Partition function

$$q^{H_R}\bar{q}^{H_L} \qquad \left(q = e^{2\pi i\tau} = e^{-2\pi\tau_2 + 2\pi i\tau_1} \right)$$

$$Z = \mathrm{Tr}_{\mathcal{H}_{a,\tilde{a}}}\left(e^{2\pi i(bF_R + \tilde{b}F_L)}\, \underbrace{e^{-2\pi i\tau_1 P} \cdot e^{-2\pi\tau_2 H}}_{q^{H_R}\bar{q}^{H_L}}\right)$$

$\mathcal{H} = \mathcal{H}^R \otimes \mathcal{H}^L$

$$= \mathrm{Tr}_{\mathcal{H}^R_a}\left(e^{2\pi i b F_R} q^{H_R}\right) \cdot \mathrm{Tr}_{\mathcal{H}^L_{\tilde{a}}}\left(e^{2\pi i\tilde{b}F_L}\bar{q}^{H_L}\right)$$

$$\mathcal{H}^R_a \ni |0\rangle_a \text{ multiplied by } \begin{cases} \psi_{a+r} & r \leq -\frac{1}{2} \\ \bar{\psi}_{-a+r} & r \leq -\frac{1}{2} \end{cases}$$

$$|0\rangle_a \leftrightarrow \begin{cases} e_{r+a} & r \geq \frac{1}{2} \text{ empty} \\ e_{r+a} & r \leq -\frac{1}{2} \text{ filled} \end{cases} \qquad \left(\begin{array}{l} H_R = -\frac{1}{24} + \frac{a^2}{2} \\ F_R = a \end{array} \right)$$

$$Z^R_{a,b}(\tau) = \mathrm{Tr}_{\mathcal{H}^R_a}\left(e^{2\pi i b F_R} q^{H_R}\right)$$

$$= e^{2\pi i b a} q^{-\frac{1}{24}+\frac{a^2}{2}} \prod_{r\geq\frac{1}{2}}\Big(1 + \overbrace{e^{2\pi i b} q^{r+a}}^{\text{creating a particle}}\Big) \prod_{r\leq-\frac{1}{2}}\Big(1 + \overbrace{e^{-2\pi i b} q^{-r-a}}^{\text{making a hole}}\Big)$$

$$= e^{2\pi i b a} q^{-\frac{1}{24}+\frac{a^2}{2}} \prod_{n=1}^{\infty}\left(1 + e^{2\pi i b} q^{n+a-\frac{1}{2}}\right)\left(1 + e^{-2\pi i b} q^{n-a-\frac{1}{2}}\right)$$

$$= q^{\frac{a^2}{2}} e^{2\pi i b a} \frac{\prod_{n=1}^{\infty}(1-q^n)(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}})}{q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)}$$

$$z = e^{2\pi i b} q^a = e^{2\pi i(b+\tau a)}$$

☆ $q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = \eta(\tau)$ Dedekind's eta

☆ $\prod_{n=1}^{\infty} (1-q^n)(1+z q^{n-\frac{1}{2}})(1+z^{-1} q^{n-\frac{1}{2}})$

$(z = e^{2\pi i v})$

$= \vartheta\left[{0 \atop 0}\right](v,\tau)$ ← Jacobi's Theta function

$:= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i v n}$ (or $\vartheta_3(v,\tau)$)

a (non-trivial) identity.

$$\therefore\ Z^R_{a,b}(\tau) = q^{\frac{a^2}{2}} e^{2\pi i b a} \frac{\vartheta\left[{0 \atop 0}\right](b+\tau a, \tau)}{\eta(\tau)}$$

$$= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2} e^{2\pi i b(n+a)}$$

$$= \frac{1}{\eta(\tau)} \vartheta\left[{a \atop b}\right](0,\tau)$$

$$\left(\vartheta\left[{a \atop b}\right](v,\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2} e^{2\pi i (v+b)(n+a)} \right)$$

<u>Note</u> $Z^R_{a,b}(\tau) = \underset{\substack{\uparrow\\ \text{inv. under}\\ a\to a+1}}{Z^R_{a+1,b}(\tau)} = \underbrace{e^{-2\pi i a}}_{\substack{\uparrow\\ \text{not invariant}\\ \text{under } b\to b+1}} Z_{a,b+1}(\tau) = Z_{-a,-b}(\tau)$

