Correlation functions

Op

$$\psi_-(t,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \underbrace{\psi_r(t)}_{=\, e^{iHt}\psi_r e^{-iHt} = \psi_r e^{-irt}} e^{ir\sigma} = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r e^{-ir(t-\sigma)}$$

$$\bar\psi_-(t,\sigma) = \sum_r \bar\psi_r(t) e^{ir\sigma} = \sum_r \bar\psi_r e^{-ir(t-\sigma)}$$

$$\tilde\psi_+(t,\sigma) = \sum_r \tilde\psi_r(t) e^{-ir\sigma} = \sum_r \tilde\psi_r e^{-ir(t+\sigma)}$$

$$\bar\psi_+(t,\sigma) = \sum_r \bar{\tilde\psi}_r(t) e^{-ir\sigma} = \sum_r \bar{\tilde\psi}_r e^{-ir(t+\sigma)}$$

$$\langle 0|T(\psi_-(1)\bar\psi_-(2))|0\rangle$$

$$\overset{t_1>t_2}{=} \sum_{r_1 r_2} e^{-ir_1(t_1-\sigma_1)} e^{-ir_2(t_2-\sigma_2)} \langle 0|\psi_{r_1}\bar\psi_{r_2}|0\rangle \qquad \begin{cases} \delta_{r_1+r_2,0} & r_2<0 \\ 0 & r_2>0 \end{cases}$$

$$= \sum_{r_1>\frac{1}{2}} e^{-ir_1(t_1-\sigma_1)+ir_1(t_2-\sigma_2)} = \sum_{n=0}^{\infty} e^{-i(n+\frac{1}{2})(t-\sigma)} \qquad \begin{cases} t = t_1-t_2 \\ \sigma = \sigma_1-\sigma_2 \end{cases}$$

<u>Note</u> $t = e^{-i\epsilon}|t| \quad \therefore \left|e^{-i(t-\sigma)}\right| < 1$

$$= \frac{e^{-\frac{i}{2}(t-\sigma)}}{1-e^{-i(t-\sigma)}} = \frac{e^{i\frac{t_1-\sigma_1+t_2-\sigma_2}{2}}}{e^{i(t_1-\sigma_1)} - e^{i(t_2-\sigma_2)}} = \frac{(z_1 z_2)^{\frac{1}{2}}}{z_1 - z_2}$$

$$\left(z_i := e^{i(t_i-\sigma_i)} \right)$$

$$\underset{t_2>t_1}{=} \sum_{r_1 r_2} e^{-ir_1(t_1-\sigma_1)} e^{-ir_2(t_2-\sigma_2)} \langle 0 | (-\overline{\psi}_{r_2} \psi_{r_1}) | 0 \rangle \quad \begin{cases} -\delta_{r_1+r_2,0} & r_1<0 \\ 0 & r_1>0 \end{cases}$$

$$= -\sum_{r_2>0} e^{ir_2(t_1-\sigma_1)-ir_2(t_2-\sigma_2)}$$

$$= -\frac{(z_1 z_2)^{\frac{1}{2}}}{z_2 - z_1} = \frac{(z_1 z_2)^{\frac{1}{2}}}{z_1 - z_2}$$

$$\therefore \langle 0 | T(\psi_-(1) \overline{\psi}_-(2)) | 0 \rangle = \frac{(z_1 z_2)^{\frac{1}{2}}}{z_1 - z_2}$$

similarly

$$\langle 0 | T(\psi_+(1) \overline{\psi}_+(1)) | 0 \rangle = \frac{(\tilde{z}_1 \tilde{z}_2)^{\frac{1}{2}}}{\tilde{z}_1 - \tilde{z}_2}$$

$$\langle 0 | T(\psi_{\mp}(1) \psi_{\mp}(2)) | 0 \rangle = \langle 0 | T(\overline{\psi}_{\mp}(1) \overline{\psi}_{\mp}(2)) | 0 \rangle$$

$$= \langle 0 | T(\overset{(-)}{\psi}_-(1) \overset{(-)}{\psi}_+(2)) | 0 \rangle = 0$$

