$J \neq 0$: extremely difficult to compute $Z, \langle \sigma_i \rangle$

$-\langle \sigma_i \sigma_j \rangle -$

(even when $H=0$)

Exception $d=1$
possible $d=2$
not (yet) done $d \geq 3$

Mean field approximation (Weiss)

Replace $\sum_{\langle i,j \rangle} \sigma_i \sigma_j$ by $\sum_{\langle i,j \rangle} \frac{\sigma_i \langle \sigma_j \rangle + \langle \sigma_i \rangle \sigma_j}{2}$

and assume $\langle \sigma_i \rangle = \langle \sigma_j \rangle = \langle \sigma \rangle$

$z :=$ # of nearest sites $\left(\begin{array}{c} z = 2d \text{ for } d\text{-dim} \\ \text{Square lattice} \end{array} \right)$

Then $\sum_{\langle i,j \rangle} \sigma_i \sigma_j \longrightarrow \frac{z}{2} \langle \sigma \rangle \sum_{i} \sigma_i$

$$\mathcal{H}_{m.f.} = -J \frac{z}{2} \langle \sigma \rangle \sum_i \sigma_i - \overset{\mu_B}{\overset{\parallel}{m_0}} H \sum_i \sigma_i$$

$$= - \underbrace{\left(J \frac{z}{2} \langle \sigma \rangle + m_0 H \right)}_{=: m_0 H'} \sum_i \sigma_i$$

Then, as before

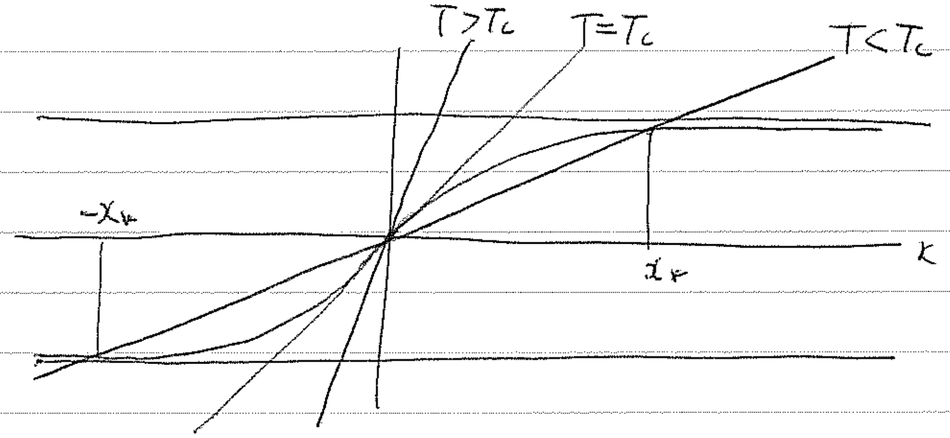
$$\cancel{\langle} M \cancel{\rangle} = m_0 \langle \sigma_i \rangle = m_0 \tanh \left(\frac{m_0 H'}{kT} \right)$$

$$= m_0 \tanh \left(\frac{J \frac{z}{2}}{kT} \frac{M}{m_0} + \frac{m_0 H}{kT} \right)$$

$$\frac{M}{m_0} = \tanh\Big(\underbrace{\frac{\overset{kT_c}{Jz}}{3kT}\,\frac{M}{m_0} + \frac{m_0 H}{kT}}_{x}\Big)$$

$$\frac{M}{m_0} = \frac{T}{T_c}\left(x - \frac{m_0 H}{kT}\right) = \tanh(x)$$

$\underline{H=0}$ $\qquad \frac{T}{T_c}\,x = \tanh(x)$



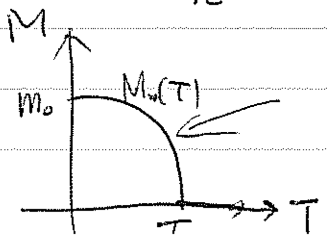
$T>T_c$: $x=0$ $\quad \therefore\ M=0$

$T<T_c$: $M=0$ or $\boxed{M = \pm m_0 \frac{T}{T_c} x_*}$ ~~Spontaneous magnetization~~

