Matching of correlators

We have seen $:e^{\sqrt{2}iX}: \longleftrightarrow \bar{\psi}_- \psi_+$

$:e^{-\sqrt{2}iX}: \longleftrightarrow \bar{\psi}_+ \psi_-$

$:e^{\sqrt{2}i\hat{X}}: \longleftrightarrow \psi_- \psi_+$

$:e^{-\sqrt{2}i\hat{X}}: \longleftrightarrow \bar{\psi}_+ \bar{\psi}_-$

They must have the same correlation functions.

Recall

$$\langle :e^{ik_1 X(1)}: \cdots :e^{ik_s X(s)}: \rangle = \prod_{i<j} \frac{|z_i - z_j|^{k_i k_j}}{|z_i z_j|^{\frac{k_i k_j}{2}}}$$

etc, for bosons

and

$$\langle \psi_-(1) \cdots \psi_-(n) \bar{\psi}_-(1') \cdots \bar{\psi}_-(n') \rangle$$

$$= (z_1 \cdots z_n z_1' \cdots z_n')^{\frac{1}{2}} \cdot \frac{\prod_{i<j} (z_i - z_j)(z_i' - z_j')}{\prod_{i,j} (z_i - z_j')}$$

etc, for fermions.

$$\left\langle e^{\sqrt{2}iX(1)} \cdots e^{\sqrt{2}iX(n)}\, e^{-\sqrt{2}iX(1')} \cdots e^{-\sqrt{2}iX(n')} \right\rangle$$

$$= \prod_{i<j} \frac{|z_i - z_j|^2}{|z_i z_j|} \cdot \prod_{i<j} \frac{|z_i' - z_j'|^2}{|z_i' z_j'|} \cdot \prod_{i,j} \frac{|z_i - z_j'|^{-2}}{|z_i z_j'|^{-1}}$$

$$= \underbrace{\frac{\prod_{i,j} |z_i z_j'|}{\prod_{i<j} |z_i z_j z_i' z_j'|}}_{= \prod_i |z_i z_i'|} \cdot \frac{\prod_{i<j} |z_i - z_j|^2 \cdot |z_i' - z_j'|^2}{\prod_{i,j} |z_i - z_j'|^2}$$

On the other hand

$$\left\langle \overline{\psi}_-(1)\psi_+(1) \cdots \overline{\psi}_-(n)\psi_+(n)\, \overline{\psi}_+(1')\psi_-(1') \cdots \overline{\psi}_+(n')\psi_-(n') \right\rangle$$

$$= \left\langle \overline{\psi}_-(1) \cdots \overline{\psi}_-(n)\psi_-(1') \cdots \psi_-(n')\ \psi_+(1) \cdots \psi_+(n)\overline{\psi}_+(1') \cdots \overline{\psi}_+(n') \right\rangle$$

$$= \left[(z_1 \cdots z_n z_1' \cdots z_n')^{\frac{1}{2}} \frac{\prod_{i<j} (z_i - z_j)(z_i' - z_j')}{\prod_{i,j} (z_i - z_j')} \right] \times \text{c.c.}$$

$$= \prod_i |z_i z_i'| \cdot \frac{\prod_{i<j} |z_i - z_j|^2 \cdot |z_i' - z_j'|^2}{\prod_{i,j} |z_i - z_j'|^2}$$

Complete match!

Perturbation

mass term $\Delta L_m = -m\bar{\psi}\psi = -m\bar{\psi}_+\psi_- - m\bar{\psi}_-\psi_+$

$$\longleftrightarrow \; -m :e^{-\sqrt{2}iX}: + m :e^{\sqrt{2}iX}: \;=\; -2m :\cos(\sqrt{2}X):$$

four-fermi term $\Delta L_4 = -\frac{g}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) = -g\bar{\psi}_-\psi_-\bar{\psi}_+\psi_+$

$$\longleftrightarrow \; -g\,\frac{\partial_t X - \partial_\sigma X}{\sqrt{2}}\cdot\frac{-\partial_t X - \partial_\sigma X}{\sqrt{2}} \;=\; \frac{g}{2}\left\{(\partial_t X)^2 - (\partial_\sigma X)^2\right\}$$

$\therefore$ We have the duality between interacting field theories :

Fermionic - side :

$$L_f = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{g}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$$

--- massive Thirring model.

