$$H = \sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r\,\frac{\bar{\psi}_{-r}\psi_r-\psi_r\bar{\psi}_{-r}}{2}\right)+\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r\,\frac{\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r-\tilde{\psi}_r\bar{\tilde{\psi}}_{-r}}{2}\right)$$

$$= \sum_{r>0}\left(r\bar{\psi}_{-r}\psi_r-\frac{r}{2}\right)+\sum_{r<0}\left(-r\psi_r\bar{\psi}_{-r}+\frac{r}{2}\right)$$

$$+\sum_{r>0}\left(r\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r-\frac{r}{2}\right)+\sum_{r<0}\left(-r\tilde{\psi}_r\bar{\tilde{\psi}}_{-r}+\frac{r}{2}\right)$$

$$= \sum_{r\in\mathbb{Z}+\frac{1}{2}} r:\bar{\psi}_{-r}\psi_r: + \sum_{r\in\mathbb{Z}+\frac{1}{2}} r:\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r: \underbrace{-\sum_{r\in\mathbb{Z}+\frac{1}{2}}|r|}_{=-\frac{1}{12}}$$

$$\left.\begin{aligned}
:\bar{\psi}_{-r}\psi_r: &= \begin{cases}\bar{\psi}_{-r}\psi_r & r>0\\ -\psi_r\bar{\psi}_{-r} & r<0\end{cases}\\
:\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r: &= \begin{cases}\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r & r>0\\ -\tilde{\psi}_r\bar{\tilde{\psi}}_{-r} & r<0\end{cases}
\end{aligned}\right\}\ \text{annihilates } |0\rangle.$$

$$P = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\left(i\bar{\psi}_-\partial_\sigma\psi_- + i\bar{\psi}_+\partial_\sigma\psi_+\right)$$

$$= \sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(-r\,\frac{\bar{\psi}_{-r}\psi_r-\psi_r\bar{\psi}_{-r}}{2}\right)+\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r\,\frac{\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r-\tilde{\psi}_r\bar{\tilde{\psi}}_{-r}}{2}\right)$$

$$= \sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(-r:\bar{\psi}_{-r}\psi_r: + r:\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r:\right)$$

$$\therefore\ P|0\rangle = 0.$$

$$F_R = \frac{1}{2\pi}\int d\sigma\, \bar{\psi}_- \psi_- = \sum_r \frac{\bar{\psi}_{-r}\psi_r - \psi_r \bar{\psi}_{-r}}{2}$$

$$= \sum_{r>0}\left(\bar{\psi}_{-r}\psi_r - \frac{1}{2}\right) + \sum_{r<0}\left(-\psi_r\bar{\psi}_{-r} + \frac{1}{2}\right)$$

$$= \sum_{r>0} \underbrace{\bar{\psi}_{-r}\psi_r}_{=\,:\bar{\psi}_{-r}\psi_r:} + \sum_{r<0} \underbrace{(-\psi_r\bar{\psi}_{-r})}_{=\,:\bar{\psi}_{-r}\psi_r:}$$

$$= \sum_r :\bar{\psi}_{-r}\psi_r:$$

$$F_R|0\rangle = 0$$

$$F_L = \frac{1}{2\pi}\int d\sigma\, \bar{\psi}_+ \psi_+ = \sum_r \frac{\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r - \tilde{\psi}_r \bar{\tilde{\psi}}_{-r}}{2}$$

$$= \sum_r \left(\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r - \frac{1}{2}\right) + \sum_{r<0}\left(-\tilde{\psi}_r\bar{\tilde{\psi}}_{-r} + \frac{1}{2}\right)$$

$$= \sum_r :\bar{\tilde{\psi}}_{-r}\tilde{\psi}_r:$$

$$F_L|0\rangle = 0$$

$$\boxed{|0\rangle \text{ has zero } F_L, F_R \text{ charges.}}$$

Other states are obtained from $|0\rangle$ by multiplying

$$\psi_{-r},\ \bar\psi_{-r},\ \tilde\psi_{-r},\ \bar{\tilde\psi}_{-r} \quad \text{with } r>0$$

e.g. $\psi_{-\frac12}|0\rangle$, $\tilde\psi_{-\frac32}|0\rangle$, $\bar\psi_{-\frac72}\tilde\psi_{-\frac92}|0\rangle$, $\ldots$