Left moving sector:

$$Z^L_{\tilde{a},\tilde{b}}(\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}^L_{\tilde{a}}}\left(e^{2\pi i \tilde{b} F_L} \bar{q}^{H_L}\right)$$

$\mathcal{H}^L_{\tilde{a}} \ni \underbrace{|\tilde{0}\rangle_{\tilde{a}}}$ multiplied by $\left.\begin{matrix}\tilde{\psi}_{-a+r} \\ \bar{\tilde{\psi}}_{+a+r}\end{matrix}\right\} r \leq -\frac{1}{2}$

filled with $\tilde{e}_{-\tilde{a}+r}$ $\quad r \leq -\frac{1}{2}$

empty in $\tilde{e}_{-\tilde{a}+r}$ $\quad r \geq \frac{1}{2}$

$H_L = -\frac{1}{24} + \frac{\tilde{a}^2}{2}$, $\quad F_L = -\tilde{a}$

$$= e^{-2\pi i \tilde{b}\tilde{a}} \bar{q}^{-\frac{1}{24}+\frac{\tilde{a}^2}{2}} \prod_{r \geq \frac{1}{2}} (\underset{\text{emp.}}{1} + \underset{\text{filled}}{e^{2\pi i \tilde{b}} \bar{q}^{r-\tilde{a}}}) \cdot \prod_{r \leq -\frac{1}{2}} (\underset{\text{filled}}{1} + \underset{\text{emp.}}{e^{-2\pi i \tilde{b}} \bar{q}^{-r+\tilde{a}}})$$

$$= \bar{q}^{-\frac{1}{24}+\frac{\tilde{a}^2}{2}} e^{-2\pi i \tilde{b}\tilde{a}} \prod_{n=1}^{\infty} (1 + e^{2\pi i \tilde{b}} \bar{q}^{n-\tilde{a}-\frac{1}{2}})(1 + e^{-2\pi i \tilde{b}} \bar{q}^{n+\tilde{a}-\frac{1}{2}})$$

$$= \frac{\vartheta\left[\begin{smallmatrix}\tilde{a}\\ -\tilde{b}\end{smallmatrix}\right](0,\bar{\tau})}{\eta(\bar{\tau})} = \left(\frac{\vartheta\left[\begin{smallmatrix}\tilde{a}\\ \tilde{b}\end{smallmatrix}\right](0,\tau)}{\eta(\tau)}\right)^*$$

Total

$$\therefore Z = Z^R_{a,b}(\tau)\, Z^L_{\tilde{a},\tilde{b}}(\bar{\tau}) = \frac{\vartheta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](0,\tau) \cdot \overline{\vartheta\left[\begin{smallmatrix}\tilde{a}\\ \tilde{b}\end{smallmatrix}\right](0,\tau)}}{|\eta(\tau)|^2}$$

Path-integral interpretation of $Z = \mathrm{Tr}_{\mathcal{H}}\Big(\underbrace{e^{2\pi i(bF_R + \tilde{b}F_L)}\, e^{-2\pi i \tau_1 P}}_{U_g}\, e^{-\overset{\beta}{\overset{\parallel}{2\pi\tau_2}} H}$

$$\cancel{\phi}(\sigma,\tau) = \pm g\, \phi(\sigma, \tau+\beta) = \pm U_g^{-1} \phi(\sigma, \tau+\beta) U_g$$

$$\psi_-(\sigma,\tau) = -e^{-2\pi i a}\psi_-(\sigma+2\pi, \tau) = -e^{2\pi i b}\psi_-(\sigma+2\pi\tau_1, \tau+2\pi\tau_2)$$

$$\psi_+(\sigma,\tau) = -e^{-2\pi i \tilde{a}}\psi_+(\sigma+2\pi, \tau) = -e^{2\pi i \tilde{b}}\psi_+(\sigma+2\pi\tau_1, \tau+2\pi\tau_2)$$

$$Z = \int \mathcal{D}\psi_\pm \mathcal{D}\bar{\psi}_\pm\, e^{-S_E(\psi_\pm, \bar{\psi}_\pm)}$$

above boundary condition.

