What is this GSO projected theory?

recall $$Z_{a,b} = \left|\frac{\vartheta\left[{a \atop b}\right](0,\tau)}{\eta(\tau)}\right|^2 = \frac{1}{|\eta|^2}\left|\sum_{n\in\mathbb{Z}} q^{\frac{(n+a)^2}{2}} e^{2\pi i b(n+a)}\right|^2$$

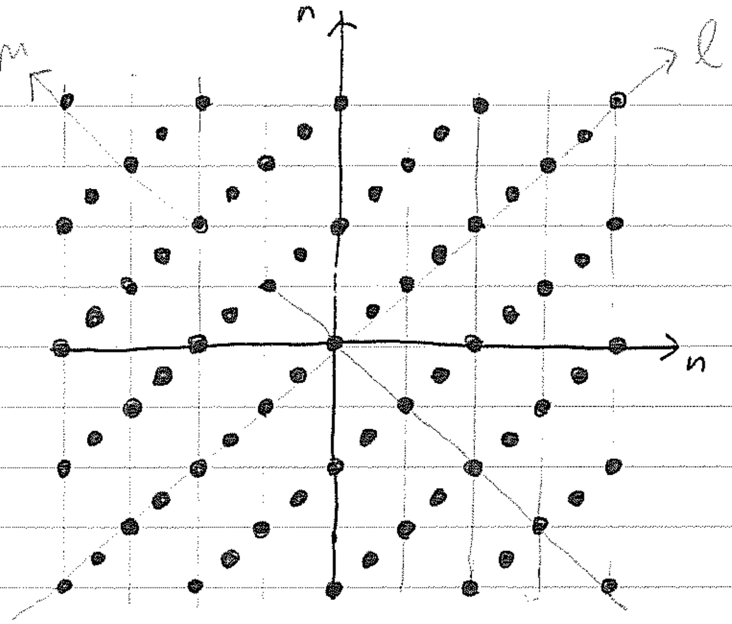
$$\left.\begin{aligned} Z_{0,0} &= \frac{1}{|\eta|^2}\left|\sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}\right|^2 = \frac{1}{|\eta|^2}\sum_{n,\tilde{n}\in\mathbb{Z}} q^{\frac{n^2}{2}}\bar{q}^{\frac{\tilde{n}^2}{2}} \\ Z_{0,\frac{1}{2}} &= \frac{1}{|\eta|^2}\left|\sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}(-1)^n\right|^2 = \frac{1}{|\eta|^2}\sum_{n,\tilde{n}\in\mathbb{Z}} q^{\frac{n^2}{2}}\bar{q}^{\frac{\tilde{n}^2}{2}}(-1)^{n+\tilde{n}} \end{aligned}\right\} \Rightarrow$$

$$\therefore\ \frac{1}{2}Z_{0,0} + \frac{1}{2}Z_{0,\frac{1}{2}} = \frac{1}{|\eta|^2}\sum_{\substack{n,\tilde{n}\in\mathbb{Z} \\ n+\tilde{n}\ \text{even}}} q^{\frac{n^2}{2}}\bar{q}^{\frac{\tilde{n}^2}{2}}.$$

$$\left.\begin{aligned} Z_{\frac{1}{2},0} &= \frac{1}{|\eta|^2}\left|\sum_{p\in\mathbb{Z}} q^{\frac{(p+\frac{1}{2})^2}{2}}\right|^2 = \frac{1}{|\eta|^2}\sum_{p,\tilde{p}\in\mathbb{Z}} q^{\frac{(p+\frac{1}{2})^2}{2}}\bar{q}^{\frac{(\tilde{p}+\frac{1}{2})^2}{2}} \\ Z_{\frac{1}{2},\frac{1}{2}} &= \frac{1}{|\eta|^2}\left|\sum_{p\in\mathbb{Z}} q^{\frac{(p+\frac{1}{2})^2}{2}} e^{\pi i(p+\frac{1}{2})}\right|^2 = \frac{1}{|\eta|^2}\sum_{p,\tilde{p}\in\mathbb{Z}} q^{\frac{(p+\frac{1}{2})^2}{2}}\bar{q}^{\frac{(\tilde{p}+\frac{1}{2})^2}{2}}(-1)^{p+\tilde{p}} \end{aligned}\right\} \Rightarrow$$

$$\therefore\ \frac{1}{2}Z_{\frac{1}{2},0} + \frac{1}{2}Z_{\frac{1}{2},\frac{1}{2}} = \frac{1}{|\eta|^2}\sum_{\substack{p,\tilde{p}\in\mathbb{Z} \\ p+\tilde{p}\ \text{even}}} q^{\frac{(p+\frac{1}{2})^2}{2}}\bar{q}^{\frac{(\tilde{p}+\frac{1}{2})^2}{2}}.$$

$$\therefore\ Z = \sum_{\substack{n,\tilde{n}\in\mathbb{Z},\ n+\tilde{n}\ \text{even} \\ \text{or} \\ n,\tilde{n}\in\mathbb{Z}+\frac{1}{2},\ n+\tilde{n}\ \text{odd}}} q^{\frac{n^2}{2}}\bar{q}^{\frac{\tilde{n}^2}{2}} \Big/ |\eta(\tau)|^2$$



