Definition of Energy-Momentum (E-M) tensor

We have already introduced E-M tensor as the Noether currents for time and space translations.

But here is more general, more precise, and <u>quantum</u> definition of E-M tensor.

First, we formulate a 2d QFT on a general 2d Riemannian manifold (Σ, g). Then, partition and correlation functions depends on metric g:

- $$Z(\Sigma, g) = \int_{\mathcal{F}(\Sigma)} \mathcal{D}_g X \, e^{-S_E(g, X)}$$

- $$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\Sigma, g} = \frac{1}{Z(\Sigma, g)} \int_{\mathcal{F}(\Sigma)} \mathcal{D}_g X \, e^{-S_E(g, X)} \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s)$$

The energy momentum tensor $T_{\mu\nu}$ is defined as the responce to the variation $g \to g + \delta g$

$$\delta Z(\Sigma, g) = Z(\Sigma, g) \cdot \left\langle \frac{1}{4\pi} \int_\Sigma \sqrt{g} \, d^2\sigma \, \delta g^{\mu\nu} T_{\mu\nu} \right\rangle$$

if $\mathcal{O}_1, \cdots, \mathcal{O}_s$ depends on g

$$\delta \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{\Sigma, g} = \langle \delta(\mathcal{O}_1 \cdots \mathcal{O}_s) \rangle_{\Sigma, g}$$

$$+ \langle \mathcal{O}_1 \cdots \mathcal{O}_s \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \delta g^{\mu\nu} T_{\mu\nu} \rangle_{\Sigma, g}$$

$$- \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{\Sigma, g} \cdot \langle \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \delta g^{\mu\nu} T_{\mu\nu} \rangle_{\Sigma, g}.$$

i.e.

$$\boxed{\text{“} \delta\left(\mathcal{D}_g X e^{-S_E(g, X)}\right) = \mathcal{D}_g X e^{-S_E(g, X)} \cdot \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \delta g^{\mu\nu} T_{\mu\nu} \text{”}}$$

- From the way it is defined, $T_{\mu\nu} = T_{\nu\mu}$ (symmetric).

- Classically, (i.e. if we ignore g-dependence of the measure $\mathcal{D}_g X$),

$$\boxed{\delta S_E = -\frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \delta g^{\mu\nu} T_{\mu\nu}}$$

e.g. Scalar theory

$$S_E(g, X) = \frac{1}{2\pi} \int_\Sigma \sqrt{g}\, d^2\sigma \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu X \partial_\nu X + U(X) \right\}$$

$$\delta S_E(g, X) = \frac{1}{2\pi} \int_\Sigma \Big[\sqrt{g}\, d^2\sigma \left\{ \frac{1}{2} \delta g^{\mu\nu} \partial_\mu X \partial_\nu X \right\}$$

$$+ \underbrace{\delta\sqrt{g}}_{= -\frac{1}{2}\sqrt{g}\, \delta g^{\mu\nu} g_{\mu\nu}} d^2\sigma \left\{ \frac{1}{2} g^{\rho\lambda} \partial_\rho X \partial_\lambda X + U(X) \right\} \Big]$$

$$\text{det}\, g \cdot \mathrm{Tr}\, g^{-1} \delta g$$

$$\left[(\because)\ \delta\sqrt{g} = \delta\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}^{-1}\left(\delta \det g\right) = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\delta g_{\nu\mu} \right.$$
$$\left. = -\frac{1}{2}\sqrt{g}\,\delta g^{\mu\nu} g_{\nu\mu} \right]$$

$$\therefore\ T_{\mu\nu} = -\partial_\mu X \partial_\nu X + g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho X \partial_\sigma X + U(X)\right)$$

[agrees with the previous "definition" (as Noether current)]

e.g. Dirac fermion

$$S_E = \frac{1}{2\pi}\int_\Sigma d^2\sigma \left(\bar{\psi}_-(\partial_\tau - i\partial_\sigma)\psi_- + \bar{\psi}_+(\partial_\tau + i\partial_\sigma)\psi_+ \right)$$

on flat space.

- How to formulate it on a curved (Σ, g) ?
- How to vary the metric? for a "fixed" $\psi_\pm$?

—— later.

(after variational calculus)

any QFT

We would like the theory to be invariant under general coordinate transf. i.e. diffeomorphism $f : \Sigma \to \Sigma$;

$$Z(\Sigma, f^* g) = Z(\Sigma, g)$$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{\Sigma, f^* g} = \langle f^* \mathcal{O}_1 \cdots f^* \mathcal{O}_s \rangle_{\Sigma, g}$$

For an infinitesimal diffeo (a vector field $\epsilon = \epsilon^\mu \frac{\partial}{\partial x^\mu}$),

$$\delta g = L_\epsilon g \qquad \left[\begin{array}{l} \delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \\ \text{or } \ \delta g^{\mu\nu} = -\nabla^\mu \epsilon^\nu - \nabla^\nu \epsilon^\mu \end{array} \right],$$