Their energies, momenta, charges are found by

$$[H,\psi_r]=-r\psi_r,\quad [H,\bar\psi_r]=-r\bar\psi_r,\quad [H,\tilde\psi_r]=-r\tilde\psi_r,\quad [H,\bar{\tilde\psi}_r]=-r\bar{\tilde\psi}_r$$

$$[P,\psi_r]=r\psi_r,\quad [P,\bar\psi_r]=r\bar\psi_r,\quad [P,\tilde\psi_r]=-r\tilde\psi_r,\quad [P,\bar{\tilde\psi}_r]=-r\bar{\tilde\psi}_r$$

$$[F_R,\psi_r]=-\psi_r,\quad [F_R,\bar\psi_r]=\bar\psi_r,\quad [F_R,\tilde\psi_r]=[F_R,\bar{\tilde\psi}_r]=0$$

$$[F_L,\psi_r]=[F_L,\bar\psi_r]=0,\quad [F_L,\tilde\psi_r]=-\tilde\psi_r,\quad [F_L,\bar{\tilde\psi}_r]=\bar{\tilde\psi}_r$$

ψ_{-r} creates energy r, momentum $-r$, F_R-charge -1, F_L-charge 0

$\bar\psi_{-r}$ creates " r " $-r$ " $+1$, " 0

$\tilde\psi_{-r}$ creates " r, " $+r$, " 0, " -1

$\bar{\tilde\psi}_{-r}$ creates " r, " $+r$, " 0, " $+1$.

e.g. $\psi_{-\frac12}|0\rangle$ has $H=\frac12-\frac1{12}=\frac5{12}$, $P=-\frac12$, $F_R=-1$, $F_L=0$

$\bar\psi_{-\frac32}\tilde\psi_{-\frac12}|0\rangle$ has $H=\frac32+\frac12-\frac1{12}=\frac{23}{12}$, $P=\frac{-3}{2}+\frac12=-1$, $F_R=+1$, $F_L=-1$.

Another interpretation

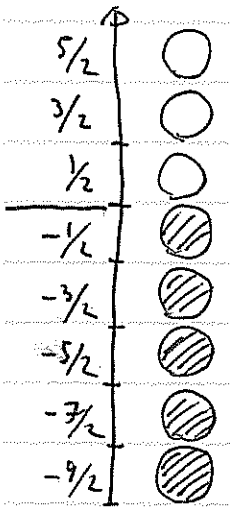
[ignore the left movers for simplicity]

$\forall r$ (positive or negative), $\overline{\Psi}_{-r}$ creates a particle $\underline{\underline{e_r}}$

of energy r, momentum $-r$, F_R-charge $+1$, (F_L-charge 0).

$\overline{\Psi}_{-r}^2 = 0 \iff e_r$ is a fermion (subject to Pauli's exclusion)

$\{\Psi_r, \overline{\Psi}_{-r}\} = 0 \iff \Psi_r$ annihilates e_r



$\overline{\Psi}_r |0\rangle = 0 \ \forall r>0 \iff |0\rangle$ is filled with e_{-r} $\forall r>0$

$\Psi_r |0\rangle = 0 \ \forall r>0 \iff |0\rangle$ is not filled with e_r $\forall r>0$