P.I. $$S = \frac{1}{2\pi}\int \left(i\bar{\psi}_-(\partial_t+\partial_\sigma)\psi_- + i\bar{\psi}_+(\partial_t-\partial_\sigma)\psi_+ \right) d^2\sigma$$

$$S_E = -iS\Big|_{t\to -i\tau} = \frac{-i}{2\pi}\int (-i)\,d\sigma\, d\tau \left[i\bar{\psi}_-(i\partial_\tau+\partial_\sigma)\psi_- + i\bar{\psi}_+(i\partial_\tau-\partial_\sigma)\psi_+ \right]$$

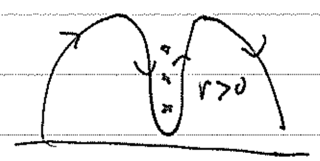
$$= \int d\sigma\, d\tau \left[\bar{\psi}_- \underbrace{\frac{1}{2\pi}(\partial_\tau - i\partial_\sigma)}_{A_-}\psi_- + \bar{\psi}_+ \underbrace{\frac{1}{2\pi}(\partial_\tau + i\partial_\sigma)\psi_+}_{A_+} \right]$$

$$\langle \psi_-(1)\bar{\psi}_-(2)\rangle = A_-^{-1}(1,2) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{2\pi}\,\frac{2\pi}{ik+r}\,e^{ik\overset{\tau_1-\tau_2}{\tau}+ir\overset{\sigma_1-\sigma_2}{\sigma}}$$

$$= \int_{\mathbb{R}}\frac{dk}{2\pi i}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{e^{ik\tau+ir\sigma}}{k-ir}$$

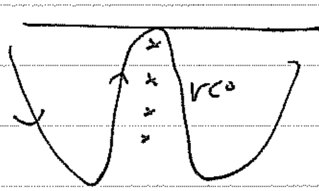
$$z = e^{\tau-i\sigma} = z_1/z_2$$

$\tau>0$



$$= \sum_{r>0} e^{-r\tau+ir\sigma} = \frac{z^{-\frac{1}{2}}}{1-z^{-1}} = \frac{z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}}{z_1-z_2}$$

$\tau<0$



$$= \sum_{r<0}(-1)\,e^{-r\tau+ir\sigma} = -\frac{z^{\frac{1}{2}}}{1-z} = -\frac{z_2^{\frac{1}{2}} z_1^{\frac{1}{2}}}{z_2-z_1}$$

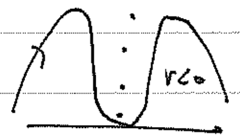
$$\therefore\ \langle \psi_-(1)\bar{\psi}_-(2)\rangle = \frac{z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}}{z_1-z_2}$$

$$\langle \psi_+(1)\bar\psi_+(2)\rangle = A_+^{-1}(1,2) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{r\in\mathbb{Z}+\frac12}\frac{1}{2\pi}\frac{2\pi}{ik-r}\, e^{ik\overbrace{\tau}^{\tau_1-\tau_2}+ir\overbrace{\sigma}^{\sigma_1-\sigma_2}}$$

$$= \int_{\mathbb{R}}\frac{dk}{2\pi i}\sum_{r\in\mathbb{Z}+\frac12}\frac{e^{ik\tau+ir\sigma}}{k+ir}$$

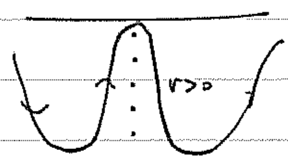
$$\bar z = e^{\tau+i\sigma} = \bar z_1/\bar z_2$$

$\overset{\tau>0}{=}$



$$= \sum_{r<0} e^{r\tau+ir\sigma} = \frac{\bar z^{-\frac12}}{1-\bar z^{-1}} = \frac{(\bar z_1\bar z_2)^{\frac12}}{\bar z_1-\bar z_2}$$

$\overset{\tau<0}{=}$



$$= \sum_{r>0}(-1)\, e^{r\tau+ir\sigma} = -\frac{\bar z^{\frac12}}{1-\bar z} = -\frac{(\bar z_2\bar z_1)^{\frac12}}{\bar z_2-\bar z_1}$$

$$\therefore\ \langle \psi_+(1)\bar\psi_+(2)\rangle = \frac{(\bar z_1\bar z_2)^{\frac12}}{\bar z_1-\bar z_2}$$

$$\langle \psi_-(1)\psi_-(2)\bar\psi_-(3)\bar\psi_-(4)\rangle$$

$$= \psi_-(1)\psi_-(2)\bar\psi_-(3)\bar\psi_-(4) + \psi_-(1)\psi_-(2)\bar\psi_-(3)\bar\psi_-(4)$$

