Partition function

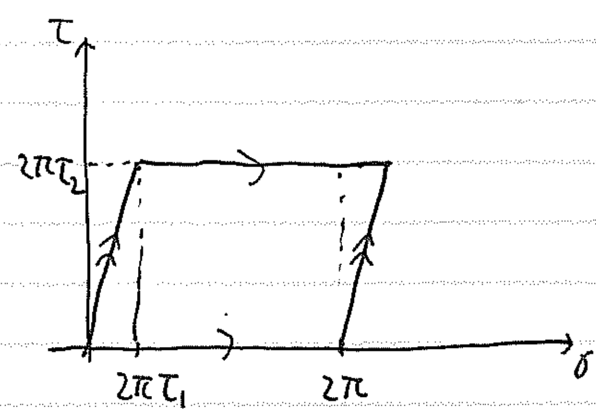
$$Z = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \int_{X(\sigma+2\pi,\tau) = X(\sigma,\tau+\beta) = X(\sigma,\tau)} \mathcal{D}X\, e^{-S_E(X)}$$

One could also insert a symmetry $g : \mathcal{H} \to \mathcal{H}$

$$Z_g = \mathrm{Tr}_{\mathcal{H}}\left(g \cdot e^{-\beta H} \right)$$

Take $g = e^{-2\pi i \tau_1 \overset{\text{w.s. momentum}}{P}}$: shift of σ by $2\pi\tau_1$

$$Z(\tau_1, \tau_2) = \mathrm{Tr}_{\mathcal{H}}\, e^{-2\pi i \tau_1 P} e^{-2\pi \tau_2 H}$$



$$= \int \mathcal{D}X\, e^{-S_E(X)}$$

$$X(\sigma,\tau) = X(\sigma+2\pi,\tau) = X(\sigma+2\pi\tau_1, \tau+2\pi\tau_2)$$

$$\boxed{\text{Path-integral over function } X \text{ on torus } \{(\sigma,\tau)\}/(\sigma,\tau)\sim(\sigma+2\pi,\tau)\sim(\sigma+2\pi\tau_1,\tau+2\pi\tau_2)}$$

$$\tau_2 H + i\tau_1 P = -i\underbrace{(\tau_1 + i\tau_2)}_{\tau}\underbrace{\frac{H-P}{2}}_{H_R} + i\underbrace{(\tau_1 - i\tau_2)}_{\bar\tau}\underbrace{\frac{H+P}{2}}_{H_L}$$

$$= -i\tau H_R + i\bar\tau H_L$$

For the massless scalar theory (σ-model with target $= \mathbb{R}$)

$$H_R = \frac{1}{4}p_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \frac{1}{24}$$

$$H_L = \frac{1}{4}p_0^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24}$$

$$Z = \mathrm{Tr}_{\mathcal{H}}\, e^{2\pi i \tau H_R - 2\pi i \bar{\tau} H_L}$$

$$= \mathrm{Tr}_{\mathcal{H}}\, q^{H_R}\bar{q}^{H_L} \qquad \begin{cases} q := e^{2\pi i\tau} = e^{-2\pi\tau_2 + 2\pi i \tau_1} \\ \bar{q} = e^{-2\pi i\bar{\tau}} = e^{-2\pi\tau_2 - 2\pi i\tau_1} \end{cases}$$

$$= (q\bar{q})^{-\frac{1}{24}}\, \mathrm{Tr}_{\mathcal{H}_0}(q\bar{q})^{\frac{1}{4}p_0^2} \cdot \prod_{n=1}^{\infty} \mathrm{Tr}_{\mathcal{H}_n}\left(q^{\alpha_{-n}\alpha_n} \cdot \bar{q}^{\tilde{\alpha}_{-n}\tilde{\alpha}_n}\right)$$

- $$\mathcal{H}_n = \mathcal{H}_n^R \otimes \mathcal{H}_n^L = \langle |0\rangle, \alpha_{-n}|0\rangle, \alpha_{-n}^2|0\rangle, \cdots \rangle \otimes \langle |0\rangle, \tilde{\alpha}_{-n}|0\rangle, \tilde{\alpha}_{-n}^2|0\rangle, \cdots \rangle$$

$$\mathrm{Tr}_{\mathcal{H}_n}\left(q^{\alpha_{-n}\alpha_n}\bar{q}^{\tilde{\alpha}_{-n}\tilde{\alpha}_n}\right) = (1 + q^n + q^{2n} + \cdots)\cdot(1 + \bar{q}^n + \bar{q}^{2n} + \cdots)$$

$$= \frac{1}{(1-q^n)(1-\bar{q}^n)}$$

- $$\mathrm{Tr}_{\mathcal{H}_0}(q\bar{q})^{\frac{p_0^2}{4}} = \mathrm{Tr}_{\mathcal{H}_0}\, e^{-2\pi\tau_2 \cdot \frac{p_0^2}{2}} = V\int \frac{dk}{2\pi}\, e^{-2\pi\tau_2 \frac{k^2}{2}} = \frac{V}{2\pi}\frac{1}{\sqrt{\tau_2}}$$

