In path integral

$$S_E = \frac{1}{2\pi}\int_{-T}^{T} d\tau \int_0^{2\pi} d\sigma \left(\frac{1}{2}(\partial_\tau X)^2 + \frac{1}{2}(\partial_\sigma X)^2 \right)$$

$$\underset{\substack{\uparrow \\ X=0 \text{ at } \tau=\pm T}}{=} \frac{1}{2}\int_{-T}^{T} d\tau \int_0^{2\pi} d\sigma\, X \underbrace{\left(-\frac{\partial_\tau^2 + \partial_\sigma^2}{2\pi} \right)}_{A} X$$

$\langle X(\sigma_1,\tau_1)\, X(\sigma_2,\tau_2) \rangle$ = inverse of the Kinetic operator A

$$= \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{2\pi}{\left(\frac{\pi m}{2T}\right)^2 + n^2} \frac{1}{2\pi T} \sin\left(\frac{\pi m(\tau_1+T)}{2T}\right) e^{in\sigma_1} \cdot \sin\left(\frac{\pi m(\tau_2+T)}{2T}\right) e^{-in\sigma_2}$$

$$= \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{4T} \cdot \frac{1}{\left(\frac{\pi m}{2T}\right)^2 + n^2} \left(e^{\frac{i\pi m(\tau_1-\tau_2)}{2T}} + e^{-\frac{i\pi m(\tau_1-\tau_2)}{2T}} - e^{\frac{i\pi m(\tau_1+\tau_2+2T)}{2T}} - e^{-\frac{i\pi m(\tau_1+\tau_2+2T)}{2T}} \right) e^{in(\sigma_1-\sigma_2)}$$

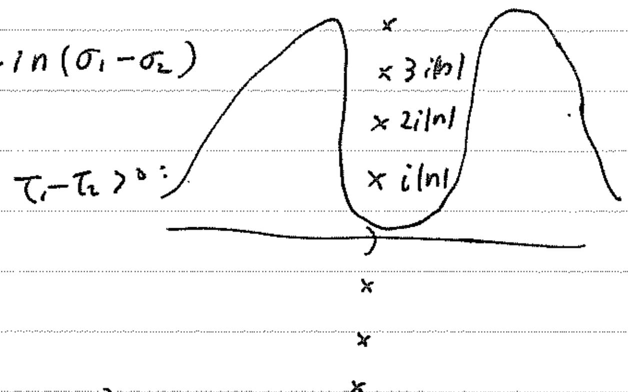
$$= \sum_{n\neq 0} \sum_{m=1}^{\infty} \frac{1}{2\pi} \cdot \frac{\pi}{2T} \cdot \frac{1}{\underbrace{\left(\frac{\pi m}{2T}\right)}_{k_m}^2 + n^2} \left(e^{ik_m(\tau_1-\tau_2)} + e^{-ik_m(\tau_1-\tau_2)} \cancel{- (-1)^m e^{ik_m(\tau_1+\tau_2)}} \cancel{- (-1)^m e^{-ik_m(\tau_1+\tau_2)}} \right) e^{in(\sigma_1-\sigma_2)}$$

$$+ \sum_{m=1}^{\infty} \frac{T}{\pi^2} \frac{1}{m^2} \left(e^{\frac{i\pi m(\tau_1-\tau_2)}{2T}} + e^{-\frac{i\pi m(\tau_1-\tau_2)}{2T}} - e^{\frac{i\pi m(\tau_1+\tau_2)}{2T}}(-1)^m - e^{-\frac{i\pi m(\tau_1+\tau_2)}{2T}}(-1)^m \right)$$

$$\boxed{n \neq 0 \quad \text{Contribution at} \quad T \to \infty}$$

$$\to \sum_{n\neq 0} \int_0^\infty \frac{dk}{2\pi} \frac{1}{k^2+n^2} \left(e^{ik(\tau_1-\tau_2)} + e^{-ik(\tau_1-\tau_2)} \right) e^{in(\sigma_1-\sigma_2)}$$

$$= \sum_{n\neq 0} \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{1}{k^2+n^2} e^{ik(\tau_1-\tau_2)+in(\sigma_1-\sigma_2)}$$



