$\boxed{e^{ikX}}$ ... transforms as $e^{ikX} \to e^{ik\epsilon} \cdot e^{ikX}$

under X-translation $X \to X + \epsilon$

$\Longleftrightarrow$ it carries target momentum k

or creates

Sigma model on $S^1_{2\pi R}$ $\Rightarrow$ k must be quantized $k = \frac{l}{R}$ $(l \in \mathbb{Z})$.

$\boxed{e^{i\hat{k}\hat{X}}}$ ($\hat{X}$ ... T-dual variable) $\hat{k} = \frac{m}{\hat{R}} = Rm$ $(m \in \mathbb{Z})$

... it carries (or creates) dual momentum $\hat{k} = Rm$

$\Updownarrow$ winding # m.

... it is an operator that creates winding # m

"Vortex Operator"

We consider $\langle e^{ik_1X(1)} \cdots e^{ik_sX(s)} \rangle$

or $\langle e^{ik_1X(1)} \cdots e^{ik_sX(s)} e^{i\hat{k}_{s+1}\hat{X}(s+1)} \cdots e^{i\hat{k}_{s+r}\hat{X}(s+r)} \rangle$

momentum / winding # conservation $\Rightarrow$ non vanishing only if

$k_1 + \cdots + k_s = 0, \quad \hat{k}_{s+1} + \cdots + \hat{k}_{s+r} = 0.$

The Operator $:e^{ikX(t,\sigma)}:$

$$\begin{cases} \text{defined for } \forall k \text{ for massless scalars } (R\to\infty) \\ \text{defined for } k=\frac{l}{R}\ (l\in\mathbb{Z}) \text{ for } \sigma\text{-model on } S^1_{2\pi R} \end{cases}$$

massless scalar

$$X(t,\sigma) = x_0 + tp_0 + \frac{i}{\sqrt{2}}\sum_{m\neq 0}\frac{1}{n}\left(\alpha_n z^{-n} + \tilde{\alpha}_n \tilde{z}^{-n}\right)$$

$$z = e^{i(t-\sigma)}, \quad \tilde{z} = e^{i(t+\sigma)}$$

- We know what $:\alpha_{n_1}\cdots\alpha_{n_s}:$ is.
- define $:f(x_0,p_0): =$ symmetrized $f(x_0,p_0)$

e.g. $:x_0 p_0: = :p_0 x_0: = \frac{x_0 p_0 + p_0 x_0}{2} = x_0 p_0 - \frac{i}{2} = p_0 x_0 + \frac{i}{2}$

$$:e^{ikX(t,\sigma)}: = e^{ik\frac{i}{\sqrt{2}}\sum_{n<0}\frac{1}{n}\left(\alpha_n z^{-n} + \tilde{\alpha}_n \tilde{z}^{-n}\right)}$$

$$\times :e^{ik(x_0+tp_0)}:$$

$$\times e^{ik\frac{i}{\sqrt{2}}\sum_{n>0}\frac{1}{n}\left(\alpha_n z^{-n} + \tilde{\alpha}_n \tilde{z}^{-n}\right)}$$

$$:e^{ik(x_0+tp_0)}: = e^{ik(x_0+tp_0)} = e^{ikx_0}\cdot e^{iktp_0}\cdot e^{-i\frac{(ik)^2}{2}t}$$

$$e^{A+B} = e^A\cdot e^B\cdot e^{-\frac{1}{2}[A,B]} \quad \text{if } [A,[A,B]] = [B,[A,B]] = 0$$

Hausdorff's formula

remark $\quad |k'\rangle \leftrightarrow e^{ik'x_0}$

$$e^{ikx_0}|k'\rangle \leftrightarrow e^{ikx_0}\cdot e^{ik'x_0} = e^{i(k+k')x_0} \leftrightarrow |k+k'\rangle$$

$$\therefore\ e^{ikx_0}|0\rangle = |k\rangle \xrightarrow{\text{c.c.}} \langle 0|e^{-ikx_0} = \langle k| .$$