$$= -\frac{(z_1z_3)^{\frac{1}{2}}}{z_1-z_3}\cdot\frac{(z_2z_4)^{\frac{1}{2}}}{z_2-z_4} + \frac{(z_1z_4)^{\frac{1}{2}}}{z_1-z_4}\cdot\frac{(z_2z_3)^{\frac{1}{2}}}{z_2-z_3}$$

$$\langle \psi_-(1)\cdots\psi_-(n)\,\bar\psi_-(1')\cdots\bar\psi_-(n')\rangle$$

$$= \psi_-(1)\cdots\psi_-(n)\,\bar\psi_-(1')\cdots\bar\psi_-(n') + \text{permutation}$$

$$= (-1)^{(n-1)+(n-2)+\cdots+1}\left(\psi_-(1)\bar\psi_-(1')\psi_-(2)\bar\psi_-(2')\cdots\psi_-(n)\bar\psi_-(n') + \text{perm.}\right)$$

$$= (-1)^{\frac{n(n-1)}{2}}\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma)}\frac{(z_1z'_{\sigma(1)})^{\frac{1}{2}}}{z_1-z'_{\sigma(1)}}\cdots\frac{(z_nz'_{\sigma(n)})^{\frac{1}{2}}}{z_n-z'_{\sigma(n)}}$$

$$= (-1)^{\frac{n(n-1)}{2}}(z_1\cdots z_n z'_1\cdots z'_n)^{\frac{1}{2}}\det\left(\frac{1}{z_i-z'_j}\right)_{1\le i,j\le n}$$

$$\underset{\text{identity}}{\overset{\uparrow}{=}} (z_1\cdots z_n z'_1\cdots z'_n)^{\frac{1}{2}}\frac{\prod_{1\le i<j\le n}(z_i-z_j)(z'_i-z'_j)}{\prod_{1\le i,j\le n}(z_i-z'_j)}$$

Similarly for $\langle \psi_+(1)\cdots\psi_+(n)\bar\psi_+(1')\cdots\bar\psi_+(n')\rangle$ $\left[\text{replace } z_i \to \bar z_i,\ z'_i \to \bar z'_i\right]$

Back to Partition function

$$Z_{a,b}(\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}_{a,a}}\, e^{2\pi i (bF_R + bF_L)}\, q^{H_R}\bar q^{H_L} = \left|\frac{\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0,\tau)}{\eta(\tau)}\right|^2$$

We have seen it is invariant under

$$T:\ \tau \to \tau+1 \quad \underline{\text{with}} \quad (a,b) \mapsto (a,\, b-a\pm\tfrac{1}{2})$$

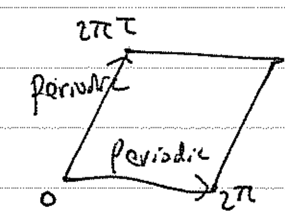
$$S:\ \tau \to -1/\tau \quad \underline{\text{with}} \quad (a,b) \mapsto (-b,\, a)$$

But $Z_{a,b}$ for <u>fixed (a,b)</u> is <u>NOT</u> modular invariant

$$\boxed{\text{How can we obtain a modular inv. partition function?}}$$

Note: $(a,b) = (\frac{1}{2},\frac{1}{2})$ is by itself invariant!

$$\begin{cases} T: (\frac{1}{2},\frac{1}{2}) \to (\frac{1}{2},\, \frac{1}{2}-\frac{1}{2}\pm\frac{1}{2}) \equiv (\frac{1}{2},\frac{1}{2}) \\ S: (\frac{1}{2},\frac{1}{2}) \to (-\frac{1}{2},\frac{1}{2}) \equiv (\frac{1}{2},\frac{1}{2}) \end{cases}$$



But "unfortunately" $\underline{Z_{\frac{1}{2},\frac{1}{2}} = 0}$

One way to understand it: $a=b=\frac{1}{2} \Leftrightarrow$ Periodic in all directions

$$\psi(\sigma+2\pi, t) = \psi(\sigma,t) = \psi(\sigma+2\pi\tau_1,\, t+2\pi\tau_2)$$

$$Z_{\frac{1}{2},\frac{1}{2}} = \int \mathcal{D}\psi\, e^{-S_E} = \underbrace{\left(\int d\psi^{(0)}_{\pm}\, d\bar\psi^{(0)}_{\pm}\right)}_{=\,0} \int \underbrace{\mathcal{D}'\psi}_{\text{noncons}t}\, e^{-S_E(\psi')}$$

~~There is no~~

constant mode does not enter

periodic B.C.