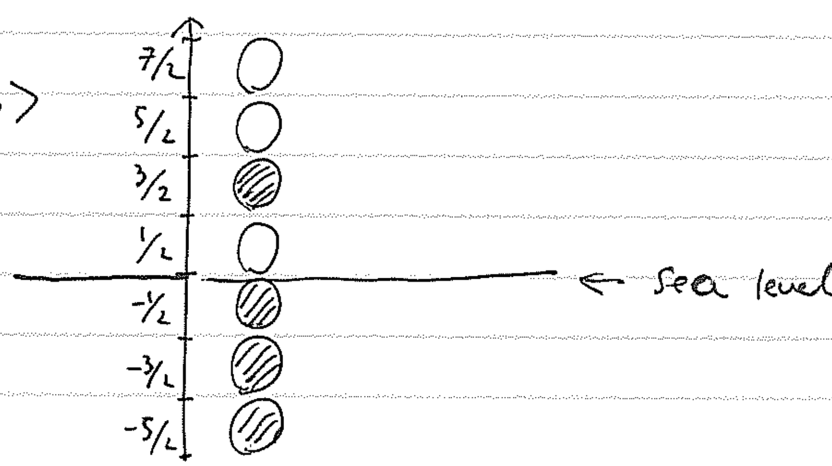
i.e. $|0\rangle$ is filled with all negative energy particles and is empty in all positive energy particles.

Dirac's Sea

$r>0$

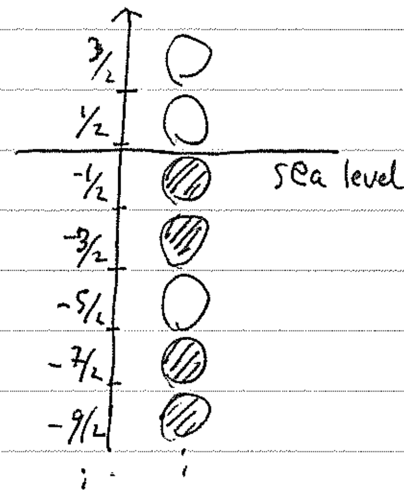
$\overline{\Psi}_{-r}|0\rangle$: the state in which e_r is filled.

e.g. $\overline{\Psi}_{-\frac{3}{2}}|0\rangle$



$r>0$ $\quad \psi_{-r}|0\rangle$: the state in which e_{-r} is emptied

e.g. $\psi_{-5/2}|0\rangle$



i.e. ψ_{-r} makes a <u>hole</u> at e_{-r}.

Twisted Boundary Conditions

Consider the system with action

$$S = \frac{1}{2\pi}\int dt\,d\sigma \left\{ i\bar{\psi}_-(\partial_t + \partial_\sigma + ia)\psi_- + i\bar{\psi}_+(\partial_t - \partial_\sigma - i\tilde{a})\psi_+ \right\}$$

for some real #'s $a, \tilde{a}$.

Two approaches

① : $S = S_{a=\tilde{a}=0} + \underbrace{\Delta S}_{\text{look at the effect of this term}}$

② : Change of variables $\psi_-(t,\sigma) \to e^{-ia\sigma}\psi_-(t,\sigma)$

$\psi_+(t,\sigma) \to e^{-i\tilde{a}\sigma}\psi_+(t,\sigma)$

(undo the $a, \tilde{a}$ terms)

② is the main focus ($\Longrightarrow$ twisted boundary condition)

But let us take ① for the moment.

(let's forget about left movers $\psi_+, \bar{\psi}_+$ for the moment)

$$\Delta S = -\frac{1}{2\pi}\int dt\,d\sigma\, \bar{\psi}_- a \psi_- \iff \Delta H = aF_R$$

$$H = H_0 + \Delta H = \left[\sum_{r \in \mathbb{Z}+\frac{1}{2}} \left(r :\bar{\psi}_{-r}\psi_r: + \cancel{r :\tilde{\bar{\psi}}_{-r}\tilde{\psi}_r:} \right) - \frac{1}{12} \right] + a \overbrace{\sum_r :\bar{\psi}_{-r}\psi_r:}^{F_R}$$

forget

$|0\rangle$ has $E_0 = -\frac{1}{12}$, $F_R = 0$ $\quad \therefore E = -\frac{1}{12}$

$\psi_{-\frac{1}{2}}|0\rangle$ has $E_0 = -\frac{1}{12} + \frac{1}{2}$, $F_R = -1$, $\quad \therefore E = -\frac{1}{12} + \frac{1}{2} - a$

$\bar{\psi}_{-\frac{1}{2}}|0\rangle$ has $E_0 = -\frac{1}{12} + \frac{1}{2}$, $F_R = +1$, $\quad \therefore E = -\frac{1}{12} + \frac{1}{2} + a$