Now let us compute

$$\langle 0|T(:e^{ik_1X(1)}:\ :e^{ik_2X(2)}:)|0\rangle$$

$$\overset{t_1>t_2}{=} \langle 0| e^{ik_1\frac{i}{\sqrt{2}}\sum_{n<0}\frac{1}{n}\cancel{(\alpha_n z_1^{-n}+\tilde\alpha_n\tilde z_1^{-n})}}\cdot e^{ik_1x_0}\cdot e^{ik_1t_1\overset{k_2}{\cancel{p_0}}}\cdot e^{i\frac{k_1^2}{2}t_1}$$

$$\times\, e^{ik_1\frac{i}{\sqrt{2}}\sum_{n>0}\frac{1}{n}(\alpha_n z_1^{-n}+\tilde\alpha_n\tilde z_1^{-n})}\cdot e^{ik_2\frac{i}{\sqrt{2}}\sum_{m<0}\frac{1}{m}(\alpha_m z_2^{-m}+\tilde\alpha_m\tilde z_2^{-m})}$$

$$\times\, e^{ik_2x_0}\, e^{ik_2t_2p_0}\, e^{i\frac{k_2^2}{2}t_2}\cdot e^{ik_2\frac{i}{\sqrt{2}}\sum_{m>0}\frac{1}{m}\cancel{(\alpha_m z_2^{-m}+\tilde\alpha_m\tilde z_2^{-m})}}|0\rangle$$

$$= e^{ik_1t_1k_2 + i\frac{k_1^2}{2}t_1 + i\frac{k_2^2}{2}t_2}$$

$$\times \langle -k_1| e^{ik_1\frac{i}{\sqrt{2}}\sum_{n>0}\frac{1}{n}(\alpha_n z_1^{-n}+\tilde\alpha_n\tilde z_1^{-n})}\cdot e^{ik_2\frac{i}{\sqrt{2}}\sum_{m<0}\frac{1}{m}(\alpha_m z_2^{-m}+\tilde\alpha_m\tilde z_2^{-m})}|k_2\rangle$$

$$\star\qquad e^{\text{annih}}\cdot e^{\text{cre}} \underset{\substack{\uparrow\\ \text{Hausdorff}}}{=} e^{\text{cre}}\cdot e^{\text{annih}}\cdot e^{[\text{annih},\,\text{cre}]}$$

$$[\text{annih},\ \text{cre}] = k_1k_2\left(\frac{i}{\sqrt{2}}\right)^2\sum_{\substack{n>0\\ m<0}}\frac{1}{nm}\underbrace{\left[\alpha_n z_1^{-n}+\tilde\alpha_n\tilde z_1^{-n},\ \alpha_m z_2^{-m}+\tilde\alpha_m\tilde z_2^{-m}\right]}_{n\delta_{n+m,0}z_1^{-n}z_2^{-m}+n\delta_{n+m,0}\tilde z_1^{-n}\tilde z_2^{-m}}$$

$$= \frac{k_1k_2}{2}\sum_{n=1}^{\infty}\frac{-1}{n}\left(z_1^{-n}z_2^{n}+\tilde z_1^{-n}\tilde z_2^{+n}\right) = \frac{k_1k_2}{2}\left[\log(1-z_2/z_1)+\log(1-\tilde z_2/\tilde z_1)\right]$$

$$\langle -k_1 | e^{cre} e^{annih} | k_2 \rangle = \langle -k_1 | k_2 \rangle = \delta(k_1+k_2).$$

$$\therefore \star = e^{ik_1k_2t_1 + i\frac{k_1^2}{2}t_1 + i\frac{k_2^2}{2}t_2}\,\delta(k_1+k_2)\cdot e^{\frac{k_1k_2}{2}\left(\log(1-z_2/z_1)+\log(1-\tilde{z}_2/\tilde{z}_1)\right)}$$

$$= \underbrace{e^{\frac{ik_1k_2}{2}(t_1-t_2)}}_{=\left(z_1\tilde{z}_1/z_2\tilde{z}_2\right)^{\frac{k_1k_2}{4}}}\delta(k_1+k_2)\,(1-z_2/z_1)^{\frac{k_1k_2}{2}}(1-\tilde{z}_2/\tilde{z}_1)^{\frac{k_1k_2}{2}}$$