Cut-off volume of $\mathbb{R}$

$$\therefore\ Z = (q\bar{q})^{-\frac{1}{24}} \cdot \frac{V}{2\pi\sqrt{\tau_2}} \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-\bar{q}^n)}$$

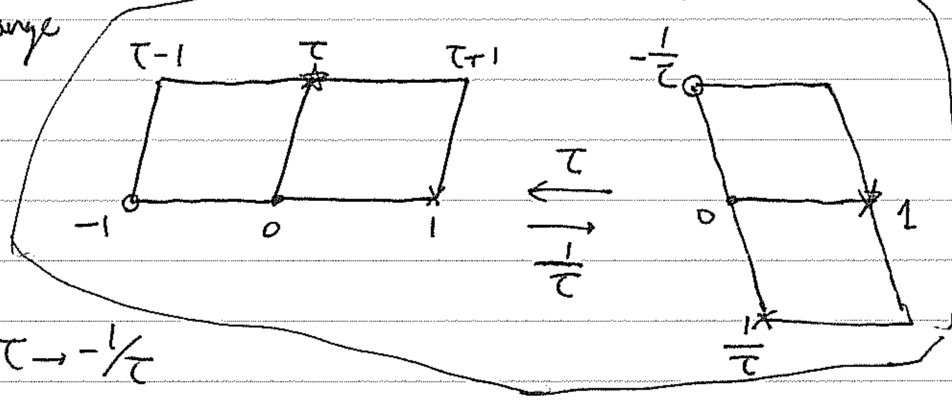
$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \qquad \text{Dedekind eta function}$$

$$Z = \frac{V}{2\pi\sqrt{\tau_2}} |\eta(\tau)|^{-2}$$

property of $\eta(\tau)$:
$$\left.\begin{array}{l} \cdot\ \eta(\tau+1) = e^{\pi i/12}\eta(\tau) \\ \cdot\ \eta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\eta(\tau) \end{array}\right\}$$

$\Rightarrow \quad Z(\tau,\bar\tau) = \frac{V}{2\pi}\left(\tau_2^{\frac{1}{2}}|\eta(\tau)|^2\right)^{-1}$ is invariant under

$\underbrace{\tau \to \tau+1}_{\text{does not change the torus}}$ & $\underbrace{\tau \to -1/\tau}_{\text{rotation \& Scaling}}$ — $\boxed{\text{modular invariant}}$



$\tau \to \tau+1$ & $\tau \to -1/\tau$

generates an action of a group $PSL(2,\mathbb{Z})$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ ad-bc=1 \right\} \Big/ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \tau \mapsto \tau+1 \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \mapsto -\frac{1}{\tau} \Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For Sigma Model with target $S^1_{2\pi R} = \mathbb{R}/2\pi\mathbb{Z}$

$$H_R = \frac{1}{4}(p-w)^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \frac{1}{24} \qquad p = \frac{\ell}{R}, \quad w = Rm$$

$$H_L = \frac{1}{4}(p+w)^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n - \frac{1}{24} \qquad \ell, m \in \mathbb{Z}$$

$$Z = \mathrm{Tr}_{\mathcal{H}}\, q^{H_R}\bar{q}^{H_L} = \sum_{\ell, m \in \mathbb{Z}} \mathrm{Tr}_{\mathcal{H}_{(\ell,m)}}\, q^{H_R}\bar{q}^{H_L}$$

$$= \sum_{\ell, m} q^{\frac{1}{4}\left(\frac{\ell}{R} - Rm\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{\ell}{R} + Rm\right)^2} \cdot \underbrace{\prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-\bar{q}^n)} \cdot (q\bar{q})^{-\frac{1}{24}}}_{1/|\eta(\tau)|^2}$$

☆ Note : as $V = 2\pi R \to \infty$, only $m=0$ survives and the sum

$\sum_\ell$ becomes integral $R \sum_\ell \frac{1}{R} = R \cdot \int_{-\infty}^{\infty} dk = V \int_{-\infty}^{\infty} \frac{dk}{2\pi}$

$\Rightarrow$ reduces to the previous formula

☆ Modular invariant (inv under $\tau \to \tau+1, -1/\tau$)