$\psi_{-\frac{1}{2}}\psi_{-\frac{3}{2}}|0\rangle$: $E_0 = -\frac{1}{12} + \frac{1}{2} + \frac{3}{2}$, $F_R = -2$ $\quad \therefore E = -\frac{1}{12} + \frac{1}{2} + \frac{3}{2} - 2a$

$\bar{\psi}_{-\frac{1}{2}}\bar{\psi}_{-\frac{3}{2}}|0\rangle$: $E_0 = -\frac{1}{12} + \frac{1}{2} + \frac{3}{2}$, $F_R = +2$ $\quad \therefore E = -\frac{1}{12} + \frac{1}{2} + \frac{3}{2} + 2a$

$\vdots$

<u>If $-\frac{1}{2} < a < \frac{1}{2}$</u> $\quad |0\rangle$ has the lowest energy (it is the ground state).

<u>If $a = \frac{1}{2}$</u> $\quad E_{\psi_{-\frac{1}{2}}|0\rangle} = -\frac{1}{12} + \frac{1}{2} - \frac{1}{2} = -\frac{1}{12} = E_{|0\rangle}$

$|0\rangle$ & $\psi_{-\frac{1}{2}}|0\rangle$ have the same (lowest) energy.

--- Two ground states.

<u>If $a = -\frac{1}{2}$</u> $\quad E_{\bar{\psi}_{-\frac{1}{2}}|0\rangle} = -\frac{1}{12} + \frac{1}{2} - \frac{1}{2} = -\frac{1}{12} = E_{|0\rangle}$

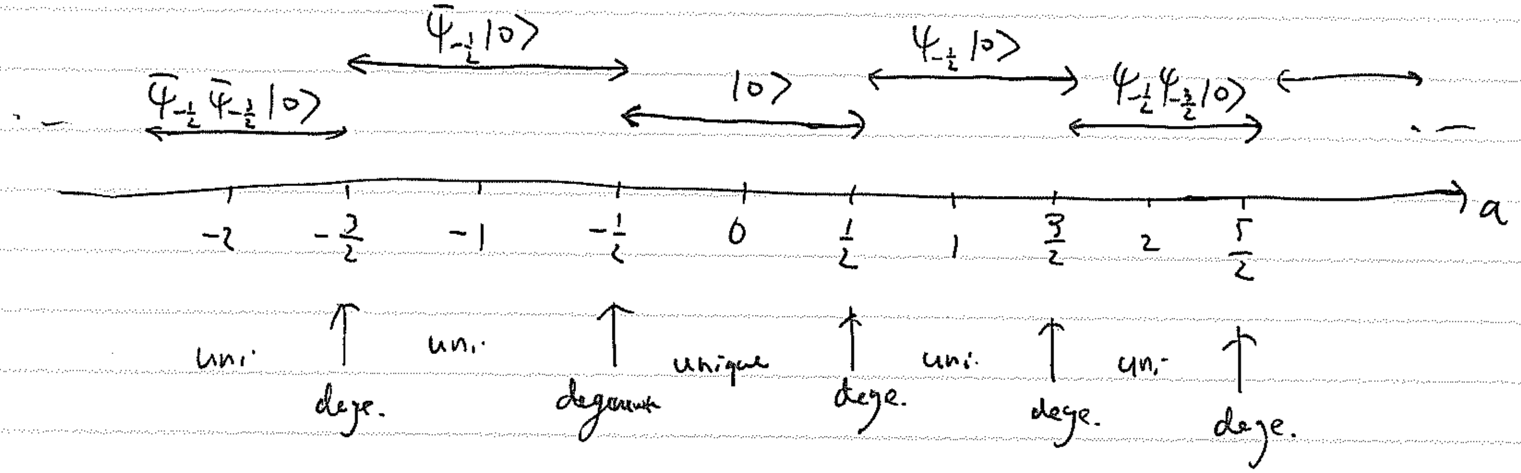
$|0\rangle$, $\bar{\psi}_{-\frac{1}{2}}|0\rangle$ are the ground states.