$$= \delta(k_1+k_2)\,\frac{(z_1-z_2)^{\frac{k_1k_2}{2}}(\tilde{z}_1-\tilde{z}_2)^{\frac{k_1k_2}{2}}}{(z_1\tilde{z}_1)^{\frac{k_1k_2}{4}}(z_2\tilde{z}_2)^{\frac{k_1k_2}{4}}}$$

same expression also for $t_2 > t_1$

$$\therefore \langle 0 | T(:e^{ik_1X(1)}::e^{ik_2X(2)}:) | 0 \rangle$$

$$= \delta(k_1+k_2)\,\frac{\left\{(z_1-z_2)(\tilde{z}_1-\tilde{z}_2)\right\}^{\frac{k_1k_2}{2}}}{(z_1\tilde{z}_1\cdot z_2\tilde{z}_2)^{\frac{k_1k_2}{4}}}$$

In Path-Integral

$$\left\langle \prod_{j=1}^{s} e^{ik_j \cdot X(\sigma_j)} \right\rangle = \frac{\int \mathcal{D}X \, e^{-S_E + \sum_{j=1}^{s} ik_j \cdot X(\sigma_j)}}{\int \mathcal{D}X \, e^{-S_E}}$$

$$S_E = \frac{1}{2}\int_{-T}^{T} d\tau \int_0^{2\pi} d\sigma \; XAX \qquad A = -\frac{1}{2\pi}\left(\partial_\tau^2 + \partial_\sigma^2\right)$$

$$S_E - \sum_{j=1}^{s} ik_j \cdot X(\sigma_j) = \frac{1}{2}\int d^2\sigma \left(XAX - 2i \sum_{j=1}^{s} k_j \delta^{(2)}(\sigma - \sigma_j) X(\sigma) \right)$$

$$= \frac{1}{2}\int d^2\sigma \Big[\Big(X - i\sum_{j=1}^{s} k_j \cdot A^{-1}\delta_{\sigma_j} \Big) A \Big(X - i\sum_{j=1}^{s} k_j A^{-1}\delta_{\sigma_j} \Big)$$

$$+ \sum_{i,j=1}^{s} k_i k_j \, \delta_{\sigma_i} A^{-1} \delta_{\sigma_j} \Big]$$

$$= \frac{1}{2}\int d^2\sigma \, (X - \cdots) A (X - \cdots) + \frac{1}{2}\sum_{i,j=1}^{s} k_i k_j \underbrace{A^{-1}(\sigma_i, \sigma_j)}_{\shortparallel \atop \langle X(\sigma_i) X(\sigma_j) \rangle}$$

$$\therefore \left\langle \prod_{j=1}^{s} e^{ik_j X(\sigma_j)} \right\rangle = \frac{\cancel{\int \mathcal{D}X \, e^{-\frac{1}{2}\int (X-\cdots)A(X-\cdots)}} - \frac{1}{2}\sum_{i,j=1}^{s} k_i k_j \langle X(\sigma_i) X(\sigma_j) \rangle}{\cancel{\int \mathcal{D}X \, e^{-\frac{1}{2}\int XAX}}}$$

$$= \prod_{i,j=1}^{s} e^{-\frac{k_i k_j}{2} \langle X(\sigma_i) X(\sigma_j) \rangle}$$

Problem $\langle X(\sigma) X(\sigma') \rangle$ is $\overset{\text{diverges}}{\text{singular}}$ if $\sigma = \sigma'$

$\therefore \displaystyle\prod_{i,j} e^{-\frac{k_i k_j}{2} \langle X(\sigma_i) X(\sigma_j) \rangle}$ is $\underbrace{\text{singular}}_{\text{diverges or vanishes}}$ from $i=j$ factors

Point-splitting regularization $\sigma_i \to \sigma_i' = \sigma_i + \epsilon_i$ close but distinct

$$\prod_{i,j} e^{-\frac{k_i k_j}{2} \langle X(\sigma_i) X(\sigma_j) \rangle} \longrightarrow \prod_{i,j} e^{-\frac{k_i k_j}{2} \langle X(\sigma_i) X(\sigma_j') \rangle}$$

recall $\langle X(\sigma) X(\sigma') \rangle = \dfrac{T}{2} \underbrace{- \log|z - z'| + \frac{1}{2} \log|z\bar{z}'|}_{\text{exercise} \to \approx -\log \mathrm{dist}(\sigma, \sigma')}$