$T \lesssim T_c \Rightarrow x_* \ll 1$

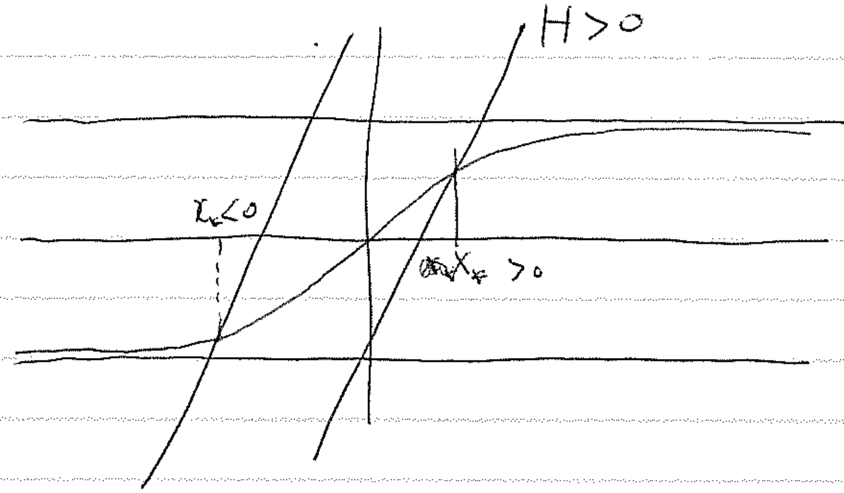
$$\frac{T}{T_c} x_* = x_* - \frac{1}{3}x_*^3 + \cdots \quad \therefore\ x_* \simeq \sqrt{3\,\frac{T_c - T}{T_c}}$$

$$M_*(T) = m_0 \frac{T}{T_c} x_* \sim m_0\sqrt{3}\,(T_c - T)^{\frac{1}{2}}$$



Spontaneous magnetization
(magnetization even for $H=0$)

$T > T_c$
<u>$H \neq 0$</u>



$$M = m_0 \tanh(x_*) \quad > 0 \quad \text{if } H > 0$$
$$\qquad\qquad\qquad\qquad < 0 \quad \text{if } H < 0$$

$\therefore$ $T > T_c$: paramagnetic

$T < T_c$: ferromagnetic

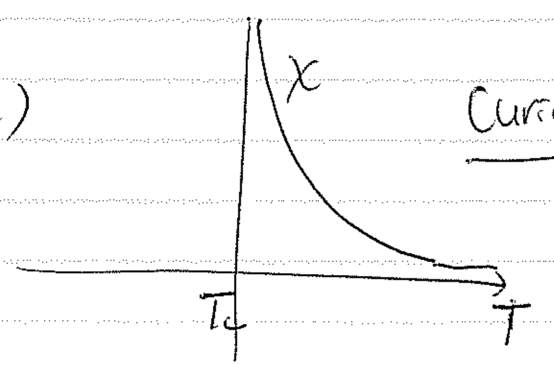
T_c : Curie temperature (iron (Fe): $T_c = 770\,°C$ (1043K))

If $H \ll \frac{kT_c}{m_0} \Rightarrow x_* \ll 1$ $\qquad \frac{T}{T_c} x_* - \frac{m_0 H}{kT_c} = x_* + \cancel{\cdots}$

$$(T - T_c) x_* \sim \frac{m_0 H}{k}$$

$$\therefore \quad M = m_0 \tanh(x_*) \sim \frac{m_0^2 H}{k(T - T_c)}$$

$$\chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} = \frac{m_0^2}{k(T - T_c)}$$