$n, \tilde n \in \mathbb{Z}$, $n+\tilde n$ even

or $n, \tilde n \in \mathbb{Z}+\frac{1}{2}$, $n+\tilde n$ odd

$$\Updownarrow$$

$$n = \frac{l}{2} - m \ , \ \tilde n = \frac{l}{2} + m$$

$$(\, l, m \in \mathbb{Z} \,)$$

$l \in 2\mathbb{Z} \Leftrightarrow , n, \tilde n \in \mathbb{Z}$ & $n + \tilde n = l$ even

$l \in 2\mathbb{Z}+1 \Leftrightarrow n, \tilde n \in \mathbb{Z}+\frac{1}{2}$ & $n + \tilde n = l$ odd.

$$Z = \frac{1}{|\eta|^2} \sum_{l,m \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{l}{2} - m\right)^2} \bar q^{\frac{1}{2}\left(\frac{l}{2} + m\right)^2}$$

$$= \frac{1}{|\eta|^2} \sum_{l,m \in \mathbb{Z}} q^{\frac{1}{4}\left(\frac{l}{\sqrt{2}} - \sqrt{2} m\right)^2} \bar q^{\frac{1}{4}\left(\frac{l}{\sqrt{2}} + \sqrt{2} m\right)^2}$$

This is equal to the partition function for $S^1_{2\pi R}$ - sigma model with $\boxed{R = \sqrt{2}}$.

$\therefore$ $\boxed{\text{GSO projected Dirac fermion} \cong S^1_{2\pi R}, R=\sqrt{2} \text{ sigma model.}}$

Fermion - Boson correspondence

(or Bosonization of fermion)

Precise map between states

$$\mathrm{Tr}_{\mathcal{H}_{GSO}}\left(e^{2\pi i(bF_R+\tilde{b}F_L)}q^{H_R}\bar{q}^{H_L}\right)$$

$$=\mathrm{Tr}_{\mathcal{H}_{NS\text{-}NS}}\left(\frac{1+(-1)^F}{2}e^{2\pi i(bF_R+\tilde{b}F_L)}q^{H_R}\bar{q}^{H_L}\right)+\mathrm{Tr}_{\mathcal{H}_{R\text{-}R}}\left(\frac{1+(-1)^F}{2}e^{2\pi i(bF_R+\hat{b}F_L)}q^{H_R}\bar{q}^{H_L}\right)$$

$\Big\|$ ← do the same as before → $\Big\|$

$$\frac{1}{|\eta|^2}\sum_{\substack{n,\tilde{n}\in\mathbb{Z}\\ n+\tilde{n}\text{ even}}} q^{\frac{n^2}{2}}q^{\frac{\tilde{n}^2}{2}}e^{2\pi i bn}e^{-2\pi i\tilde{b}n}\quad,\frac{1}{|\eta|^2}\sum_{\substack{n,\tilde{n}\in\mathbb{Z}+\frac{1}{2}\\ n+\tilde{n}\text{ odd}}} q^{\frac{n^2}{2}}\bar{q}^{\frac{\tilde{n}^2}{2}}e^{2\pi i bn}e^{-2\pi i\hat{b}\tilde{n}}$$

$$=\frac{1}{|\eta|^2}\sum_{l,m\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{l}{2}-m\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{l}{2}+m\right)}e^{2\pi i b\left(\frac{l}{2}-m\right)}e^{-2\pi i b\left(\frac{l}{2}+m\right)}$$

$$\Rightarrow\quad\boxed{\begin{aligned}F_R &\longleftrightarrow \frac{l}{2}-m=\frac{1}{\sqrt{2}}(p-w)\\ F_L &\longleftrightarrow -\frac{l}{2}+m=-\frac{1}{\sqrt{2}}(p+w)\end{aligned}}$$

also

$$\boxed{\begin{aligned}&NS\text{-}NS \Leftrightarrow \frac{l}{2}\pm m\in\mathbb{Z}\Leftrightarrow l\text{ even}\\ &R\text{-}R \Leftrightarrow \frac{l}{2}\pm m\in\mathbb{Z}+\frac{1}{2}\Leftrightarrow l\text{ odd}\end{aligned}}$$

$$\therefore\ \mathcal{H}_{NS\text{-}NS}\longleftrightarrow\bigoplus_{l\text{ even},\,\forall m}\mathcal{H}_{(l,m)}$$

$$\mathcal{H}_{R\text{-}R}\longleftrightarrow\bigoplus_{l\text{ odd},\,\forall m}\mathcal{H}_{(l,m)}$$