Put $\cdot$ $\psi_-(\sigma,\tau) = e^{i\alpha\sigma + i\beta\tau}\psi'_-(\sigma,\tau)$

$$\psi'_-(\sigma,\tau) = -\psi'_-(\sigma+2\pi,\tau) = -\psi'_-(\sigma+2\pi\tau_1, \tau+2\pi\tau_2)$$ antiperiodic in both

$$\Longleftrightarrow \quad \alpha = a, \quad \beta = -\frac{b+\tau_1 a}{\tau_2}$$

$\cdot$ $\psi_+(\sigma,\tau) = e^{i\tilde{\alpha}\sigma + i\tilde{\beta}\tau}\psi'_+(\sigma,\tau)$

ψ'_+ antiperiodic in both

$$\Longleftrightarrow \quad \tilde{\alpha} = \tilde{a}, \quad \tilde{\beta} = -\frac{\tilde{b}+\tau_1 \tilde{a}}{\tau_2}$$

$$S_E = -iS\Big|_{t\to -i\tau} = -\frac{i}{2\pi}\int d\sigma(-id\tau)\Big[\, i\bar\psi_-(i\partial_\tau+\partial_\sigma)\overbrace{\psi_-}^{\overset{e^{i\alpha\sigma+i\beta\tau}\psi_-'}{\parallel}} + i\bar\psi_+(i\partial_\tau-\partial_\sigma)\overbrace{\psi_+}^{\overset{e^{i\tilde\alpha\sigma+i\tilde\beta\tau}\psi_+'}{\parallel}}\Big]$$

$$= \frac{1}{2\pi}\int d\sigma d\tau\Big[\,\bar\psi_-'(\partial_\tau - i\partial_\sigma \underbrace{+i\beta+\alpha}_{-i\frac{b+\tau a}{\tau_2}})\psi_-' + \bar\psi_+'(\partial_\tau+i\partial_\sigma \underbrace{+i\tilde\beta-\tilde\alpha}_{-i\frac{\tilde b+\bar\tau\tilde a}{\tau_2}})\psi_+'\Big]$$

$$= \frac{1}{2\pi}\int d\sigma d\tau\Big[\,\bar\psi_-'(D_\tau - iD_\sigma)\psi_-' + \bar\psi_+'(\tilde D_\tau + i\tilde D_\sigma)\psi_+'\Big]$$

$$D_\mu = \partial_\mu + iA_\mu \qquad \tilde D_\mu = \partial_\mu + i\tilde A_\mu$$

$$A_\tau - iA_\sigma = -\frac{b+\tau a}{\tau_2} \qquad \tilde A_\tau + i\tilde A_\sigma = -\frac{\tilde b+\bar\tau\tilde a}{\tau_2}$$

$$\boxed{\text{Fermion Coupled to gauge fields } A_\mu \text{ (for Right)},\ \tilde A_\mu \text{ (for Left)}}$$

$$A_\mu = \tilde A_\mu \iff \tilde b = b,\ \tilde a = a$$

In that case,

$$Z = \left|\frac{\vartheta\left[{a \atop b}\right](0,\tau)}{\eta(\tau)}\right|^2$$

It is invariant under $b\to b+1$ (of course also under $a\to a+1$)

Modular invariance?

Using modular transformation property of $\eta(\tau)$ & $\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](v,\tau)$, we find

$$\cdot\ Z^R_{a,b}(\tau+1) = Z^R_{a,b+a\pm\frac{1}{2}}(\tau)\cdot e^{\pi i(-a^2\mp a-\frac{1}{2})}$$

$$\cdot\ Z^R_{a,b}(-1/\tau) = Z^R_{b,-a}(\tau)\cdot e^{2\pi i ab}$$

$$\therefore\ \left|Z^R_{a,b}(\tau+1)\right|^2 = \left|Z^R_{a,b+a\pm\frac{1}{2}}(\tau)\right|^2 \text{ but } \neq \left|Z^R_{a,b}(\tau)\right|^2 \quad \text{---}(1)$$

$$\left|Z^R_{a,b}(-1/\tau)\right|^2 = \left|Z^R_{b,-a}(\tau)\right|^2 \text{ but } \neq \left|Z^R_{a,b}(\tau)\right|^2 \quad \text{---}(2)$$

Can we understand this?