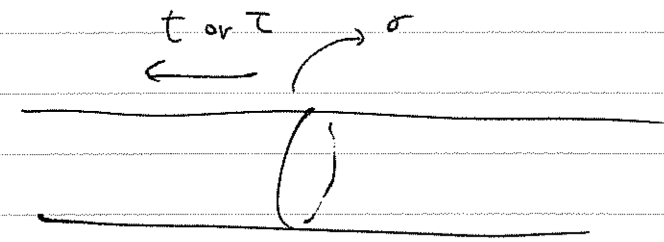
$$Z = \sum_{\ell, m} e^{-2\pi\tau_2 \frac{1}{2}\left\{\left(\frac{\ell}{R}\right)^2 + (Rm)^2\right\} - 2\pi i \tau_1 \cdot \ell m} \Big/ |\eta(\tau)|^2$$

use the Poisson resum twice

$$\sum_{k \in \mathbb{Z}} e^{-\frac{2\pi A}{2}(k+b)^2} = \frac{1}{\sqrt{A}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{2\pi}{2A}\ell^2 \pm 2\pi i \ell b}$$

Correlation functions

We will consider cylinder



But before that : in QM , recall

$$\int_{\substack{X(t_f)=X_f \\ X(t_i)=X_i}} \mathcal{D}X \, e^{i\int_{t_i}^{t_f} L\,dt} \, O_1(t_1)\, O_2(t_2)$$

$$\overset{t_1>t_2}{=} \langle x_f | e^{-i(t_f-t_1)\hat H} \hat O_1 \, e^{-i(t_1-t_2)\hat H} \hat O_2 \, e^{-i(t_2-t_i)\hat H} | x_i \rangle$$

$$\overset{t_2>t_1}{=} \langle x_f | e^{-i(t_f-t_2)\hat H} \hat O_2 \, e^{-i(t_2-t_1)\hat H} \hat O_1 \, e^{-i(t_1-t_i)\hat H} | x_i \rangle$$

$$= \langle x_f | e^{-it_f\hat H} \underbrace{T\left(\hat O_1(t_1)\, \hat O_2(t_2)\right)}_{\text{time ordered product}} e^{+it_f\hat H} | x_i \rangle$$

$$\hat O_{1,2}(t_{1,2}) = e^{it_{1,2}\hat H} \hat O_{1,2} \, e^{-it_{1,2}\hat H}$$

$$\left[\text{ Same for } O_1(t_1) \cdots O_s(t_s) \rightsquigarrow T\left(\hat O_1(t_1) \cdots \hat O_s(t_s)\right) \right]$$

Take $t_f = T(1-i\epsilon)$, $t_i = -T(1-i\epsilon)$ & $T \to \infty$

$$= \sum_{n,m} \langle x_f | e^{-iT(1-i\epsilon)\hat H} | n \rangle \langle n | T(\hat O_1(t_1)\, \hat O_2(t_2)) | m \rangle \langle m | e^{-iT(1-i\epsilon)\hat H} | x_i \rangle$$

$$= \sum_{n,m} e^{-iT(1-i\epsilon)E_n} \langle x_f | n \rangle \langle n | T(\hat O_1(t_1)\, \hat O_2(t_2)) | m \rangle e^{-iT(1-i\epsilon)E_m} \langle m | x_i \rangle$$

$$= ☆$$

$|\Omega\rangle$ ground state, $E_\Omega < E_n$ $\forall$ other n

$$e^{-iT(1-i\epsilon)E_n} \ll e^{-iT(1-i\epsilon)E_\Omega} \quad \text{as } T\to\infty$$

$$\text{☆} \xrightarrow{T\to\infty} \langle x_f|\Omega\rangle\langle\Omega|T(\hat{O}_1(t_1)\hat{O}_2(t_2))|\Omega\rangle\langle\Omega|x_i\rangle \cdot e^{-2iT(1-i\epsilon)E_\Omega}$$

Similary
$$\int_{\substack{x(T)=x_f \\ x(-T)=x_i}} \mathcal{D}x\, e^{i\int_{-T}^{T} L\,dt} \xrightarrow{T\to\infty} \langle x_f|\Omega\rangle\langle\Omega|x_i\rangle e^{-2iT(1-i\epsilon)E_\Omega}$$

$$\langle O_1(t_1)\,O_2(t_2)\rangle := \lim_{T\to\infty} \frac{\int_{\cdot\cdot} \mathcal{D}x\, e^{i\int_{-T}^{T} L\,dt}\, O_1(t_1)\,O_2(t_2)}{\int_{\cdot\cdot} \mathcal{D}x\, e^{i\int_{-T}^{T} L\,dt}}$$

$$= \langle\Omega|T(\hat{O}_1(t_1)\,\hat{O}_2(t_2))|\Omega\rangle$$

Similarly
$$\langle O_1(t_1)\cdots O_s(t_s)\rangle = \langle\Omega|T(\hat{O}_1(t_1)\cdots\hat{O}_s(t_s))|\Omega\rangle$$

NB Assumed $\exists$ normalizable ground state $|\Omega\rangle$

Let's compute $\langle X(t_1,\sigma_1)X(t_2,\sigma_2)\rangle = \langle\Omega|T(X(t_1,\sigma_1)X(t_2,\sigma_2))|\Omega\rangle$

for free massless scalar in both Operator & Path-integral.

