrightarrow{\ell = m'} \quad w' = R' \cdot m'$$

Winding # (Topological charge) $\longleftrightarrow$ Momentum (Noether charge)

$$w = R \cdot m \quad \xleftrightarrow{m = \ell'} \quad p' = \ell'/R'$$

$\therefore$ Currents

$$J^\mu = \begin{cases} \partial_t X \\ -\partial_\sigma X \end{cases} \longleftrightarrow J'^\mu_w = \begin{cases} \partial_\sigma X' \\ \partial_t X' \end{cases}$$

$$J^\mu_w = \begin{cases} \partial_\sigma X \\ -\partial_t X \end{cases} \longleftrightarrow J'^\mu = \begin{cases} \partial_t X' \\ -\partial_\sigma X' \end{cases}$$

Conservation eqn

$$\partial_\mu J^\mu = 0 \text{ (EOM)} \longleftrightarrow \partial_\mu J'^\mu_w \text{ (Identity)}$$

$$\partial_\mu J^\mu_w = 0 \text{ (Identity)} \longleftrightarrow \partial_\mu J'^\mu \text{ (EOM)}$$

$$X(t,\sigma) = x_0 + tp + \sigma w + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n e^{-in(t-\sigma)} + \tilde{\alpha}_n e^{-in(t+\sigma)}\right)$$

$$\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma}\right)X = \frac{1}{\sqrt{2}}(p-w) + \sum_{n\neq 0} \alpha_n e^{-in(t-\sigma)}$$

$$\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \sigma}\right)X = \frac{1}{\sqrt{2}}(p+w) + \sum_{n\neq 0} \tilde{\alpha}_n e^{-in(t+\sigma)}$$

Dictionary Continued :

R		$R' = 1/R$
$\partial_t X$	$\longleftrightarrow$	$\partial_\sigma X'$
$\partial_\sigma X$	$\longleftrightarrow$	$\partial_t X'$
$\therefore \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma}\right) X$	$\longleftrightarrow$	$-\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma}\right) X'$
$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \sigma}\right) X$	$\longleftrightarrow$	$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \sigma}\right) X'$

Therefore

α_n	$\longleftrightarrow$	$-\alpha_n'$
$\tilde{\alpha}_n$	$\longleftrightarrow$	$\tilde{\alpha}_n$

Analogy — electric-magnetic duality in 3+1 dim

gauge coupling e	gauge coupling $e' = \frac{1}{e}$

$$\mathbb{E} = -\nabla V - \frac{\partial \mathbb{A}}{\partial t} \longleftrightarrow \mathbb{B}' = \nabla \times \mathbb{A}'$$

$$\mathbb{B} = \nabla \times \mathbb{A} \longleftrightarrow -\mathbb{E}' = +\nabla V' + \frac{\partial \mathbb{A}'}{\partial t}$$

Maxwell equation (with $\rho = \mathbb{J} = 0$)

$$\left.\begin{array}{l} \nabla \cdot \mathbb{E} = 0 \\ \nabla \times \mathbb{B} - \frac{\partial \mathbb{E}}{\partial t} = 0 \end{array}\right\} \text{EOM for } (V, \mathbb{A}) \longleftrightarrow \left.\begin{array}{l} \nabla \cdot \mathbb{B}' = 0 \\ -\nabla \times \mathbb{E}' - \frac{\partial \mathbb{B}'}{\partial t} = 0 \end{array}\right\} \text{Bianchi Identity}$$

$$\left.\begin{array}{l} \nabla \cdot \mathbb{B} = 0 \\ \nabla \times \mathbb{E} + \frac{\partial \mathbb{B}}{\partial t} = 0 \end{array}\right\} \text{Bianchi Identity} \longleftrightarrow \left.\begin{array}{l} -\nabla \cdot \mathbb{E}' = 0 \\ \nabla \times \mathbb{B}' - \frac{\partial \mathbb{E}'}{\partial t} = 0 \end{array}\right\} \text{EOM for } (V', \mathbb{A}')$$

Electric charge | Magnetic charge

$$Q_e = \int_S \mathbb{E} \cdot d\mathbf{a} \longleftrightarrow Q'_m = \int_S \mathbb{B}' \cdot d\mathbf{a}$$

(conservation requires EOM) | (Conservation is identity)

Magnetic charge | Electric charge

$$Q_m = \int_S \mathbb{B} \cdot d\mathbf{a} \qquad -Q'_e = -\int_S \mathbb{E}' \cdot d\mathbf{a}$$

(conservation ~~requires~~ is Identity | (conservation requires EOM)

Digression : Open strings

$$\Sigma = \underset{t}{\mathbb{R}} \times \underset{\sigma}{[0, \pi]} \qquad 0 \le \sigma \le \pi$$

Need boundary condition at $\sigma = 0, \pi$.

① Neumann b.c. $\partial_\sigma X = 0$ at $\sigma = 0$

$X(\sigma=0)$ can be anywhere

② Dirichlet b.c. $\partial_t X = 0$ at $\sigma = 0$

$X(\sigma=0)$ is fixed at some position.

T-duality

S^1_R	$S^1_{1/R}$
Neumann b.c. $\partial_\sigma X = 0$	Dirichlet b.c. $\partial_t \hat{X} = 0$
Dirichlet b.c. $\partial_t X = 0$	Neumann b.c. $\partial_\sigma \hat{X} = 0$