Dirac Fermion in 1+1 dimensions

$\gamma^\mu \partial_\mu = \not{\partial}$

$$S = \frac{1}{2\pi}\int dt\, d\sigma \left[i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \right]$$

$\gamma^{\mu=0,1}$ ... Gamma matrices in 1+1 dimensions

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = \begin{cases} 1 & \mu=\nu=t=0 \\ -1 & \mu=\nu=\sigma=1 \\ 0 & \mu\neq\nu \end{cases}$$

soln: $\gamma^0 = \gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \gamma^\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\bar{\psi}_\pm = \psi_\pm^*$

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \qquad \bar{\psi} = \psi^\dagger \gamma^0 = (\bar{\psi}_-, \bar{\psi}_+)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\bar{\psi}_+, \bar{\psi}_-)$$

$$\therefore \bar{\psi}\gamma^\mu \partial_\mu \psi = (\bar{\psi}_+, \bar{\psi}_-)\begin{pmatrix} 0 & \partial_t - \partial_\sigma \\ \partial_t + \partial_\sigma & 0 \end{pmatrix}\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$$

$$= \bar{\psi}_+(\partial_t - \partial_\sigma)\psi_+ + \bar{\psi}_-(\partial_t + \partial_\sigma)\psi_-$$

$$\bar{\psi}\psi = (\bar{\psi}_+, \bar{\psi}_-)\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \bar{\psi}_+\psi_- + \bar{\psi}_-\psi_+$$

$$\therefore S = \frac{1}{2\pi}\int dt\, d\sigma \Big(i\bar{\psi}_-(\partial_t + \partial_\sigma)\psi_- + i\bar{\psi}_+(\partial_t - \partial_\sigma)\psi_+ - m\bar{\psi}_+\psi_- - m\bar{\psi}_-\psi_+ \Big)$$

EOM

$$i(\partial_t + \partial_\sigma)\psi_- - m\psi_+ = 0$$

$$i(\partial_t - \partial_\sigma)\psi_+ - m\psi_- = 0$$

$\boxed{m=0} \Rightarrow$

$$(\partial_t + \partial_\sigma)\psi_- = 0$$

$$(\partial_t - \partial_\sigma)\psi_+ = 0$$

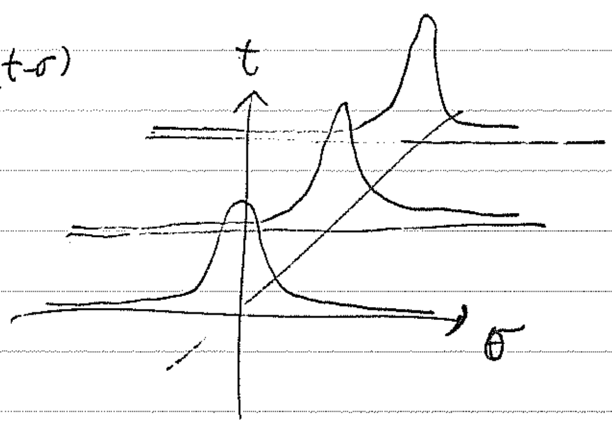
Solutions:

$\psi_-(t,\sigma) = f(t-\sigma)$ right moving

any function

$\psi_+(t,\sigma) = g(t+\sigma)$ left moving

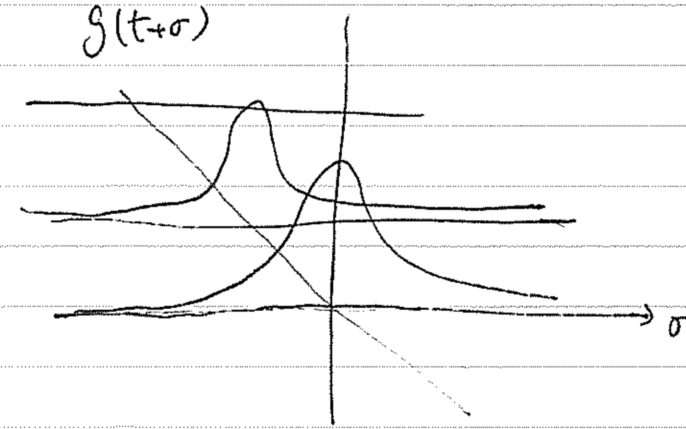
$f(t-\sigma)$



to right with the speed of light

$\therefore \boxed{\psi_-, \bar{\psi}_- \text{ -- right movers}}$

$g(t+\sigma)$



to left "