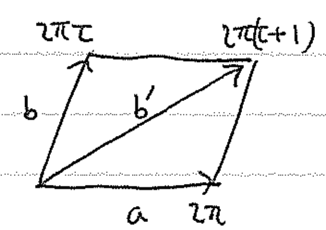
— Yes! (write t = Euclidean time τ to avoid confusion)

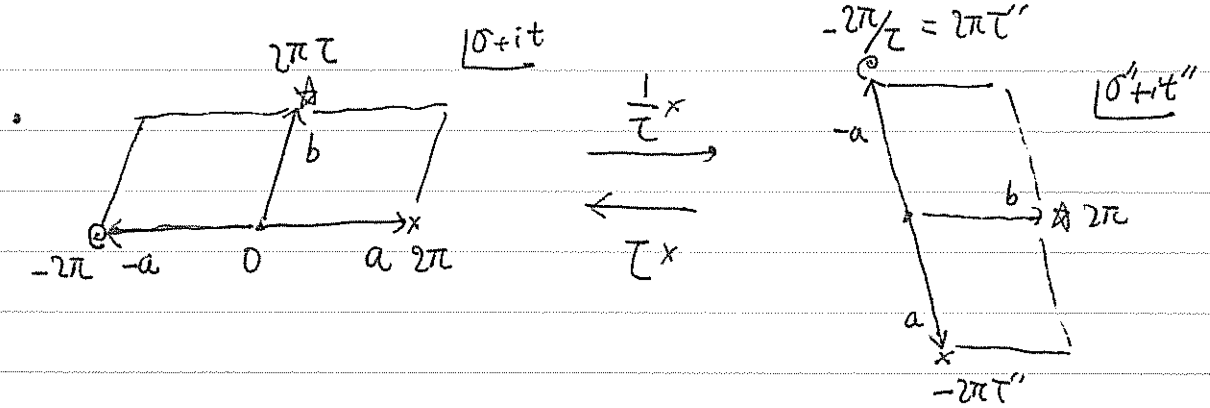
- $\psi_-(\sigma,t) = -e^{-2\pi i a}\psi_-(\sigma+2\pi,t) = -e^{2\pi i b}\psi_-(\sigma+2\pi\tau_1, t+2\pi\tau_2)$

$$= -\underbrace{e^{2\pi i b}\left(-e^{-2\pi i a}\right.}_{e^{2\pi i(b-a\pm\frac{1}{2})}}\left.\psi_-(\underbrace{\sigma+2\pi\tau_1+2\pi}_{2\pi(\tau_1+1)}, t+2\pi\tau_2)\right)$$

$\therefore$ For $\tau' = \tau+1$, we have $\boxed{a' = a,\ b' = b-a\pm\frac{1}{2}.}$

(1) Says $\left|Z^R_{a',b'}(\tau')\right|^2 = \left|Z^R_{a,b}(\tau)\right|^2$





$$\psi_-(\sigma'',t'') \overset{\times}{=\!=} -e^{-2\pi i a}\,\psi_-(\sigma''-2\pi\tau_1'',\ t''-2\pi\tau_2'')$$

$$\overset{\star}{=\!=} -e^{2\pi i b}\,\psi_-(\sigma''+2\pi,\ t'') \qquad \| \quad -e^{2\pi i a}\,\psi_-(\sigma''+2\pi\tau_1'',\ t''+2\pi\tau_2'')$$

$\therefore$ For $\tau'' = -1/\tau$, we have $\boxed{a'' = -b\ ,\ b'' = a}$

(2) says $\left|Z^R_{a'',b''}(\tau'')\right|^2 = \left|Z^R_{a,b}(\tau)\right|^2$

Thus $Z = \left|\dfrac{\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0,\tau)}{\eta(\tau)}\right|^2$ transforms

under modular transformation as it "should".

But it is <u>not</u> invariant. (for a <u>fixed</u> a, b)