Currents for $F_R \leftrightarrow \frac{1}{\sqrt{2}}(p-w)$, $F_L \leftrightarrow -\frac{1}{\sqrt{2}}(p+w)$

$$\Rightarrow \boxed{\begin{array}{l} \bar{\psi}_- \psi_- \longleftrightarrow \frac{1}{\sqrt{2}}(\partial_t X - \partial_\sigma X) \\ \\ \bar{\psi}_+ \psi_+ \longleftrightarrow -\frac{1}{\sqrt{2}}(\partial_t X + \partial_\sigma X) \end{array}}$$

$$\Rightarrow \quad \left.\begin{array}{l} \alpha_n \longleftrightarrow \sum_r \bar{\psi}_{-r+n}\psi_r \\ \\ \tilde{\alpha}_n \longleftrightarrow -\sum_r \bar{\tilde{\psi}}_{-r+n}\tilde{\psi}_r \end{array}\right|$$

$(n \neq 0)$. sum over
$r \in \mathbb{Z}+\frac{1}{2}$ in NS-NS (l eve
$r \in \mathbb{Z}$ in R-R (l odd)

Thus,

$$\alpha_{-1}|\overset{\overset{l}{\shortparallel}}{0},\overset{\overset{m}{\shortparallel}}{0}\rangle \in \mathcal{H}_{(0,0)} \longleftrightarrow \underbrace{\sum_r \bar{\psi}_{-r-1}\psi_r|0\rangle_{0,0}}_{\shortparallel} \in \mathcal{H}_{NS\text{-}NS}$$

$$\bar{\psi}_{-\frac{1}{2}}\psi_{-\frac{1}{2}}|0\rangle_{0,0}$$

$$\alpha_{-1}^2|0,0\rangle \longleftrightarrow \left(\sum_r \bar{\psi}_{-r-1}\psi_r\right)\bar{\psi}_{-\frac{1}{2}}\psi_{-\frac{1}{2}}|0\rangle_{0,0}$$

$$\shortparallel$$

$$\cancel{\cdots} + \bar{\psi}_{-\frac{3}{2}}\psi_{\frac{1}{2}} + \cancel{\bar{\psi}_{-\frac{1}{2}}\bar{\psi}_{-\frac{1}{2}}} + \bar{\psi}_{+\frac{1}{2}}\psi_{-\frac{3}{2}} + \cancel{\cdots}$$

$$= \bar{\psi}_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle_{0,0} + \bar{\psi}_{-\frac{3}{2}}\bar{\psi}_{-\frac{1}{2}}|0\rangle_{0,0}$$

$$\nparallel$$

$$\alpha_{-2}|0,0\rangle \longleftrightarrow \sum_r \bar{\psi}_{-r-2}\psi_r|0\rangle_{0,0} = \bar{\psi}_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle_{0,0} + \bar{\psi}_{-\frac{1}{2}}\psi_{-\frac{3}{2}}|0\rangle_{0,0}$$

$$\vdots$$

$\forall \prod_n \alpha_{-n}^{N_n}\tilde{\alpha}_{-n}^{\tilde{N}_n}|0,0\rangle \longleftrightarrow$ we know what this is.

$|l, m\rangle \Leftrightarrow$?

It must be a state in $\mathcal{H}_{NS-NS}$ if l even

$\mathcal{H}_{R-R}$ if l odd

s.t. $F_R = \frac{l}{2} - m$, $F_L = -\frac{l}{2} - m$, $H_R = \frac{1}{2}\left(\frac{l}{2} - m\right)^2 - \frac{1}{24}$, $H_L = \frac{1}{2}\left(\frac{l}{2} + m\right)^2 - \frac{\,}{2}$

We know $|0\rangle_{a,\tilde{a}}$ has $F_R = a$, $F_L = -\tilde{a}$, $H_R = \frac{a^2}{2} - \frac{1}{24}$, $H_L = \frac{1}{2}\tilde{a}^2 - \frac{1}{24}$.

Thus,

$$\boxed{|l, m\rangle \longleftrightarrow |0\rangle_{\frac{l}{2}-m, \frac{l}{2}+m}}$$

$|-1, 0\rangle \leftrightarrow |0\rangle_{-\frac{1}{2}, -\frac{1}{2}} = \psi_0 \overline{\tilde{\psi}}_0 |0\rangle_{\frac{1}{2}, \frac{1}{2}}$

e.g. $|1, 0\rangle \leftrightarrow |0\rangle_{\frac{1}{2}, \frac{1}{2}}$

recall the spectral flow picture.