<u>If $\frac{1}{2} < a < \frac{3}{2}$</u> $\quad \psi_{-\frac{1}{2}}|0\rangle$ is the unique ground state

<u>If $a = \frac{3}{2}$</u> $\quad \psi_{-\frac{1}{2}}|0\rangle$, $\psi_{-\frac{1}{2}}\psi_{-\frac{3}{2}}|0\rangle$ has both $E = -\frac{1}{12} + \frac{1}{2} - \frac{3}{2}$

$= -\frac{1}{12} + \frac{1}{2} + \frac{3}{2} - 2 \cdot \frac{3}{2}$.

Thus, the ground states are



Approach ②

The system is equivalent to

$$S = \frac{1}{2\pi}\int dt\,d\sigma \left\{ i\overline{\psi}_-(\partial_t+\partial_\sigma)\psi_- + i\overline{\psi}_+(\partial_t-\partial_\sigma)\psi_+ \right\}$$

with twisted B.C.
$$\begin{cases} \psi_-(t,\sigma+2\pi) = -e^{2\pi i a}\,\psi_-(t,\sigma) \\ \psi_+(t,\sigma+2\pi) = -e^{2\pi i \tilde{a}}\,\psi_+(t,\sigma) \end{cases}$$

Put $\psi_-(t,\sigma) = e^{ia\sigma}\psi_-'(t,\sigma)$ $\qquad$ $\psi_-'(t,\sigma+2\pi) = -\psi_-'(t,\sigma)$

$\psi_+(t,\sigma) = e^{i\tilde{a}\sigma}\psi_+'(t,\sigma)$ $\qquad$ $\psi_+'(t,\sigma+2\pi) = -\psi_+'(t,\sigma)$ $\Big\}$ antiperiodic B.C.

Then
$$S = \frac{1}{2\pi}\int dt\,d\sigma\left\{ i\overline{\psi}_-'(\partial_t+\partial_\sigma+ia)\psi_-' + i\overline{\psi}_+'(\partial_t-\partial_\sigma-i\tilde{a})\psi_+' \right\}$$

Note: $a \to a+1$ or $\tilde{a}\to\tilde{a}+1$ does not change the twisted B.C.

Theory must be periodic under $a\to a+1$ or $\tilde{a}\to\tilde{a}+1$

Mode expansion

$$\psi_-(t,\sigma) = \sum_{r\in\mathbb{Z}+a+\frac{1}{2}} \psi_r(t)\,e^{ir\sigma}, \quad \overline{\psi}_-(t,\sigma) = \sum_{r\in\mathbb{Z}-a+\frac{1}{2}} \overline{\psi}_r(t)\,e^{ir\sigma} \qquad \left(\overline{\psi}_r = \psi_{-r}^\dagger\right)$$

$$\psi_+(t,\sigma) = \sum_{r\in\mathbb{Z}-a+\frac{1}{2}} \tilde{\psi}_r(t)\,e^{-ir\sigma}, \quad \overline{\psi}_+(t,\sigma) = \sum_{r\in\mathbb{Z}+a+\frac{1}{2}} \overline{\tilde{\psi}}_r(t)\,e^{-ir\sigma} \qquad \left(\overline{\tilde{\psi}}_r = \tilde{\psi}_{-r}^\dagger\right)$$

[relation to $\psi'_r, \bar{\psi}'_r, \cdots$ for the antiperiodic fields $\psi_-, \bar{\psi}_-, \cdots$:

$$\psi_{r+a} = \psi'_r \quad (r \in \mathbb{Z}+\tfrac{1}{2}), \qquad \bar{\psi}_{r-a} = \bar{\psi}'_r \quad (r \in \mathbb{Z}+\tfrac{1}{2})$$

For now, focus on the right movers :

$$H_R = \sum_{r \in \mathbb{Z}+a+\frac{1}{2}} r \cdot \frac{\bar{\psi}_{-r}\psi_r - \psi_r \bar{\psi}_{-r}}{2}$$

$$= \sum_{r>0} \left(r\bar{\psi}_{-r}\psi_r - \frac{r}{2} \right) + \sum_{r<0} \left(-r\psi_r\bar{\psi}_{-r} + \frac{r}{2} \right)$$