<u>Curie-Weiss law</u>

Landau's theory of phase transitions

$$\mathcal{H} = V(\mathcal{H}_0 - \hat{M} H) \qquad \hat{M} = m_0 \sum_i \frac{\sigma_i}{V}$$

$$\mathcal{H}_0 = -J \sum_{\langle ij \rangle} \frac{\sigma_i \sigma_j}{V}$$

Partition function

free energy

$$Z(T,H) = \sum_{\text{all states}} e^{-\mathcal{H}/kT} = e^{-VF(T,H)/kT}$$

$$M(T,H) = \frac{\sum_{\text{all states}} \hat{M} e^{-\mathcal{H}/kT}}{\sum_{\text{all states}} e^{-\mathcal{H}/kT}} = \frac{\frac{kT}{V}\frac{\partial}{\partial H} Z(T,H)}{Z(T,H)}$$

$$= -\frac{\partial}{\partial H} F(T,H)$$

Ferromagnetism at $T<T_c$: $M(T,0_+) > 0 > M(T,0_-)$

$\rightarrow$ $F(T,H)$ is not smooth function of H at $H=0$,

--- singularity

Landau: Intermediate free energy (effective potential)

$$Z(T,H,M) = \sum_{\substack{\text{states with} \\ \hat{M}=M}} e^{-\mathcal{H}/kT} =: e^{-V U_{eff}(T,H,M)/kT}$$

$$Z(T,H) = e^{-VF(T,H)/kT} = \int dM \, e^{-V U_{eff}(T,H,M)/kT}$$

Or in $V \gg 1$ limit.

$$F(T,H) = U_{eff}(T,H,M_*)$$

where M_* is obtained by minimizing $U_{eff}(T,H,M)$.

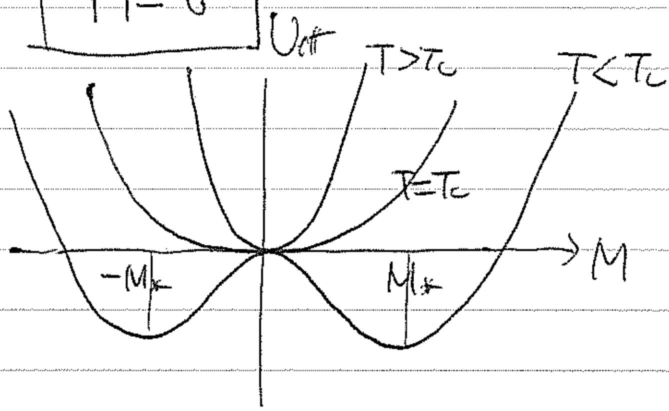
$$\left(\Rightarrow \frac{\partial U_{eff}}{\partial M}(T,H,M_*) = 0 \right)$$

Ansatz $\quad U_{eff} = U_0(T) + A(T)M^2 + B(T)M^4 + \cdots - HM$

$$A(T) \begin{cases} > 0 & T \gtrsim T_c \\ = 0 & T = T_c \\ < 0 & T < T_c \end{cases} \quad \& \ B(T) > 0$$

e.g. $A(T) = a(T - T_c)$

$\boxed{H=0}$



$T \geq T_c$: $M_{min} = 0$

$T < T_c$: $M_{min} = \pm M_*$

Spontaneous magnetization!

(breaks $M \to -M$ symmetry)

$M_* = ?$

$$0 = \frac{\partial U_{eff}}{\partial M} = 2A(T)M + 4B(T)M^3 + \cdots$$

$$M^2 = -\frac{A(T)}{2B(T)} \sim \frac{a(T_c - T)}{2B(T_c)}$$

$$\therefore \quad M_* = \sqrt{\frac{a}{2B(T_c)}}\,(T_c - T)^{\frac{1}{2}}$$

same as mean field result!

Free energy $F(T) = F(T, H=0) = U_{eff}(T, 0, M_*)$

$$\overset{T \lesssim T_c}{\simeq} U_0(T) + a(T - T_c) \cdot \frac{a(T_c - T)}{2B(T_c)} + B(T_c)\frac{a^2}{4B(T_c)^2}(T_c - T)^2$$

$$\doteq U_0 - \frac{a^2}{4B(T_c)}(T_c - T)^2$$

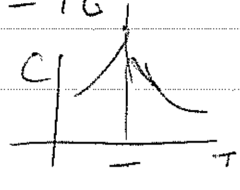
$$\overset{T \gtrsim T_c}{\sim} U_0(T)$$