$\tau_1>\tau_2$

$$= \sum_{n\neq 0} \frac{2\pi i}{2\pi} \frac{1}{2i|n|} e^{-|n|(\tau_1-\tau_2)+in(\sigma_1-\sigma_2)}$$

$$= \frac{1}{2} \sum_{n\neq 0} \frac{1}{|n|} e^{-|n|(\tau_1-\tau_2)+in(\sigma_1-\sigma_2)}$$

$$= -\frac{1}{2}\log\left(1-e^{-\tau+i\sigma}\right) - \frac{1}{2}\log\left(1-e^{-\tau-i\sigma}\right) \qquad \begin{cases} \tau := \tau_1-\tau_2 \\ \sigma := \sigma_1-\sigma_2 \end{cases}$$

$$= -\frac{1}{2}\log\left|1-e^{-\tau+i\sigma}\right|^2$$

$$= +\tau_1 - \log\left|e^{\tau_1-i\sigma_1} - e^{\tau_2-i\sigma_2}\right|$$

If $\tau_2>\tau_1$

$$= \tau_2 - \log\left|e^{\tau_1-i\sigma_1} - e^{\tau_2-i\sigma_2}\right|$$

$\boxed{n=0 \text{ Contribution}}$

$\tau = \tau_1 - \tau_2$

$$= \frac{T}{\pi^2}\left(Li_2\left(e^{\frac{i\pi\tau}{2T}}\right) + Li_2\left(e^{-\frac{i\pi\tau}{2T}}\right) + Li_2\left(-e^{\frac{i\pi(\tau_1+\tau_2)}{2T}}\right) - Li_2\left(-e^{-\frac{i\pi(\tau_1+\tau_2)}{2T}}\right) \right)$$

$$Li_2(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^2} \qquad \text{diLogarithm}$$

- $Li_2(x) + Li_2(-x) = \frac{1}{2} Li_2(x^2)$ ☹ trivial.
- $Li_2(e^{i\epsilon}) + Li_2(e^{-i\epsilon}) = \frac{\pi^2}{3} - \pi|\epsilon| + O(\epsilon^2)$ ☹ nontrivial

$$= \frac{T}{\pi^2}\left(\frac{\pi^2}{3} - \pi \cdot \left|\frac{\pi\tau}{2T}\right| + O\left(\frac{1}{T^2}\right)\right) + \frac{T}{\pi^2}\left(\frac{\pi^2}{3} - \pi \cdot \cancel{\left|\frac{\pi(\tau_1+\tau_2)}{2T}\right|} - \frac{1}{2}\left(\frac{\pi^2}{3} - \pi \cdot \cancel{\left|\frac{\pi(\tau_1+\tau_2)}{T}\right|}\right) + O\left(\frac{1}{T^2}\right)\right)$$

$$= \frac{T}{2} - \frac{|\tau|}{2} + O\left(\frac{1}{T^2}\right)$$

$$= \frac{T}{2} \mp \frac{\tau_1 - \tau_2}{2} + O\left(\frac{1}{T^2}\right) \qquad \text{if } \tau_1 > \tau_2, \ \tau_2 > \tau_1$$

$$\therefore \ \langle X(1) X(2) \rangle = \frac{T}{2} - \log\left| e^{\tau_1 - i\sigma_1} - e^{\tau_2 - i\sigma_2} \right| + \frac{1}{2}(\tau_1 + \tau_2)$$

$$= \frac{T}{2} - \log|z_1 - z_2| + \frac{1}{2}\log|z_1 z_2|$$

$$\begin{cases} z_1 = e^{\tau_1 - i\sigma_1} \\ z_2 = e^{\tau_2 - i\sigma_2} \end{cases}$$

A simpler way to understand the zero mode part

$$\langle X(1)X(2)\rangle = \sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right)\frac{dk}{2\pi}\,\frac{1}{k^2+n^2}\,e^{ik(\tau_1-\tau_2)+in(\sigma_1-\sigma_2)}$$

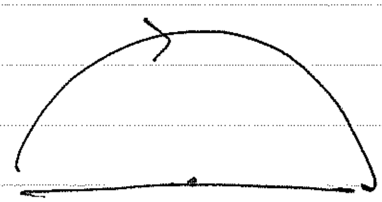
$$= \left.\sum_{n\neq 0}\int\frac{dk}{2\pi}\,\frac{1}{k^2+n^2}\,e^{ik(\tau_1-\tau_2)+in(\sigma_1-\sigma_2)}\right] \to \text{as before.}$$

$$\left.+\int\frac{dk}{2\pi}\,\frac{1}{k^2+\cancel{n^2}}\,e^{ik(\tau_1-\tau_2)+\cancel{in(\sigma_1-\sigma_2)}}\right] \to I_0$$