A ground state $|\Omega\rangle_a$ must obey

$$\boxed{\psi_r |\Omega\rangle_a = 0, \quad \bar{\psi}_r |\Omega\rangle_a = 0 \qquad \forall r > 0}$$

Let us denote by $|0\rangle_a$ the state obeying

$$\boxed{\psi_{r+a} |0\rangle_a = 0, \quad \bar{\psi}_{r-a} |0\rangle_a = 0 \qquad \forall r \geq \tfrac{1}{2}}$$

For $-\frac{1}{2} < a < \frac{1}{2}$, the two conditions are the same.

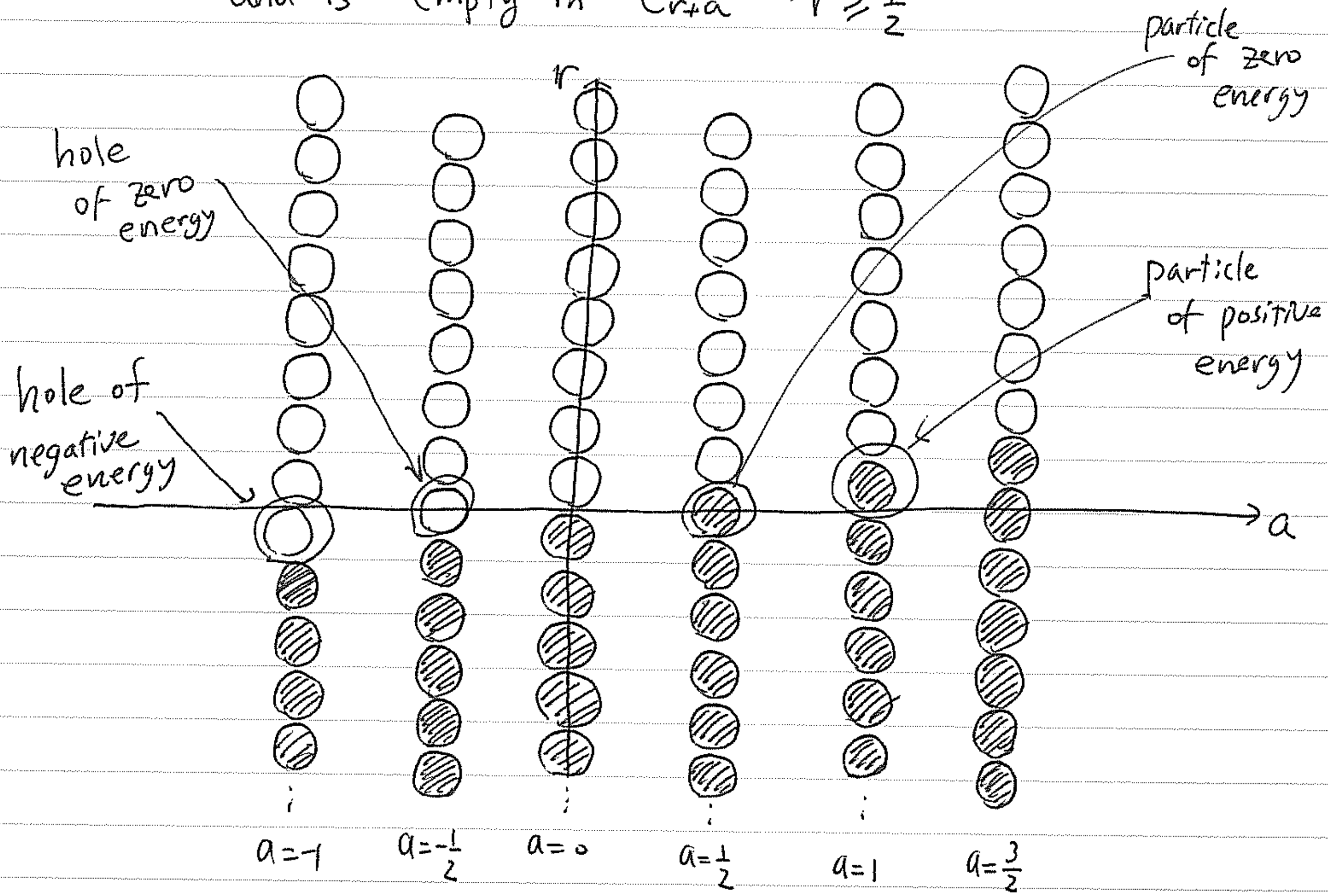
$\therefore |\Omega\rangle_a = |0\rangle_a$