$\frac{\partial F}{\partial T}$ continuous at $T = T_c$

$\frac{\partial^2 F}{\partial T^2}$: discontinuous at $T = T_c$

2nd order phase transition

$\frac{\partial^2 F}{\partial T^2} = -\frac{1}{T} C$ ← heat capacity

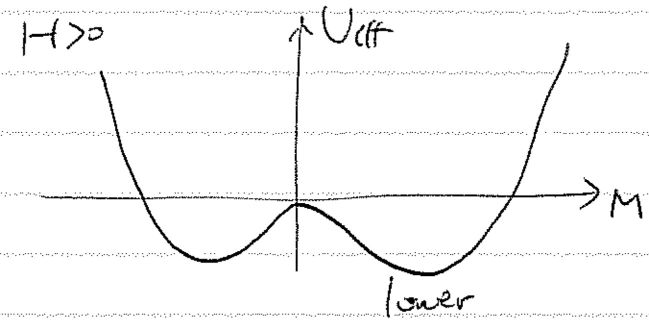


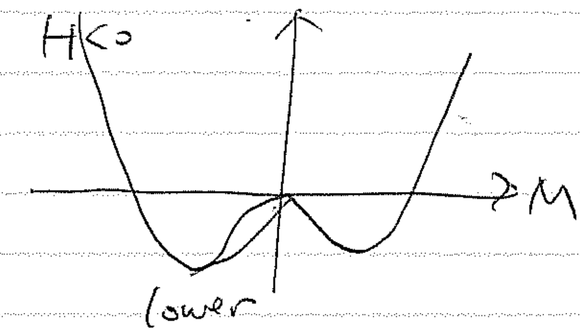
$\boxed{H \neq 0}$ $$0 = \frac{\partial U_{eff}}{\partial M} = 2A(T)M + 4B(T)M^3 + \cdots - H$$

$T > H_c$, small $|H|$ $\Rightarrow$ $$M \sim \frac{H}{2A(T)} = \frac{H}{2a(T-T_c)}$$

Curie-Weiss

$T < H_c$, small $|H|$





$M \sim \pm M_*(0)$ for $H>0$, $H<0$

↑ extrema at $H=0$

$$F(T,H) = F(T,0) - M_*(0)|H| + O(H^2)$$

↑ non-smooth,

Back to $H \to 0$

M ⋯ order parameter

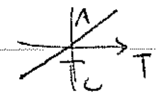
nearly all spins lined up ↑↑↑↑ ↑↑↑↑

$T<T_c$: $M \neq 0$ ordered phase ($\cancel{T<T_c}$) ↔ Spontaneous symmetry breaking

$T>T_c$: $M=0$ disordered phase ($\cancel{T>T_c}$) — spins random ↓↑↑ ↑↓

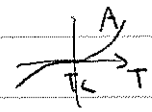
Mean field approx. $\iff$ $U_{eff} = U_0(T) + A(T)M^2 + B(T)M^4 + \cdots$

$A(T) = a(T - T_c)$, & $B(T_c) > 0$.



But it could be that

$A(T) \sim \pm |T - T_c|^{\gamma}$



$$U_{eff} = U_0(T) + A(T)M^2 + B(T)M^{2m} + \text{higher} - HM$$

$m > 2$

$\Rightarrow$ $H = 0$ & $T < T_c$: $M \sim \left(- \dfrac{A(T)}{m B(T_c)} \right)^{\frac{1}{2m-2}} \sim (T_c - T)^{\frac{\gamma}{2m-2}}$ β

$H \neq 0$, $T > T_c$: $M \sim \dfrac{H}{(T - T_c)^{\gamma}}$ $\therefore \chi \sim \dfrac{1}{(T - T_c)^{\gamma}}$

$\gamma, \beta, \cdots$ critical exponents.

<u>2d Ising model</u> $\gamma = \frac{7}{4}$, $\beta = \frac{1}{8}$ $\Longleftarrow$ $\boxed{m = 8}$

$$\left(T_c = 2.269 \cdots \frac{J}{k} = \frac{2}{\log(\sqrt{2} + 1)} \cdot \frac{J}{k} \right)$$