$\tau_1-\tau_2>0$

$$\Rightarrow I_0 = \left(\int + \int\right)\frac{dk}{2\pi}\,\frac{e^{ik(\tau)}}{k^2} \qquad \tau=\tau_1-\tau_2>0$$

$$= \cancel{\left(\int\frac{dk}{2\pi}\,\frac{e^{ik\tau}}{k^2}\right)} \to 0 \quad \text{by}$$



$$+\int_0^{\pi}\frac{d(\epsilon e^{i\theta})}{2\pi}\,\frac{e^{i\tau\epsilon e^{i\theta}}}{\epsilon^2 e^{2i\theta}} \qquad e^{i\tau\epsilon e^{i\theta}} = 1+i\tau\epsilon e^{i\theta}+O(\epsilon^2)$$

$$= -\int_0^{\pi}\frac{d(e^{-i\theta})}{2\pi\epsilon} + \int_0^{\pi}\frac{i\,d\theta}{2\pi}\cdot i\tau\frac{\epsilon\cdot\epsilon}{\epsilon^2} + O(\epsilon)$$

$$= \frac{1}{\pi\epsilon} - \frac{\tau}{2} + O(\epsilon)$$

$$\updownarrow$$

$$\frac{T}{2}$$

r.

Other 2-point functions

$$\langle \partial_\mu X(1)\, X(2)\rangle$$

$$\langle \partial_\mu X(1)\, \partial_\nu X(2)\rangle$$

$$\langle \partial_{\mu_1}\cdots\partial_{\mu_r} X(1)\, \partial_{\nu_1}\cdots\partial_{\nu_s} X(2)\rangle$$

makes sense also in σ-model on a circle.

$$\langle \partial_{z_1} X(1)\, X(2)\rangle = -\frac{1}{2}\frac{1}{z_1-z_2} + \frac{1}{4}\frac{1}{z_1}$$

$$\langle \partial_{\bar z_1} X(1)\, X(2)\rangle = -\frac{1}{2}\frac{1}{\bar z_1-\bar z_2} + \frac{1}{4}\frac{1}{\bar z_1}$$

$$\langle \partial_z X(1)\, \partial_z X(2)\rangle = -\frac{1}{2}\frac{1}{(z_1-z_2)^2}$$

$$\langle \partial_{\bar z} X(1)\, \partial_{\bar z} X(2)\rangle = -\frac{1}{2}\frac{1}{(\bar z_1-\bar z_2)^2}$$

$$\langle \partial_z X(1)\, \partial_{\bar z} X(2)\rangle = -\partial_{z_1}\partial_{\bar z_1}\langle X(1)\, X(2)\rangle$$

$$= \partial_{z_1}\partial_{\bar z_1}\log|z_1-z_2| = \frac{\pi}{2}\delta^{(2)}(z_1-z_2)$$

[c.f. $\langle X(1)\, X(2)\rangle$ is supposed to be the inverse of

$$\frac{1}{2\pi}\left(-\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\sigma^2}\right) \sim -\frac{2}{\pi}|z|^2\partial_z\partial_{\bar z}$$]

Note all these correlation functions are singular

as $(\tau_1, \sigma_1) \longrightarrow (\tau_2, \sigma_2)$

recall normal ordered product (in path integral)

$$X(1)\,X(2) = \overbrace{X(1)\,X(2)} + :X(1)\,X(2):$$

$$\partial_\mu X(1)\,X(2) = \partial_\mu \overbrace{X(1)\,X(2)} + :\partial_\mu X(1)\,X(2):$$

$$\partial_\mu X(1)\,\partial_\nu X(2) = \partial_\mu \overbrace{X(1)\,\partial_\nu} X(2) + :\partial_\mu X(1)\,\partial_\nu X(2):$$

$$X(1)\,X(2)\,\partial_\mu X(3) = \overbrace{X(1)\,X(2)}\,\partial_\mu X(3) + X(1)\,\overbrace{X(2)\,\partial_\mu} X(3)$$

$$+ \overbrace{X(1)\,X(2)\,\partial_\mu} X(3) + :X(1)\,X(2)\,\partial_\mu X(3):$$

$\vdots$

$$\boxed{\begin{array}{c} :\partial_{\mu_1} \cdots \partial_{\mu_{r_1}} X(1)\;\partial_{\nu_1} \cdots \partial_{\nu_{r_2}} X(2) \cdots \partial_{\lambda_1} \cdots \partial_{\lambda_{r_s}} X(s): \\ \text{regular as } 1 \to 2,\; 1 \to 3,\; \cdots \\ \text{i.e. } \forall\, i \to j \end{array}}$$