For other values of a, the two conditions are different.

($|0\rangle_a \longleftrightarrow |0\rangle$ in the approach 1)

$|0\rangle_a$ is the state filled by the particle e_{r+a} $\forall r \leq -\frac{1}{2}$

and is empty in e_{r+a} $\forall r \geq \frac{1}{2}$



need to fill the hole at $e_{a+\frac{1}{2}}$ of negative energy

$\therefore |\Omega\rangle_a = \overline{\psi}_{-a-\frac{1}{2}} |0\rangle_a$

$|\Omega\rangle_a = |0\rangle_a$

need to remove the particle $e_{a-\frac{1}{2}}$ of positive energy to obtain the ground state

$\therefore |\Omega\rangle_a = \psi_{a-\frac{1}{2}} |0\rangle_a$

This picture is called <u>the spectral flow</u>

$\Rightarrow$

$-\frac{1}{2} \leq a \leq \frac{1}{2}$ $\quad |\Omega\rangle_a = |0\rangle_a$ $\qquad$ $-\frac{3}{2} \leq a \leq -\frac{1}{2}$ $\quad |\Omega\rangle_a = \bar{\psi}_{-a-\frac{1}{2}}|0\rangle_a$

$\frac{1}{2} \leq a \leq \frac{3}{2}$ $\quad |\Omega\rangle_a = \psi_{a-\frac{1}{2}}|0\rangle_a$ $\qquad$ $-\frac{5}{2} \leq a \leq -\frac{3}{2}$ $\quad |\Omega\rangle_a = \bar{\psi}_{-a-\frac{1}{2}}\bar{\psi}_{-a-\frac{3}{2}}|0\rangle_a$

$\frac{3}{2} \leq a \leq \frac{5}{2}$ $\quad |\Omega\rangle_a = \psi_{a-\frac{1}{2}}\psi_{a-\frac{3}{2}}|0\rangle_a$ $\qquad$ $\vdots$

$\vdots$

What is the energy and the charge of the states $|0\rangle_a$, $|\Omega\rangle_a$.

The state $|0\rangle_a$

look at the spectral flow.

at $a=0$: $H_R = -\frac{1}{24}$, $F_R = 0$

at $a=\pm 1$: $H_R = -\frac{1}{24} + \frac{1}{2}$, $F_R = \pm 1$

at $a=\pm 2$: $H_R = -\frac{1}{24} + \frac{1}{2} + \frac{3}{2}$, $F_R = \pm 2$

$\vdots$

$a = \pm n$: $H_R = -\frac{1}{24} + \frac{1}{2} + \frac{3}{2} + \cdots + \left(n - \frac{1}{2}\right) = -\frac{1}{24} + \frac{n}{2} + \frac{n(n-1)}{2} = -\frac{1}{24} + \frac{n^2}{2}$

$F_R = \pm n$.

extrapolate $\rightsquigarrow$

$$\boxed{\begin{aligned} H_R &= -\frac{1}{24} + \frac{a^2}{2} \\ F_R &= a \end{aligned}}$$

on $|0\rangle_a$

$$\boxed{\text{Ground state(s) } |\Omega\rangle_a}$$

$-\frac{1}{2} < a < \frac{1}{2}$ $\quad |\Omega\rangle_a = |0\rangle_a \quad H_R = -\frac{1}{24} + \frac{a^2}{2}$, $F_R = a$