$\therefore \boxed{\psi_+, \bar{\psi}_+ \text{ -- left movers}}$

Symmetries, Currents an Charges

· Phase rotation $\psi_\pm \to e^{-i\alpha}\psi_\pm$, $\bar\psi_\pm \to e^{i\alpha}\bar\psi_\pm$ "vector-like".

current $\begin{cases} J_V^t = \bar\psi_-\psi_- + \bar\psi_+\psi_+ \\ J_V^\sigma = \bar\psi_-\psi_- - \bar\psi_+\psi_+ \end{cases}$ Charge $F_V = \frac{1}{2\pi}\int d\sigma\,(\bar\psi_-\psi_- + \bar\psi_+\psi_+)$

· $\boxed{m=0}$: "Axial phase rotation". $\psi_\pm \to e^{\mp i\beta}\psi_\pm$, $\bar\psi_\pm \to e^{\pm i\beta}\bar\psi_\pm$

Current $\begin{cases} J_A^t = -\bar\psi_-\psi_- + \bar\psi_+\psi_+ \\ J_V^\sigma = -\bar\psi_-\psi_- - \bar\psi_+\psi_+ \end{cases}$ Charge $F_A = \frac{1}{2\pi}\int d\sigma\,(-\bar\psi_-\psi_- + \bar\psi_+\psi_+)$

<u>Or</u> take "linear" recombination $\longrightarrow$ "Right" & "Left" phase rotation

$$R \begin{cases} \psi_- \to e^{-i\alpha_R}\psi_- & \bar\psi_- \to e^{i\alpha_R}\bar\psi_- \\ \psi_+ \to \psi_+ & \bar\psi_+ \to \bar\psi_+ \end{cases}$$

$$L \begin{cases} \psi_- \longrightarrow \psi_- & \bar\psi_- \longrightarrow \bar\psi_- \\ \psi_+ \to e^{-i\alpha_L}\psi_+ & \bar\psi_+ \to e^{i\alpha_L}\bar\psi_+ \end{cases}$$

Currents $\begin{cases} J_R^t = \bar\psi_-\psi_- \\ J_R^\sigma = \bar\psi_-\psi_- \end{cases}$ Charges $F_R = \frac{1}{2\pi}\int d\sigma\,\bar\psi_-\psi_-$

$\begin{cases} J_L^t = \bar\psi_+\psi_+ \\ J_L^\sigma = -\bar\psi_+\psi_+ \end{cases}$ $F_L = \frac{1}{2\pi}\int d\sigma\,\bar\psi_+\psi_+$

· Time and space translation $\delta\psi_\pm = \epsilon^\lambda \partial_\lambda \psi_\pm , \quad \delta\bar\psi_\pm = \epsilon^\lambda \partial_\lambda \bar\psi_\pm$

$$\delta S = \frac{1}{2\pi}\int dt\, d\sigma \left[\epsilon^\lambda \partial_\lambda \left(i\bar\psi \not\partial \psi - m\bar\psi\psi \right) + i\bar\psi \gamma^\mu \partial_\mu \epsilon^\lambda \cdot \partial_\lambda \psi \right]$$

$$\doteq \frac{1}{2\pi}\int dt\, d\sigma\, \partial_\mu \epsilon^\lambda \underbrace{\left[i\bar\psi\gamma^\mu \partial_\lambda \psi - \delta^\mu_\lambda \left(i\bar\psi\not\partial\psi - m\bar\psi\psi \right) \right]}_{T^\mu_\lambda}$$