$|2, 0\rangle \leftrightarrow |0\rangle_{1,1} = \bar{\psi}_{-\frac{1}{2}} \tilde{\psi}_{-\frac{1}{2}} |0\rangle_{0,0}$

$|3, 0\rangle \longleftrightarrow |0\rangle_{\frac{3}{2}, \frac{3}{2}} = \bar{\psi}_{-1} \tilde{\psi}_{-1} |0\rangle_{\frac{1}{2}, \frac{1}{2}}$

$|4, 0\rangle \longleftrightarrow |0\rangle_{2,2} = \bar{\psi}_{-\frac{1}{2}} \bar{\psi}_{-\frac{3}{2}} \tilde{\psi}_{-\frac{1}{2}} \tilde{\psi}_{-\frac{3}{2}} |0\rangle_{0,0}$

$\vdots$

$|0, 1\rangle \leftrightarrow |0\rangle_{-1,1} = \psi_{-\frac{1}{2}} \tilde{\psi}_{-\frac{1}{2}} |0\rangle_{0,0}$

$|0, 2\rangle \longleftrightarrow |0\rangle_{-2,2} = \psi_{-\frac{1}{2}} \psi_{-\frac{3}{2}} \tilde{\psi}_{-\frac{1}{2}} \tilde{\psi}_{-\frac{3}{2}} |0\rangle_{0,0}$

$|1, 1\rangle \longleftrightarrow |0\rangle_{-\frac{1}{2}, \frac{3}{2}} = \psi_0 \tilde{\psi}_{-1} |0\rangle_{\frac{1}{2}, \frac{1}{2}}$

Map between operators

- We already have $\boxed{\pm\frac{1}{\sqrt{2}}\left(\partial_t X \mp \partial_\sigma X\right) \longleftrightarrow \bar{\psi}_{\mp}\psi_{\mp}}$

- Look at $|l=2, m=0\rangle \longleftrightarrow \bar{\psi}_{-\frac{1}{2}}\tilde{\psi}_{-\frac{1}{2}}|0\rangle_{0,0}$

$\uparrow$ created by $:e^{i\frac{2}{\sqrt{2}}X}:$ from $|0,0\rangle$

$\uparrow$ created by $\bar{\psi}_-\psi_+$ from $|0\rangle_{0,0}$

$$:e^{\sqrt{2}iX}: \longleftrightarrow \bar{\psi}_-\psi_+$$

c.c.

$$:e^{-\sqrt{2}iX}: \longleftrightarrow \bar{\psi}_+\psi_-$$

- Look at $|l=0, m=1\rangle \longleftrightarrow \psi_{-\frac{1}{2}}\tilde{\psi}_{-\frac{1}{2}}|0\rangle_{0,0}$

$\uparrow$ created by $:e^{i\sqrt{2}\hat{X}}$ ($\hat{X}$ = T-dual variable)

$\uparrow$ created by $\psi_-\psi_+$

$$:e^{\sqrt{2}i\hat{X}}: \longleftrightarrow \psi_-\psi_+$$

c.c.

$$:e^{-\sqrt{2}i\hat{X}}: \longleftrightarrow \bar{\psi}_+\bar{\psi}_-$$

$$|4,0\rangle \longleftrightarrow \bar{\psi}_{-\frac{1}{2}}\bar{\psi}_{-\frac{3}{2}}\tilde{\psi}_{-\frac{1}{2}}\tilde{\psi}_{-\frac{3}{2}}|0\rangle_{0,0}$$

$$\rightsquigarrow \quad :e^{2\sqrt{2}iX}: \longleftrightarrow \frac{1}{4}\bar{\psi}_-(\partial_t-\partial_\sigma)\bar{\psi}_-\cdot\psi_+(\partial_t+\partial_\sigma)\psi_+$$

$$\vdots$$

What does $|\pm 1,0\rangle \longleftrightarrow |0\rangle_{\pm\frac{1}{2},\pm\frac{1}{2}}$ mean?

created by $:e^{\pm\frac{i}{\sqrt{2}}X}:$ from $|0,0\rangle$

ground states in R-R sector.

$\Rightarrow$ $:e^{\pm\frac{i}{\sqrt{2}}X}:$ creates ~~a~~ R-R groud states $|0\rangle_{\pm\frac{1}{2},\pm\frac{1}{2}}$ from the NS-NS ground state $|0\rangle_{0,0}$.

In general

$$\boxed{\mathcal{H}_{NS-NS} \underset{l\,:\,\text{odd}}{\overset{e^{i\frac{l}{\sqrt{2}}X}}{\rightleftarrows}} \mathcal{H}_{R-R}}$$