We have from $Z(\Sigma, f^* g) = Z(\Sigma, g)$

$$0 = \left\langle \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma \left(-\nabla^\mu \epsilon^\nu - \nabla^\nu \epsilon^\mu \right) T_{\mu\nu} \right\rangle = -\frac{1}{2\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \nabla^\mu \epsilon^\nu \langle T_{\mu\nu} \rangle$$

$$= \frac{1}{2\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \epsilon^\nu \nabla^\mu \langle T_{\mu\nu} \rangle \qquad \forall \epsilon$$

$$\Rightarrow \quad \nabla^\mu \langle T_{\mu\nu} \rangle = 0$$

From $\langle f^* \mathcal{O}_1 \cdots f^* \mathcal{O}_s \rangle_{\Sigma, g} = \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{\Sigma, f^* g}$,

$$\langle \delta (\mathcal{O}_1 \cdots \mathcal{O}_s) \rangle_{\Sigma, g} = -\frac{1}{2\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \nabla^\mu \epsilon^\nu \langle T_{\mu\nu} \mathcal{O}_1 \cdots \mathcal{O}_s \rangle$$

— Ward identity.

If $T_{\mu\nu}$ is away from $\mathcal{O}_1 \cdots \mathcal{O}_s$

$\Rightarrow \quad \langle \mathcal{O}_1 \cdots \mathcal{O}_s \nabla^\mu T_{\mu\nu} \rangle = 0$

$\therefore$ $\boxed{\nabla^\mu T_{\mu\nu} = 0 \quad \text{as an operator}}$

(valid if it is away from other operator)

--- "E-M conservation equation"

So far everything was about general Q.F.T.

A Definition of Conformal Field Theory

A (general covariant) Q.F.T. is conformally invariant if it is invariant under local scale transformation

$$g \longrightarrow e^\phi g \qquad \phi : \text{a function of } \Sigma$$

"Up to an overall multiplication" in the following sense :

$$Z(\Sigma, e^{\phi} g) = e^{C(g,\phi)} Z(\Sigma, g)$$

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\Sigma, e^{\phi} g} = e^{C_{1,p_1}(g,\phi) + \cdots + C_{s,p_s}(g,\phi)} \langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\Sigma, g}$$

some #'s that depends only on g and ϕ.

i.e. "$\mathcal{D}_{e^{\phi} g} X \, e^{-S_E(e^{\phi} g, X)} = e^{C(g,\phi)} \mathcal{D}_g X \, e^{-S_E(g,X)}$"

In particular, trace of E-M tensor is a C-# :

$$T^{\mu}{}_{\mu} = g^{\mu\nu} T_{\mu\nu} = \langle T^{\mu}{}_{\mu} \rangle$$

and it depends only on the metric. In addition,

- It must be a scalar
- It must have dimension 2. $\left(\underset{\substack{\uparrow \\ \text{dimensionless}}}{\delta_{\phi} C(g,\phi)} = -\frac{1}{4\pi} \int \sqrt{g}\, d^2\sigma \langle T^{\mu}{}_{\mu} \rangle \delta\phi \right)$

Assume that it is a <u>local</u> expression of the metric. Then, it has to be proportional to the <u>scalar curvature R</u>

$$T^{\mu}{}_{\mu} = \langle T^{\mu}{}_{\mu} \rangle = \underbrace{\#}_{\substack{\uparrow \\ \text{numerical constant}}} R .$$

We define the <u>central charge</u> c by

$$\boxed{T^{\mu}_{\mu} = \langle T^{\mu}_{\mu} \rangle = -\frac{c}{12} R}$$

c is a characteristic of the C.F.T.

One can actually reconstruct $C(g,\phi)$ by the requirements:

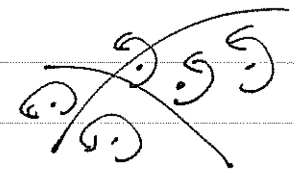
- $C(f^* g, \phi \circ f) = C(g, f)$ by general covariance
- $C(g, \phi_1 + \phi_2) = C(g, \phi_1) + C(e^{\phi_1} g, \phi_2)$ by composition
- $\delta_\phi C(g,\phi)\big|_{\phi=0} = \frac{c}{48\pi} \int \sqrt{g}\, d^2\sigma \, R \cdot \delta\phi$ by $\langle T^{\mu}_{\mu} \rangle = -\frac{c}{12} R$.

$$\Rightarrow \boxed{C(g,\phi) = \frac{c}{48\pi} \int_\Sigma \sqrt{g}\, d^2\sigma \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + R(g)\phi \right)}.$$

From now on, we derive some consequence of $T^{\mu}_{\mu} = -\frac{c}{12} R$ and various Ward identities.

Before that we need a digression on variational calculus, on Riemann surfaces.