Bosonic side :

$$L_b = \frac{1}{2}(\partial_t X)^2 - \frac{1}{2}(\partial_\sigma X)^2 - 2m:\cos(\sqrt{2}X):$$

$$+\frac{g}{2}(\partial_t X)^2 - \frac{g}{2}(\partial_\sigma X)^2 \; .$$

$$\underline{\text{NB}} \quad :e^{\pm\sqrt{2}iX(\sigma)}: = \lim_{\sigma'\to\sigma}\left[e^{\pm\sqrt{2}iX(\sigma)}\right]_{\substack{\text{point}\\ \text{-split}}} \times \text{dist}(\sigma,\sigma')^{\frac{k^2}{2}} = \text{dist}(\sigma,\sigma')^{-1}$$

$$k^2 = \sqrt{2}^2 = 2$$

has mass-dimension $= 1$.

If we write $e^{\pm\sqrt{2}iX} = \frac{1}{\mu} :e^{\pm\sqrt{2}iX}:$ with some mass parameter μ ($e^{\pm\sqrt{2}iX}$ is then dimensionless),

$$\Delta L_m \longleftrightarrow -2m\mu\cos(\sqrt{2}X).$$

$$L_b = \frac{1}{2}(1+g)\left\{(\partial_t X)^2 - (\partial_\sigma X)^2\right\} - 2m\mu\cos\left(\sqrt{2}X\right)$$

Define $\tilde{X} := \sqrt{1+g}\,X$, then it has deformed periodicity

$$\tilde{X} \equiv \tilde{X} + 2\pi\sqrt{2}\cdot\sqrt{1+g} \qquad \therefore\ \tilde{R} = \sqrt{2+2g}.$$

put $\beta := \sqrt{\frac{2}{1+g}}$, then the Lagrangian is written as

$$L_b = \frac{1}{2}(\partial_t\tilde{X})^2 - \frac{1}{2}(\partial_\sigma\tilde{X})^2 - 2m\mu\cos(\beta\tilde{X})$$

$\cdots$ <u>The Sine-Gordon model</u>

$$\left[\ \text{EOM is} \quad \partial_t^2\tilde{X} - \partial_\sigma^2\tilde{X} - 2m\mu\beta\,\underline{\sin(\beta\tilde{X})} = 0\ \right]$$

$\therefore$ We have an interacting version of Boson-fermion Correspondence

$$\boxed{\begin{array}{ll} \text{Massive Thirring model} \longleftrightarrow & \text{Sine-Gordon model} \\ \mathcal{L}_f = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{g}{4}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi & \mathcal{L}_b = \frac{1}{2}(\partial_\mu\tilde{X})^2 - 2m\mu\cos(\beta\tilde{X}) \end{array}}$$

map of parameters :

$$\beta = \sqrt{\frac{2}{1+g}}$$

$$\tilde{R} = \sqrt{2+2g}$$

At the value $g=0 \longleftrightarrow \beta=\sqrt{2}$ $(\tilde{R}=\sqrt{2})$,

it is a correspondence between

$$\boxed{\begin{array}{c} \underline{\text{Free}} \text{ massive Dirac} \\ \mathcal{L}_f = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \end{array}} \longleftrightarrow \boxed{\begin{array}{c} \underline{\text{Interacting}} \text{ sine-Gordon} \\ \mathcal{L}_b = \frac{1}{2}(\partial_\mu X)^2 - 2m\mu\cos(\sqrt{2}X). \end{array}}$$

"more conventional" notation say in Coleman (1975)

<u>Ours</u>

$$S_f = \frac{1}{2\pi}\int d^2\sigma \left(i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{g}{4}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi \right)$$

$$\updownarrow$$

$$S_b = \frac{1}{2\pi}\int d^2\sigma \left(\frac{1}{2}(\partial_\mu\tilde{X})^2 - 2m\mu\cos(\beta\tilde{X}) \right)$$