$\frac{1}{2} < a < \frac{3}{2}$ $\quad |\Omega\rangle_a = \psi_{a-\frac{1}{2}}|0\rangle_a \quad H_R = -\frac{1}{24} + \frac{a^2}{2} + (a - \frac{1}{2}) = -\frac{1}{24} + \frac{(a-1)^2}{2}$

$F_R = a - 1$

$-\frac{3}{2} < a < -\frac{1}{2}$ $\quad |\Omega\rangle_a = \bar{\psi}_{-a-\frac{1}{2}}|0\rangle_a \quad H_R = -\frac{1}{24} + \frac{a^2}{2} + (a + \frac{1}{2}) = -\frac{1}{24} + \frac{(a+1)^2}{2}$

$F_R = a + 1$

$\vdots$

$\therefore |\Omega\rangle_a$ has
$$\boxed{\begin{aligned} H_R &= -\frac{1}{24} + \frac{1}{2}(a - n_a)^2 \\ F_R &= a - n_a \end{aligned}}$$

where n_a is the closest integer to a.

- unique if $a \notin \mathbb{Z} + \frac{1}{2}$
- ambiguous if $a \notin \mathbb{Z} + \frac{1}{2}$

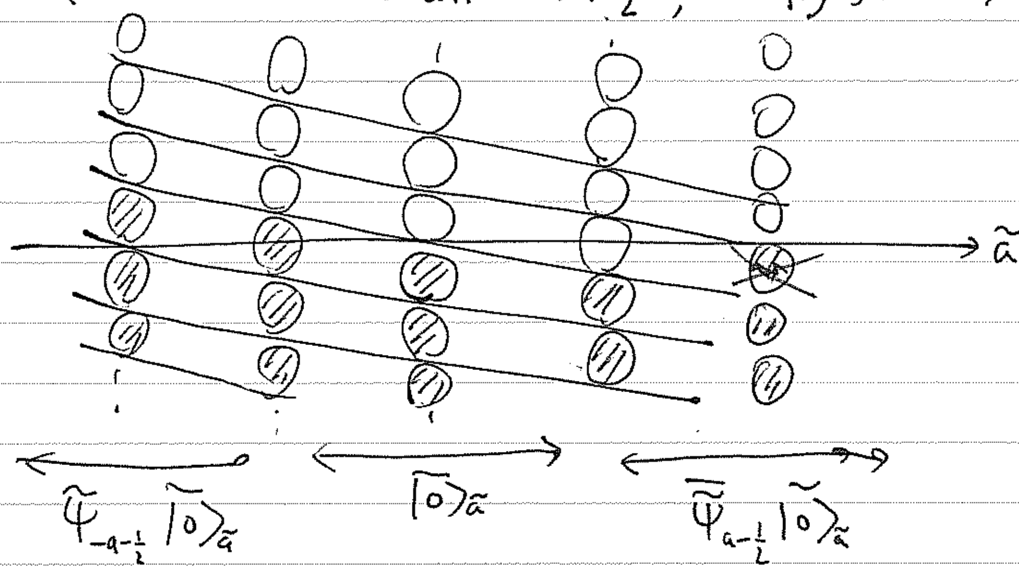
$\Updownarrow$

Two-fold degeneracy of the ground state
($(a - n_a)^2$ is unique)

Left moving sector

$|\tilde{0}\rangle_{\tilde{a}}$ ... annihilated by $\tilde{\psi}_{-a+r}$, $\bar{\tilde{\psi}}_{a+r}$ $\forall r \geq \frac{1}{2}$

(filled with e_{-a+r} $\forall r \leq -\frac{1}{2}$, empty for $\forall r \geq \frac{1}{2}$)



$|\tilde{0}\rangle_{\tilde{a}}$ has $H_L = -\frac{1}{24} + \frac{\tilde{a}^2}{2}$, $F_L = -\tilde{a}$

$|\tilde{\Omega}\rangle_{\tilde{a}}$ has $H_L = -\frac{1}{24} + \frac{(\tilde{a} - n_{\tilde{a}})^2}{2}$, $F_L = -\tilde{a} + n_{\tilde{a}}$

($n_{\tilde{a}}$ closest integer to $\tilde{a}$).