Normal Ordered product in Operator formalism

Consider $R \ll \infty$ sigma model $\left(\begin{array}{l}\text{to make the ground state } |0\rangle \\ \text{normalizable} \quad \langle 0|0\rangle = 1\end{array}\right)$

recall $\quad T(\hat{O}_1(1)\cdots\hat{O}_s(s)) = \overbrace{O_1(1)\cdots O_s(s)}$

<u>define</u> $\quad :\hat{O}_1(1)\cdots\hat{O}_s(s): \;:=\; \overbrace{:O_1(1)\cdots O_s(s):}$

e.g. $\quad :\partial_z\hat{X}(1)\,\partial_z\hat{X}(2): \;=\; \overbrace{:\partial_z X(1)\,\partial_z X(2):}$

$$= \overbrace{\partial_z X(1)\,\partial_z X(2)} - \partial_z \overline{X(1)\,\partial_z X}(2)$$

$$= T(\partial_z\hat{X}(1)\,\partial_z\hat{X}(2)) - \langle 0|T(\partial_z\hat{X}(1)\,\partial_z\hat{X}(2))|0\rangle$$

Note

$$X\Big|_{\text{Wick}} = x_0 - i\underbrace{\tau}_{\frac{1}{2}\log(z\bar{z})} p_0 + \underbrace{\sigma}_{\frac{i}{2}\log(z/\bar{z})} w + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}\right)$$

$$z\partial_z X\Big|_{\text{Wick}} = -\frac{i}{2}(p_0 - w) - \frac{i}{\sqrt{2}}\sum_{n\neq 0}\alpha_n z^{-n}$$

$$\bar{z}\partial_{\bar{z}} X\Big|_{\text{Wick}} = -\frac{i}{2}(p_0 + w) - \frac{i}{\sqrt{2}}\sum_{n\neq 0}\tilde{\alpha}_n \bar{z}^{-n}$$

We find

$$:\alpha_n\alpha_m: \; = \alpha_n\alpha_m - \langle 0|\alpha_n\alpha_m|0\rangle$$

$$= \begin{cases} \alpha_n\alpha_m & \text{if } n+m\neq 0 \text{ or } m>0 \text{ or } n<0 \\ \underbrace{\alpha_n\alpha_m - n}_{=\,\alpha_m\alpha_n} & \text{if } n+m=0 \text{ and } m<0 \ (\Leftrightarrow n>0) \end{cases}$$

ie swap the position if creation ops α_m $m<0$ is on the right of annihilation ops α_n $n>0$.

$$\alpha_1\alpha_1\alpha_{-1} = \underbrace{\alpha_1\alpha_1\alpha_{-1}}_{=\,\alpha_1} + \underbrace{\alpha_1\alpha_1\alpha_{-1}}_{=\,\alpha_1} + :\alpha_1\alpha_1\alpha_{-1}:$$

$$\therefore\ :\alpha_1\alpha_1\alpha_{-1}: \; = \alpha_1\alpha_1\alpha_{-1} - 2\alpha_1 = \alpha_{-1}\alpha_1\alpha_1$$

$$\alpha_1\alpha_2\alpha_{-1}\alpha_{-2} = :\alpha_1\alpha_2\alpha_{-1}\alpha_{-2}: + :\alpha_1\alpha_2\alpha_{-1}\alpha_{-2}: + \alpha_1\alpha_2\alpha_{-1}\alpha_{-2}$$

$$+ :\alpha_1\alpha_2\alpha_{-1}\alpha_{-2}:$$

$$\therefore\ :\alpha_1\alpha_2\alpha_{-1}\alpha_{-2}: \; = \alpha_1\alpha_2\alpha_{-1}\alpha_{-2} - \underbrace{:\alpha_2\alpha_{-2}:}_{=\,\alpha_{-2}\alpha_2} - 2\underbrace{:\alpha_1\alpha_{-1}:}_{=\,\alpha_{-1}\alpha_1} - 1\cdot 2$$

$$= \alpha_{-1}\alpha_{-2}\alpha_1\alpha_2$$

$:\alpha_{n_1}\cdots\alpha_{n_s}:$ = bring annihilation ops α_{n_i} $n_i>0$ to the right of creation ops α_{n_j} $n_j<0$.