<u>Standard Convension</u>

$$S_f = \int d^2\sigma \left(i\bar{\psi}_c\partial\!\!\!/\psi_c - m\bar{\psi}_c\psi_c - \frac{g_c}{2}\bar{\psi}_c\gamma^\mu\psi_c\bar{\psi}_c\gamma_\mu\psi_c \right)$$

$$\updownarrow$$

$$S_b = \int d^2\sigma \left(\frac{1}{2}(\partial_\mu\phi_c)^2 - 2m\mu\cos(\beta_c\phi_c) \right)$$

<u>Relation</u>:

$$\psi_c = \frac{1}{\sqrt{2\pi}}\psi, \quad \phi_c = \frac{1}{\sqrt{2\pi}}\tilde{X}$$

$$\because \frac{1}{2}\cdot\frac{g}{2\pi} = \frac{g_c}{\sqrt{2\pi}^4} \qquad \beta\tilde{X} = \beta_c\phi_c$$

$$\therefore g = \frac{1}{\pi}g_c \qquad \therefore \beta = \frac{\beta_c}{\sqrt{2\pi}}$$

$$\therefore \beta = \sqrt{\frac{2}{1+g}} \iff \beta_c = \sqrt{\frac{4\pi}{1+g_c/\pi}}$$

Conformal Transformations

A <u>conformal transformation</u> of a Riemannian manifold (M, g) is a diffeomorphism $f : M \longrightarrow M$ that transforms the metric to itself up to (position dependent) scaling

$$f^* g = e^\lambda g \qquad (\lambda \text{ a function on } M).$$

Infinitesimally, it is a vector field v s.t.

$$\mathcal{L}_v g = \lambda g \qquad \text{———} (\star)$$

↑ Lie derivative.

On Euclidean space $M = \mathbb{R}^n$, $g = \sum_\mu (dx^\mu)^2$, $(\star)$ reads

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \lambda \delta_{\mu\nu} \qquad \text{———} (\star)_E$$

We find

$$0 = \partial_\mu(\partial_\nu v_\rho) - \partial_\nu(\partial_\mu v_\rho) = \partial_\mu(\lambda \delta_{\nu\rho} - \partial_\rho v_\nu) - \partial_\nu(\lambda \delta_{\mu\rho} - \partial_\rho v_\mu)$$

$$= \partial_\rho(\partial_\nu v_\mu - \partial_\mu v_\rho) - (\delta_{\mu\rho}\partial_\nu\lambda - \delta_{\nu\rho}\partial_\mu\lambda) \qquad \text{———} (\star\star)$$

and then

$$0 = \partial_\rho\partial_\sigma(\partial_\nu v_\mu - \partial_\mu v_\nu) - \partial_\sigma\partial_\rho(\partial_\nu v_\mu - \partial_\mu v_\nu)$$

$$= \partial_\rho(\delta_{\mu\sigma}\partial_\nu\lambda - \partial_{\nu\sigma}\partial_\mu\lambda) - \partial_\sigma(\delta_{\mu\rho}\partial_\nu\lambda - \delta_{\nu\rho}\partial_\mu\lambda)$$

$\delta^{\sigma\nu}$

$$0 = \partial_\rho\partial_\mu\lambda - \partial_\rho\partial_\mu\lambda - \delta_{\mu\rho}\partial^\sigma\partial_\sigma\lambda + \partial_\rho\partial_\mu\lambda$$

$$\therefore\ (n-2)\,\partial_\rho\partial_\mu\lambda + \delta_{\mu\rho}\,\partial^\sigma\partial_\sigma\lambda = 0$$

$\delta^{\rho\mu}$

$$(n-2+n)\,\partial^\rho\partial_\rho\lambda = 0 \quad \overset{n\neq 1}{\Longrightarrow} \quad \partial^\rho\partial_\rho\lambda = 0$$