$\left(\mathrm{dist}(\sigma, \sigma') = \text{distance of } \sigma \text{ \& } \sigma' \text{ w.r.t. } ds^2 = d\sigma^2 + d\tau^2 \right)$

Renormalization of e^{ikX}

$$:e^{ikX(\sigma)}: := \lim_{\sigma' \to \sigma} e^{ikX(\sigma) - \frac{k^2}{2} \log \mathrm{dist}(\sigma, \sigma')}$$

Then

$$\left\langle \prod_{j=1}^{s} :e^{ik_j X(\sigma_j)}: \right\rangle = \prod_{i,j=1}^{s} e^{-\frac{k_i k_j}{2} \langle X(\sigma_i) X(\sigma_j') \rangle} \cdot \prod_{i=1}^{s} e^{-\frac{k_i^2}{2} \log \mathrm{dist}(\sigma_i, \sigma_i')}$$

$$= \prod_{i,j=1}^{s} e^{-\frac{k_i k_j}{2} \cdot \frac{T}{2}} \cdot \prod_{i \neq j} e^{-\frac{k_i k_j}{2} \left(-\log|z_i - z_j| + \frac{1}{2} \log|z_i \bar{z}_j| \right)}$$

$= \downarrow$

$$= e^{-\frac{T}{4}\sum_{i,j} k_i k_j} \cdot \prod_{i<j} e^{+k_i k_j\left(\log|z_i - z_j| - \frac{1}{2}|z_i z_j|\right)}$$

$$= e^{-\frac{T}{4}\left(\sum_i k_i\right)^2} \prod_{i<j} \frac{|z_i - z_j|^{k_i k_j}}{|z_i z_j|^{\frac{k_i k_j}{2}}}$$

$$\xrightarrow{T\to\infty} 2\sqrt{\frac{\pi}{T}}\, \delta(k_1 + \cdots + k_s) \cdot \prod_{i<j} \frac{|z_i - z_j|^{k_i k_j}}{|z_i z_j|^{\frac{k_i k_j}{2}}}$$

$\therefore$ With the point splitting regularization & multiplicative renormalization $\;{}^{\times}_{\times} e^{ikX(\sigma)} {}^{\times}_{\times} = \lim_{\sigma' \to \sigma} e^{ikX(\sigma)} \operatorname{dist}(\sigma, \sigma')^{-\frac{k^2}{2}}$

We find

$$\left\langle {}^{\times}_{\times} e^{ik_1 X(1)} {}^{\times}_{\times} \cdots {}^{\times}_{\times} e^{ik_s X(s)} {}^{\times}_{\times} \right\rangle = 2\sqrt{\frac{\pi}{T}}\, \delta(k_1 + \cdots + k_s) \cdot \prod_{i<j} \frac{|z_i - z_j|^{k_i k_j}}{|z_i z_j|^{\frac{k_i k_j}{2}}}$$

$s=2$ case essentially agrees with operator result for

$$\langle 0 | T\left(:e^{ik_1 X(1)}: \; :e^{ik_2 X(2)}: \right) | 0 \rangle = \delta(k_1 + k_2) \frac{|z_1 - z_2|^{k_1 k_2}}{|z_1 z_2|^{\frac{k_1 k_2}{2}}}$$

normal ordered product

With $\tilde{z}_i \to \bar{z}_i$

Sigma model on $S^1_{2\pi R} = \mathbb{R}/2\pi R\mathbb{Z}$

e^{ikX} can be defined only if $k = \frac{\ell}{R}$ $(\ell \in \mathbb{Z})$

<u>In operator</u>

$:e^{ikX(t,\sigma)}:$ defined in the same way

$$= e^{ik\frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{-1}{n}(\alpha_{-n}e^{in(t-\sigma)}+\tilde{\alpha}_{-n}e^{in(t+\sigma)})} \times :e^{ik(x_0+tp_0+\sigma w)}:$$
$$\times\ e^{ik\frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_n e^{-in(t-\sigma)}+\tilde{\alpha}_n e^{-in(t+\sigma)})}$$

w commutes with everything else

$$:e^{ik(x_0+tp_0+\sigma w)}: = e^{ik\sigma w} :e^{ik(x_0+tp_0)}: = :e^{ik(x_0+tp_0)}: e^{ik\sigma w}$$