Another way

$$Z_{\frac{1}{2},\frac{1}{2}} = \mathrm{Tr}_{\mathcal{H}_{\frac{1}{2},\frac{1}{2}}} e^{\pi i (F_R + F_L)} q^{H_R} \bar{q}^{H_L}$$

$a = \tilde{a} = \frac{1}{2}$ sector is called <u>Ramond-Ramond sector</u> ("RR" for short)

$$\psi_- = \sum_{n \in \mathbb{Z}} \psi_n(t) e^{in\sigma}, \cdots \quad \text{all modings are } \underline{\text{integers}}$$

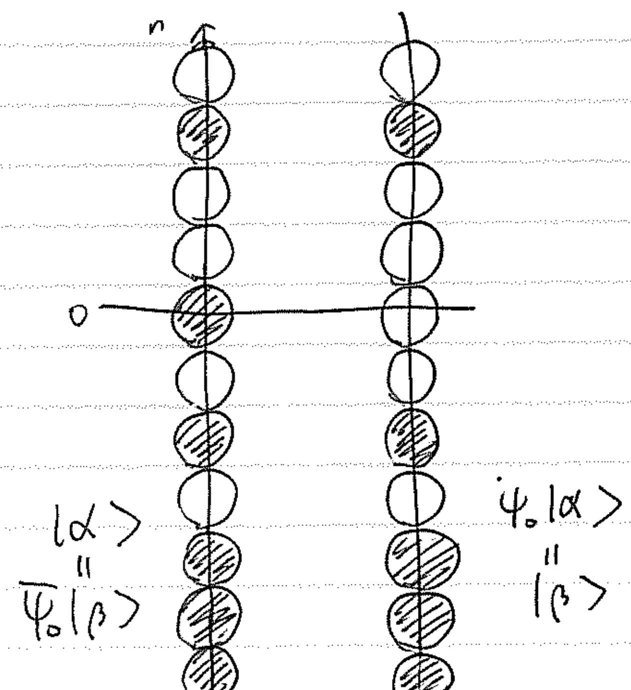
$$\psi_n, \bar{\psi}_n, \tilde{\psi}_n, \bar{\tilde{\psi}}_n \ (n \in \mathbb{Z}), \quad \{\psi_n, \bar{\psi}_m\} = \delta_{n+m,0}, \cdots \text{ etc.}$$

$$\{\psi_0, \bar{\psi}_0\} = 1 \quad \left(\{\tilde{\psi}_0, \bar{\tilde{\psi}}_0\} = 1 \right)$$

$$H_R = \sum_{n \in \mathbb{Z}} n : \bar{\psi}_{-n} \psi_n : -\frac{1}{24} + \frac{1}{2}\left(\frac{1}{2}\right)^2$$

n=0 does not contribute

$$H_L = \sum_{n \in \mathbb{Z}} n : \bar{\tilde{\psi}}_{-n} \tilde{\psi}_n : -\frac{1}{24} + \frac{1}{2}\left(\frac{1}{2}\right)^2 \qquad [H_{R,L}, \overset{(-)}{\psi_0}] = [H_{R,L}, \overset{(-)}{\tilde{\psi}_0}] = 0$$



These two states have the same energy but the charge (F_R, say) differ by ± 1.

(opposite value of $e^{\pi i (F_R + F_L)} q^{H_R} \bar{q}^{H_L}$)

$\longrightarrow Z_{\frac{1}{2},\frac{1}{2}} = 0.$

under modular transf.

What is the smallest invariant set including $(a,b)=(0,0)$?

$$\underset{\circlearrowleft_S}{(0,0)} \overset{T}{\longleftrightarrow} (0,\tfrac{1}{2}) \overset{S}{\longleftrightarrow} \underset{\circlearrowleft_T}{(\tfrac{1}{2},0)}$$

← this is the modular invariant set.

$Z_{0,0} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},0}$ is modular invariant.