(λ must be harmonic)

$$(n-2)\,\partial_\rho\partial_\mu\lambda = 0$$

If $n\neq 2$, $\quad \partial_\rho\partial_\mu\lambda = 0 \quad \boxed{\therefore\ \lambda = b + c_\mu x^\mu}$

constants

$$(\star)_\in \Rightarrow \quad \partial_\nu v_\mu + \partial_\mu v_\nu = (b + c_\rho x^\rho)\,\delta_{\mu\nu}$$

$$(\star\star) \Rightarrow \quad \partial_\nu v_\mu - \partial_\mu v_\nu = \omega_{\mu\nu} + c_\nu x_\mu - c_\mu x_\nu$$

constant, antisym

Sum & integrate:

$$2v_\mu = a_\mu + b x_\mu + \omega_{\mu\nu} x^\nu + \left(c_\rho x^\rho x^\mu - \frac{|x|^2}{2} c_\mu\right)$$

translation dilatation rotation "special conformal transformation".

<u>Standard notation:</u>

$iP_\mu = \frac{\partial}{\partial x^\mu}$ (translation), $D = x^\mu \frac{\partial}{\partial x^\mu}$ (dilatation)

$iM_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}$ (rotation)

$iK_\mu = |x|^2 \frac{\partial}{\partial x^\mu} - 2x_\mu x^\nu \frac{\partial}{\partial x^\nu}$ (special conformal transformation).

In two-dimensions, $\partial_\rho \partial_\mu \lambda = 0$ is NOT required.

λ is required to obey only $\partial^\rho \partial_\rho \lambda = 0$ (∞-many!).

Or more directly,

$$2\partial_1 v_1 = 2\partial_2 v_2 = \lambda \iff \partial_1 v_1 - \partial_2 v_2 = 0$$

$$\partial_1 v_2 + \partial_2 v_1 = 0$$

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2$$

$$\partial_{\bar{z}} v^z = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)(v^1 + iv^2) = \frac{1}{2}\underbrace{(\partial_1 v^1 - \partial_2 v^2)}_{=0} + \frac{i}{2}\underbrace{(\partial_1 v^2 - \partial_2 v^1)}_{=0}$$

$$= 0$$

$$\partial_z v^{\bar{z}} = 0$$

$\therefore$ $v^z = v(z)$ any holomorphic function of z.

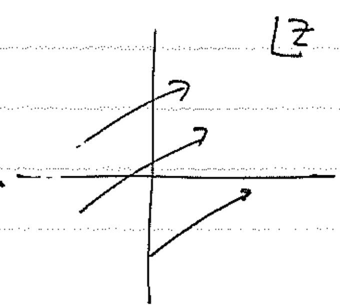
$v^{\bar{z}} = \bar{v}(\bar{z})$ any holomorphic function of $\bar{z}$

(antiholomorphic function of z).

[If v is real, then $\bar{v}(\bar{z}) = \overline{v(z)}$]

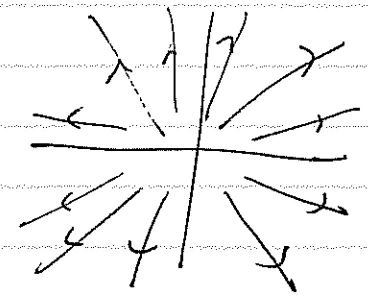
e.g. $\quad i l_n = z^{n+1}\dfrac{\partial}{\partial z}, \quad i\tilde{l}_n = \bar{z}^{n+1}\dfrac{\partial}{\partial \bar{z}}$

$i l_{-1} = \dfrac{\partial}{\partial z}, \quad i\tilde{l}_{-1} = \dfrac{\partial}{\partial \bar{z}}$ translations



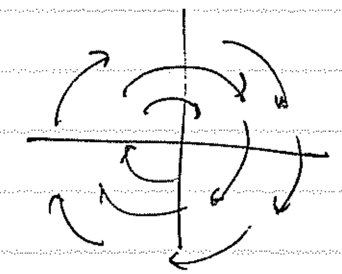
$i l_0 = z \frac{\partial}{\partial z}$, $i \tilde{l}_0 = \bar{z} \frac{\partial}{\partial \bar{z}}$