energy-momentum tensor

$$T^t_t = i\bar\psi\gamma^t\partial_t\psi - i\bar\psi\gamma^\mu\partial_\mu\psi + m\bar\psi\psi = -i\bar\psi\gamma^\sigma\partial_\sigma\psi + m\bar\psi\psi$$

$$= -i\bar\psi_-\partial_\sigma\psi_- + i\bar\psi_+\partial_\sigma\psi_+ + m\bar\psi_-\psi_+ + m\bar\psi_+\psi_-$$

$$T^\sigma_t = i\bar\psi\gamma^\sigma\partial_t\psi = i\bar\psi_-\partial_t\psi_- - i\bar\psi_+\partial_t\psi_+$$

$$T^t_\sigma = i\bar\psi\gamma^t\partial_\sigma\psi = i\bar\psi_-\partial_\sigma\psi_- + i\bar\psi_+\partial_\sigma\psi_+$$

$$T^\sigma_\sigma = i\bar\psi\gamma^\sigma\partial_\sigma\psi - i\bar\psi\gamma^\mu\partial_\mu\psi + m\bar\psi\psi = -i\bar\psi\gamma^t\partial_t\psi + m\bar\psi\psi$$

$$= -i\bar\psi_-\partial_t\psi_- - i\bar\psi_+\partial_t\psi_+ + m\bar\psi_-\psi_+ + m\bar\psi_+\psi_- .$$

$$H = \frac{1}{2\pi}\int d\sigma \left(-i\bar\psi_-\partial_\sigma\psi_- + i\bar\psi_+\partial_\sigma\psi_+ + m\bar\psi_-\psi_+ + m\bar\psi_+\psi_- \right)$$

$$P = \frac{1}{2\pi}\int d\sigma \left(i\bar\psi_-\partial_\sigma\psi_- + i\bar\psi_+\partial_\sigma\psi_+ \right)$$

<u>Note</u> $T^\mu_{\ \mu} = i\bar\psi\gamma^\mu\partial_\mu\psi - 2(i\bar\psi\not\partial\psi - m\bar\psi\psi)$

$$= -i\bar\psi\gamma^\mu\partial_\mu\psi + 2m\bar\psi\psi$$

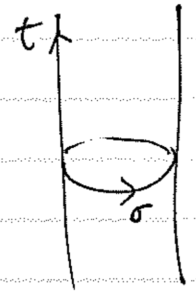
EOM
$i\gamma^\mu\partial_\mu\psi = m\psi$

$$= m\bar\psi\psi$$

$$= \begin{cases} \neq 0 & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases}$$

important !

Quantization of the $m=0$ system



$\sigma \equiv \sigma + 2\pi$ (circle)

require ANTIPERIODIC B.C. $\psi_\pm(t,\sigma+2\pi) = -\psi_\pm(t,\sigma)$

called NS-NS sector

mode expansion

$$\psi_-(t,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r(t)\, e^{ir\sigma}, \quad \bar{\psi}_-(t,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{\psi}_r(t)\, e^{ir\sigma} \quad \left(\bar{\psi}_r := \psi_{-r}^{*}\right)$$

$$\psi_+(t,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{\psi}_r(t)\, e^{-ir\sigma}, \quad \bar{\psi}_+(t,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{\tilde{\psi}}_r(t)\, e^{-ir\sigma} \quad \left(\bar{\tilde{\psi}}_r := \tilde{\psi}_{-r}^{*}\right)$$

$$S = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \int \underbrace{i\bar{\psi}_{-r}(\partial_t + ir)\psi_r \, dt}_{L_r = i\bar{\psi}_{-r}\dot{\psi}_r - r\bar{\psi}_{-r}\psi_r} + \sum_{r\in\mathbb{Z}+\frac{1}{2}} \int \underbrace{i\bar{\tilde{\psi}}_{-r}(\partial_t + ir)\tilde{\psi}_r \, dt}_{\tilde{L}_r = i\bar{\tilde{\psi}}_{-r}\dot{\tilde{\psi}}_r - r\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r}$$

Decoupled system of ∞-ly many "fermionic oscillators".