Operator formalism

NB The ground state $|\Omega\rangle = |0\rangle$ is NOT normalizable.

(Instead, it is δ-function normalizable:

$|k\rangle$ (represented by e^{ikx_0}) obey

$$\langle k|k'\rangle = \int dx_0\, e^{-ikx_0} e^{ik'x_0} = 2\pi\,\delta(k-k')\ )$$

Nevertheless, let's compute $\langle 0|T(X(1)X(2))|0\rangle$

allowing $\delta(0)$!

$t_1 > t_2$

$$\langle 0|T(X(1)X(2))|0\rangle = \langle 0|\Big(x_0 + t_1 p_0 + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\big(\alpha_n e^{-in(t_1-\sigma_1)} + \tilde{\alpha}_n e^{-in(t_1+\sigma_1)}\big)\Big)$$

$$\cdot\Big(x_0 + t_2 p_0 + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\frac{1}{m}\big(\alpha_m e^{-im(t_2-\sigma_2)} + \tilde{\alpha}_m e^{-im(t_2+\sigma_2)}\big)\Big)|0\rangle$$

Note: $\langle 0|\alpha_n = 0$ if $n<0$, $\alpha_n|0\rangle = 0$ if $n>0$.

$$= \langle 0|(x_0+t_1p_0)(x_0+t_2\cancel{p_0})|0\rangle + \left(\frac{i}{\sqrt{2}}\right)^2 \sum_{\substack{n>0\\ m<0}} \frac{1}{nm} e^{-in(t_1-\sigma_1)-im(t_2-\sigma_2)} \overset{n\delta_{n+m,0}\langle 0|0\rangle}{\langle 0|\alpha_n\alpha_m|0\rangle}$$

$$+ \left(\sigma_1,\sigma_2 \to -\sigma_1,-\sigma_2,\ \alpha_n\alpha_m \to \tilde{\alpha}_n\tilde{\alpha}_m\right)$$

$$x_0 p_0 - i$$

$$= \langle 0 | x_0^2 + t (p_0 x_0) | 0 \rangle$$

$$-\frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{n(-n)} \left(e^{-in(t-\sigma)} + e^{-in(t+\sigma)} \right) \qquad \begin{cases} t := t_1 - t_2 \\ \sigma := \sigma_1 - \sigma_2 \end{cases}$$

$$= -\delta''(0) + \delta(0) \left[-it_1 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{e^{-in(t-\sigma)}}{n} + \frac{e^{-in(t+\sigma)}}{n} \right) \right]$$

$$\begin{cases} t = |t| e^{-i\epsilon} \Rightarrow \left| e^{-i(t\pm\sigma)} \right| < 1 \\ \therefore \sum_{n=1}^{\infty} \frac{e^{-in(t\pm\sigma)}}{n} = -\log\left(1 - e^{-i(t\pm\sigma)}\right) \end{cases}$$

$$= -\delta''(0) + \delta(0) \left[-it_1 - \frac{1}{2} \log\left(1 - e^{-i(t-\sigma)}\right) - \frac{1}{2} \log\left(1 - e^{-i(t+\sigma)}\right) \right]$$

$$= -\delta''(0) + \delta(0) \left[-\frac{1}{2} \right] \log\left[e^{+2it_1} \left(1 - e^{-i(t_1 - t_2 - \sigma_1 + \sigma_2)}\right)\left(1 - e^{-i(t_1 - t_2 + \sigma_1 - \sigma_2)}\right) \right]$$

$$= -\delta''(0) + \delta(0) \left[-\frac{1}{2} \log\left(e^{i(t_1-\sigma_1)} - e^{i(t_2-\sigma_2)}\right)\left(e^{i(t_1+\sigma_1)} - e^{i(t_2+\sigma_2)}\right) \right]$$

Same expression for $t_2 > t_1$

$$\therefore \langle 0 | T(X(1) X(2)) | 0 \rangle = -\delta''(0)$$

$$+ \delta(0) \left[-\frac{1}{2} \log (z_1 - z_2)(\tilde{z}_1 - \tilde{z}_2) \right] \qquad \begin{aligned} z_i &= e^{i(t_i - \sigma_i)} \\ \tilde{z}_i &= e^{i(t_i + \sigma_i)} \end{aligned}$$