- $i(l_0 + \tilde{l}_0) = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}$



dilatation

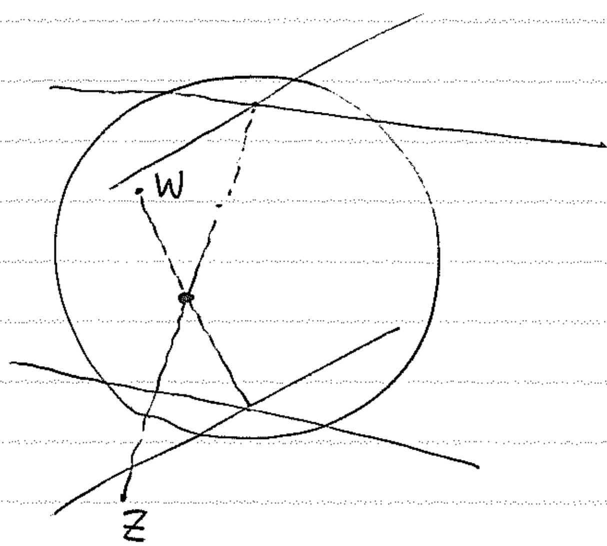
- $(l_0 - \tilde{l}_0) = -iz \frac{\partial}{\partial z} + i\bar{z} \frac{\partial}{\partial \bar{z}}$



rotation

$i l_1 = z^2 \frac{\partial}{\partial z}$, $i \tilde{l}_1 = \bar{z}^2 \frac{\partial}{\partial \bar{z}}$ ··· special conformal transformation

cf.



$W = Z^{-1}$: stereographic projection to the opposite plane.

$$\frac{\partial}{\partial W} = \frac{\partial z}{\partial W} \frac{\partial}{\partial z} = -W^{-2} \frac{\partial}{\partial z} = -z^2 \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \bar{W}} = -\bar{z}^2 \frac{\partial}{\partial \bar{z}}$$

$z^2 \frac{\partial}{\partial z}$, $\bar{z}^2 \frac{\partial}{\partial \bar{z}}$ are translations in W-plane.

But there are others $z^{n+1}\frac{\partial}{\partial z}$, $\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$ $(n \neq 0, \pm 1)$

They form an algebra

$$[l_n, l_m] = (n-m)\, l_{n+m}$$

$$[\tilde{l}_n, \tilde{l}_m] = (n-m)\, \tilde{l}_{n+m}$$

$$[l_n, \tilde{l}_m] = 0$$

$\left.\begin{matrix} l_0, l_1, l_{-1} \\ \tilde{l}_0, \tilde{l}_1, \tilde{l}_{-1} \end{matrix}\right\}$ form a subalgebra.

They have no poles on the sphere

e.g. $\frac{\partial}{\partial z} = -w^2\frac{\partial}{\partial w}$, $z\frac{\partial}{\partial z} = -w\frac{\partial}{\partial w}$, $z^2\frac{\partial}{\partial z} = -\frac{\partial}{\partial w}$.

Other $l_n, \tilde{l}_n$'s have poles e.g. $\left\{\begin{matrix} z^{-1}\frac{\partial}{\partial z} \\ z^3\frac{\partial}{\partial z} = -w^{-1}\frac{\partial}{\partial w} \end{matrix}\right.$

Real vector fields from

$l_0, l_1, l_{-1}, \tilde{l}_0, \tilde{l}_1, \tilde{l}_{-1}$ generates $SL(2,\mathbb{C})$ action

$$z \mapsto \left(\frac{az+b}{cz+d}\right) \qquad (ad-bc=1).$$

- $c=0$, $d=1=a$, $b \leftrightarrow$ translation
- $b=c=0$, $a \neq d \leftrightarrow$ rotation or dilatation
- $a=0$, $b=1=c$, $d \leftrightarrow$ special conformal transformation.

Definition of Energy-Momentum (E-M) tensor

We have already introduced E-M tensor as the Noether currents for time and space translations.

But here is more general, more precise, and <u>quantum</u> definition of E-M tensor.