Quantization : quantize each & sum them up.

$$L_r = i\bar{\psi}_{-r}\dot{\psi}_r - r\bar{\psi}_{-r}\psi_r$$

$$\{\psi_r, \bar{\psi}_{-r}\} = 1, \quad \{\psi_r, \psi_r\} = \{\bar{\psi}_{-r}, \bar{\psi}_{-r}\} = 0$$

$$H_r = r\,\frac{\bar{\psi}_{-r}\psi_r - \psi_r\bar{\psi}_{-r}}{2} = r\bar{\psi}_{-r}\psi_r - \frac{r}{2}$$

$$= -r\psi_r\bar{\psi}_{-r} + \frac{r}{2}$$

denote $|0\rangle'_r$ = the state annihilated by ψ_r

$$\begin{tikzcd} 0 & {|0\rangle'_r} \arrow[l, "\psi_r"'] \arrow[r, shift right, "\bar{\psi}_{-r}"'] & {\bar{\psi}_{-r}|0\rangle'_r} \arrow[l, shift right, "\psi_r"'] \arrow[r, "\bar{\psi}_{-r}"'] & 0 \end{tikzcd}$$

$$H_r = \begin{pmatrix} -\frac{r}{2} & 0 \\ 0 & +\frac{r}{2} \end{pmatrix} \quad \text{wrt} \quad (|0\rangle'_r, \bar{\psi}_{-r}|0\rangle'_r)$$

<u>If $r > 0$</u> the ground state is $|0\rangle'_r$ (energy $-\frac{r}{2}$)

<u>If $r < 0$</u> the ground state is $\bar{\psi}_{-r}|0\rangle'_r$ (energy $= \frac{r}{2}$)

Denote the ground state $|0\rangle_r$

$$\left.\begin{array}{lll} r>0: & \psi_r|0\rangle_r = 0 & E_r = -\frac{r}{2} \\ r<0: & \bar{\psi}_{-r}|0\rangle_r = 0 & E_r = \frac{r}{2} \end{array}\right\} = -\frac{|r|}{2}$$

Collect everything

$$|0\rangle := \bigotimes_{r\in\mathbb{Z}+\frac{1}{2}} |0\rangle_r \otimes \bigotimes_{r\in\mathbb{Z}+\frac{1}{2}} \widetilde{|0\rangle}_r$$

$$\{\psi_r, \bar{\psi}_s\} = \delta_{r+s,0}\,, \quad \{\psi_r, \psi_s\} = \{\bar{\psi}_r, \bar{\psi}_s\} = 0$$

$$\{\tilde{\psi}_r, \bar{\tilde{\psi}}_s\} = \delta_{r+s,0}\,, \quad \{\tilde{\psi}_r, \tilde{\psi}_s\} = \{\bar{\tilde{\psi}}_r, \bar{\tilde{\psi}}_s\} = 0$$

$$\{\psi_r, \tilde{\psi}_s\} = \{\psi_r, \bar{\tilde{\psi}}_s\} = \{\bar{\psi}_r, \tilde{\psi}_s\} = \{\bar{\psi}_r, \bar{\tilde{\psi}}_s\} = 0.$$

$$\boxed{|0\rangle \text{ is annihilated by all } \psi_r, \bar{\psi}_r, \tilde{\psi}_r, \bar{\tilde{\psi}}_r \text{ with } r>0.}$$

$$E_{|0\rangle} = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \left(-\frac{|r|}{2}\right) + \sum_{r\in\mathbb{Z}+\frac{1}{2}} \left(-\frac{|r|}{2}\right) = -\sum_{r\in\mathbb{Z}+\frac{1}{2}} |r|$$

$$= -2\,\cancel{\Sigma}\left(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \cdots\right) \qquad ???$$

$$\zeta(s,x) = \sum_{n=0}^{\infty} (n+x)^{-s} \qquad \zeta(-1,x) = -\frac{1}{2}\left(x-\frac{1}{2}\right)^2 + \frac{1}{24}$$

$$E_{|0\rangle} = -2\,\zeta\left(-1,\frac{1}{2}\right) = -\frac{1}{12} \qquad \left(\text{again!}\right)$$