Claim $Z = \frac{1}{2}\left(Z_{0,0} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},0} + \overset{0}{\overset{\parallel}{Z_{\frac{1}{2},\frac{1}{2}}}}\right)$

Can be interpreted as the partition function of some Q.F.T. (i.e. is of the form $\mathrm{Tr}_{\mathcal{H}}(e^{-2\pi i \tau_1 P} e^{-2\pi \tau_2 H})$)

recall $Z_{a,b} = \mathrm{Tr}_{\mathcal{H}_{a,a}}\left(e^{2\pi i b (F_R + F_L)} q^{H_R} \bar{q}^{H_L}\right)$

$$\therefore Z_{0,0} = \mathrm{Tr}_{\mathcal{H}_{0,0}}\left(q^{H_R} \bar{q}^{H_L}\right)$$

$$Z_{0,\frac{1}{2}} = \mathrm{Tr}_{\mathcal{H}_{0,0}}\left(e^{\pi i (F_R + F_L)} q^{H_R} \bar{q}^{H_L}\right)$$

$$Z_{\frac{1}{2},0} = \mathrm{Tr}_{\mathcal{H}_{\frac{1}{2},\frac{1}{2}}}\left(q^{H_R} \bar{q}^{H_L}\right)$$

$$Z_{\frac{1}{2},\frac{1}{2}} = \mathrm{Tr}_{\mathcal{H}_{\frac{1}{2},\frac{1}{2}}}\left(e^{\pi i (F_R + F_L)} q^{H_R} \bar{q}^{H_L}\right)$$

Recall $|0\rangle_{a,\tilde{a}} \in \mathcal{H}_{a,\tilde{a}}$ has $F_R = a$, $F_L = -\tilde{a}$ & $\psi_r, \bar{\psi}_r, \tilde{\psi}_r, \bar{\tilde{\psi}}_r$ has integral F_R, F_L charges

$\therefore$ $F_R + F_L$ eigenvalues are $\underline{\in \mathbb{Z}}$ if $a = \tilde{a}$

One may write $e^{\pi i (F_R + F_L)} = (-1)^{F_R + F_L} =: (-1)^F$

it has only $+1$ or -1 eigenvalues.

$$Z = \underset{\substack{\mathcal{H}_{0,0} \\ \shortparallel \\ \mathcal{H}_{NS\text{-}NS}}}{\mathrm{Tr}} \left(\frac{1+(-1)^F}{2} q^{H_R} \bar{q}^{H_L} \right) + \underset{\substack{\mathcal{H}_{\frac{1}{2},\frac{1}{2}} \\ \shortparallel \\ \mathcal{H}_{R,R}}}{\mathrm{Tr}} \left(\frac{1+(-1)^F}{2} q^{H_R} \bar{q}^{H_L} \right)$$

$$\boxed{\frac{1+(-1)^F}{2} = \text{projection to } (-1)^F = +1 \text{ eigenspace.}}$$

If we define $\mathcal{H} = \{ (-1)^F = +1 \text{ states in } \mathcal{H}_{NS\text{-}NS} \oplus \mathcal{H}_{R\text{-}R} \}$

Then $Z = \mathrm{Tr}_{\mathcal{H}} (q^{H_R} \bar{q}^{H_L})$.

This projection is called (non-chiral) <u>GSO projection</u>.

<u>G</u>liozzi - <u>S</u>cherk - <u>O</u>live (1976)

$|0\rangle_{0,0} \in \mathcal{H}_{NS\text{-}NS}$ & $|0\rangle_{\frac{1}{2},\frac{1}{2}} \in \mathcal{H}_{R\text{-}R}$

are both allowed (they have $F = F_R + F_L = 0$)

$\therefore (-1)^F = 1$

$\psi_{-\frac{1}{2}}|0\rangle_{0,0} \in \mathcal{H}_{NS\text{-}NS}$, $\psi_0|0\rangle_{\frac{1}{2},\frac{1}{2}} \in \mathcal{H}_{R\text{-}R}$

are not allowed (they have $(-1)^F = -1$)

any even multiples of $\psi_{-r}, \bar{\psi}_{-r}, \tilde{\psi}_{-r}, \tilde{\bar{\psi}}_{-r}$ $r \geq 0$

on $|0\rangle_{0,0} \in \mathcal{H}_{NS\text{-}NS}$ & $|0\rangle_{\frac{1}{2},\frac{1}{2}} \in \mathcal{H}_{R\text{-}R}$

are allowed

and any odd multiples of them on $|0\rangle_{0,0}$, $|0\rangle_{\frac{1}{2},\frac{1}{2}}$

are not allowed

Select only "bosonic state"