First, we formulate a 2d QFT on a general 2d Riemannian manifold (Σ, g). Then, partition and correlation functions depends on metric g:

- $$Z(\Sigma, g) = \int_{\mathcal{F}(\Sigma)} \mathcal{D}_g X \, e^{-S_E(g, X)}$$

- $$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\Sigma, g} = \frac{1}{Z(\Sigma, g)} \int_{\mathcal{F}(\Sigma)} \mathcal{D}_g X \, e^{-S_E(g, X)} \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s)$$

The energy momentum tensor $T_{\mu\nu}$ is defined as the responce to the variation $g \to g + \delta g$

$$\delta Z(\Sigma, g) = Z(\Sigma, g) \cdot \left\langle \frac{1}{4\pi} \int_\Sigma \sqrt{g} \, d^2\sigma \, \delta g^{\mu\nu} T_{\mu\nu} \right\rangle$$

$$\delta\langle O_1\cdots O_s\rangle_{\Sigma,g} = \langle \delta(O_1\cdots O_s)\rangle_{\Sigma,g}$$

← if $O_1,\cdots,O_s$ depends on g

$$+\left\langle O_1\cdots O_s \frac{1}{4\pi}\int_\Sigma \sqrt{g}\,d^2\sigma\,\delta g^{\mu\nu}\,T_{\mu\nu}\right\rangle_{\Sigma,g}$$

$$-\langle O_1\cdots O_s\rangle_{\Sigma,g}\cdot\left\langle \frac{1}{4\pi}\int_\Sigma \sqrt{g}\,d^2\sigma\,\delta g^{\mu\nu}\,T_{\mu\nu}\right\rangle_{\Sigma,g}.$$

i.e.

$$\boxed{\text{“}\ \delta\left(\mathcal{D}_g X e^{-S_E(g,X)}\right) = \mathcal{D}_g X e^{-S_E(g,X)}\cdot\frac{1}{4\pi}\int_\Sigma \sqrt{g}\,d^2\sigma\,\delta g^{\mu\nu}\,T_{\mu\nu}\ \text{”}}$$

- From the way it is defined, $T_{\mu\nu}=T_{\nu\mu}$ (symmetric).

- Classically, (i.e. if we ignore g-dependence of the measure $\mathcal{D}_g X$),

$$\boxed{\delta S_E = -\frac{1}{4\pi}\int_\Sigma \sqrt{g}\,d^2\sigma\,\delta g^{\mu\nu}\,T_{\mu\nu}}$$

e.g. Scalar theory

$$S_E(g,X) = \frac{1}{2\pi}\int_\Sigma \sqrt{g}\,d^2\sigma\left\{\frac{1}{2}g^{\mu\nu}\partial_\mu X\partial_\nu X + U(X)\right\}$$

$$\delta S_E(g,X) = \frac{1}{2\pi}\int_\Sigma\Big[\sqrt{g}\,d^2\sigma\left\{\frac{1}{2}\delta g^{\mu\nu}\partial_\mu X\partial_\nu X\right\}$$

$$+\underbrace{\delta\sqrt{g}}_{\shortparallel\ -\frac{1}{2}\sqrt{g}\,\delta g^{\mu\nu}g_{\mu\nu}}\,d^2\sigma\left\{\frac{1}{2}g^{\rho\lambda}\partial_\rho X\partial_\lambda X + U(X)\right\}\Big]$$

$\det g \cdot \mathrm{Tr}\, g^{-1}\delta g$

$$\left[\begin{aligned} (?)\ \delta\sqrt{g} &= \delta\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}^{-1}\left(\delta \det g\right) = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\,\delta g_{\nu\mu} \\ &= -\frac{1}{2}\sqrt{g}\,\delta g^{\mu\nu}\, g_{\nu\mu} \end{aligned}\right]$$

$$\therefore\ T_{\mu\nu} = -\partial_\mu X \partial_\nu X + g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho X \partial_\sigma X + U(X)\right)$$

[agrees with the previous "definition" (as Noether current)]

e.g. Dirac fermion

$$S_E = \frac{1}{2\pi}\int_\Sigma d^2\sigma\left(\bar{\psi}_-(\partial_\tau - i\partial_\sigma)\psi_- + \bar{\psi}_+(\partial_\tau + i\partial_\sigma)\psi_+\right)$$

on flat space.

- How to formulate it on a curved (Σ, g) ?
- How to vary the metric? for a "fixed" $\psi_\pm$?

—— later.

(after variational calculus)