The ground state $|\Omega\rangle$ is $|0,0\rangle$ & it is normalizable

$$\therefore \langle\Omega| T(:e^{ik_1X(1)}::e^{ik_2X(2)}:)|\Omega\rangle \qquad k_1 = \frac{\ell_1}{R},\ k_2 = \frac{\ell_2}{R}$$

= computed in the same way as before,

$e^{ik\sigma w}$ does not contribute : $e^{ik\sigma w}|\Omega\rangle = |\Omega\rangle$

since $w=0$ on $|0,0\rangle$.

$$= \delta_{\ell_1+\ell_2,0}\ \frac{(z_1-z_2)^{\frac{k_1k_2}{2}}(\tilde{z}_1-\tilde{z}_2)^{\frac{k_1k_2}{2}}}{(z_1\tilde{z}_1)^{\frac{k_1k_2}{4}}(z_2\tilde{z}_2)^{\frac{k_1k_2}{4}}}$$

In path integral

- There is no winding # since $X(\sigma, \tau=\pm T) \equiv 0$
- There is a winding in the τ direction

$$X(\sigma, \tau=T) - X(\sigma, \tau=-T) = 2\pi R \cdot m \qquad (m \in \mathbb{Z})$$

but one can show that it doesn't make any change.

$$\therefore \left\langle \prod_{j=1}^{s} {}^{\times}_{\times} e^{i k_j X(j)} {}^{\times}_{\times} \right\rangle \qquad k_j = \frac{l_j}{R} \quad (l_j \in \mathbb{Z}) \text{ of course}$$

$$= \underbrace{\lim_{T \to \infty} e^{-\frac{T}{4}\left(\sum_i k_i\right)^2}}_{\delta_{l_1 + \cdots + l_s, 0}} \prod_{i<j} \frac{|z_i - z_j|^{k_i k_j}}{|z_i z_j|^{\frac{k_i k_j}{2}}}$$

"Vortex operator"

The operator $e^{ik'X'}$, $X' = $ T-dual variable

$$\left[\text{recall} \quad \begin{array}{l} \partial_t X' = \partial_\sigma X \\ \partial_\sigma X' = \partial_t X \end{array} \quad , \quad X' \equiv X' + \frac{2\pi}{R} \right]$$

$e^{ik'X'}$ well-defined only if $k' = Rm$ $(m \in \mathbb{Z})$

$$X'(t,\sigma) = \underbrace{\hat{x}_0}_{} + t\,\underbrace{W}_{} + \sigma\,\underbrace{p_0}_{} + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \left(\underbrace{-\alpha_n}_{} e^{-in(t-\sigma)} + \tilde{\alpha}_n e^{-in(t+\sigma)} \right)$$

$$:e^{ik'X'(t,\sigma)}: = e^{ik'\frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{-1}{n} \left(-\alpha_{-n} e^{in(t-\sigma)} + \tilde{\alpha}_{-n} e^{in(t+\sigma)} \right)}$$

$$\times :e^{ik'(\hat{x}_0 + tW + \sigma p_0)}:$$

$$\times e^{ik'\frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\alpha_n e^{-in(t-\sigma)} + \tilde{\alpha}_n e^{-in(t+\sigma)} \right)}$$

$$e^{ik'\hat{x}_0} \cdot e^{ik'tW} \cdot e^{i\frac{k'^2}{2}t} \cdot e^{ik'\sigma p_0}$$

$\langle e^{ik_1'X'(1)} e^{ik_2X(2)} \rangle = ? = 0$ by $\left.\begin{array}{l}\text{momentum} \\ \text{winding \#}\end{array}\right\}$ conservation!

need other operators $\mathcal{O}$ to have non-zero $\langle \mathcal{O} e^{ik_1'X'(1)} e^{ik_2X(2)} \rangle$

Consider

$$\langle l, m | T(:e^{ik_1'X'(1)}::e^{ik_2X(2)}:) | 0,0 \rangle$$

This can be non-zero only if $k_1' = Rm$, $k_2 = \frac{l}{R}$.

$t_1 > t_2$ case

$$= \underbrace{\langle \ell, m| e^{i\underset{Rm}{k_1}\hat{x}_0}}_{\langle \ell,0|} \cdot e^{ik_1' t_1 \overset{0}{w}} \cdot e^{i\frac{k_1'^2}{2}t_1} \cdot e^{ik_1'\sigma_1 \overset{\ell/R = k_2}{p_0}} \cdot e^{ik_1'\frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\alpha_n z_1^{-n} + \tilde{\alpha}_n \tilde{z}_1^{-n}\right)}$$

$$\cdot e^{ik_2\frac{i}{\sqrt{2}}\sum_{m=1}^{\infty}\frac{-1}{m}\left(\alpha_{-m} z_2^{m} + \tilde{\alpha}_{-m}\tilde{z}_2^{m}\right)} \cdot \underbrace{e^{i\underset{\ell/R}{k_2}\hat{x}_0} \cdot e^{ik_2 t_2 \overset{0}{p_0}} \cdot e^{i\frac{k_2^2}{2}t_2} \cdot e^{ik_2\sigma_2 \overset{0}{w}} |0,0\rangle}_{|\ell,0\rangle e^{i\frac{k_2^2}{2}t_2}}$$

$$= e^{i\frac{k_1'^2}{2}t_1 + ik_1'k_2\sigma_1 + i\frac{k_2^2}{2}t_2} \times$$

$$\times \langle \ell, 0| e^{ik_1'\frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{n}\left(-\alpha_n z_1^{-n} + \tilde{\alpha}_n\tilde{z}_1^{-n}\right)} \cdot e^{ik_2\frac{i}{\sqrt{2}}\sum_{m=1}^{\infty}\frac{-1}{m}\left(\alpha_{-m}z_2^{m} + \tilde{\alpha}_{-m}\tilde{z}_2^{m}\right)} |\ell, 0\rangle$$

different from the previous case only by the sign $\alpha_n \to -\alpha_n$

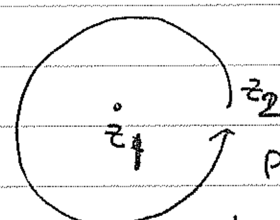
$$\therefore \exp\left(\frac{k_1'k_2}{2}\log(1 - z_2/z_1)\right) \longrightarrow \exp\left(-\frac{k_1'k_2}{2}\log(1 - z_2/z_1)\right)$$

$$= \underbrace{e^{i\frac{k_1'^2}{2}t_1 + ik_1'k_2\sigma_1 + i\frac{k_2^2}{2}t_2}}_{e^{\frac{k_1'^2}{4}\log(z_1\tilde{z}_1) + \frac{k_1'k_2}{2}\log(\tilde{z}_1/z_1) + \frac{k_2^2}{4}\log(z_2\tilde{z}_2)}} \cdot e^{-\frac{k_1'k_2}{2}\left(\log(z_1 - z_2) - \log z_1\right)} \cdot e^{\frac{k_1'k_2}{2}\left(\log(\tilde{z}_1 - \tilde{z}_2) - \log\tilde{z}_1\right)}$$

$$= (z_1\tilde{z}_1)^{\frac{k_1'^2}{4}} \cdot (z_2\tilde{z}_2)^{\frac{k_2^2}{4}} \cdot \left(\frac{\tilde{z}_1 - \tilde{z}_2}{z_1 - z_2}\right)^{\frac{k_1'k_2}{2}}$$

Wick rotation

$$= |z_1|^{\frac{R^2m^2}{2}} \cdot |z_2|^{\frac{\ell^2}{2R^2}} \cdot \underbrace{\left(\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}\right)^{\frac{\ell m}{2}}}_{e^{-\ell m \arg(z_1 - z_2)}}$$



of $e^{i\frac{\ell}{R}X}$

z_2 rotates, phase changes by $-2\pi \ell m$

$X(z_2) \to X(z_2) + 2\pi R m$