Calculus on Riemann Surfaces



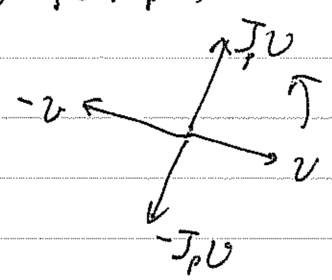
Assume Σ is oriented (i.e. a notion of "positive orientation" is defined at every point and depends continuously)

In 2d, a metric g defines an almost complex structure on Σ.

i.e. a linear transformation $J_p : T_p\Sigma \to T_p\Sigma$ s.t. $J_p^2 = -id_p$

For $\forall v \in T_p\Sigma$, define $J_p v \in T_p\Sigma$ as the vector

- Orthogonal to v : $g_p(v, J_p v) = 0$
- Same norm as v : $g_p(v, v) = g_p(J_p v, J_p v)$
- rotation from v to $J_p v$ is positive :



It is integrable, i.e. one can find a complex coordinate

$$z = x^1 + i x^2 \quad \text{s.t.} \quad J\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial x^2}, \quad J\left(\frac{\partial}{\partial x^2}\right) = -\frac{\partial}{\partial x^1}$$

(Of course that is not unique), but different choice is related by a holomorphic coordinate change.

In short, a metric determines a complex structure

For a complex coordinate $z = x^1 + i x^2$, we put $\bar{z} = x^1 - i x^2$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$$

so that $\begin{pmatrix} \dfrac{\partial z}{\partial z} = 1 & \dfrac{\partial \bar{z}}{\partial z} = 0 \\ \dfrac{\partial z}{\partial \bar{z}} = 0 & \dfrac{\partial \bar{z}}{\partial \bar{z}} = 1 \end{pmatrix}$

If z is holomorphic w.r.t. the complex str. defined by g,

$$g_{zz} = \frac{1}{4} g\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right) = \frac{1}{4}\left(\underbrace{g_{11} - g_{22}}_{=0} - 2i\underbrace{g_{12}}_{=0}\right) = 0$$

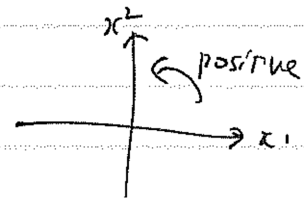
$$g_{\bar{z}\bar{z}} = \frac{1}{4} g\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right) = \frac{1}{4}\left(g_{11} - g_{22} + 2i g_{12}\right) = 0$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{4}\left(g_{11} + g_{22}\right) \neq 0$$

As $g = g_{\mu\nu} dx^\mu dx^\nu$ is valid for <u>any</u> even complex coordinate system, we have

$$g = g_{z\bar{z}}\, dz\, d\bar{z} + g_{\bar{z}z}\, d\bar{z}\, dz = g_{z\bar{z}}\left(dz\, d\bar{z} + d\bar{z}\, dz\right).$$

<u>Example</u> $\Sigma = \mathbb{R}^2$, $g = (dx^1)^2 + (dx^2)^2$



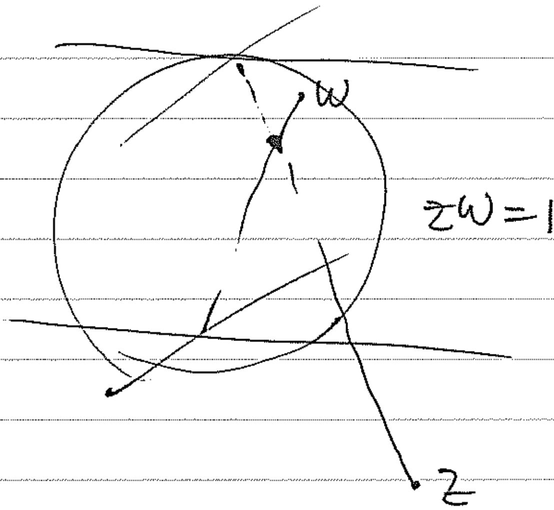
$z = x^1 + i x^2$ is a holomorphic coordinate.

$$\left(g_{11} = g_{22} = 1,\ g_{12} = 0\right)$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{4}\left(g_{11} + g_{22}\right) = \frac{1}{2}$$

$$g = \frac{1}{2}\left(dz\, d\bar{z} + d\bar{z}\, dz\right).$$

example Sphere (of radius $\frac{1}{2}$):

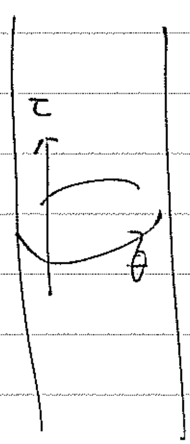


Start with the holomorphic coordinates z & w related by $zw=1$

$$g = \frac{1}{2}\frac{dz d\bar{z} + d\bar{z} dz}{(|z|^2+1)^2} = \frac{1}{2}\frac{dw d\bar{w} + d\bar{w} dw}{(|w|^2+1)^2}.$$

example Cylinder $\quad \zeta = \tau + i\theta \qquad \theta \equiv \theta + 2\pi$

$-\infty < \tau < +\infty$



$$g = (d\tau)^2 + (d\theta)^2 = \frac{1}{2}\left(d\zeta d\bar{\zeta} + d\bar{\zeta} d\zeta\right)$$

$$g_{\zeta\bar{\zeta}} = \frac{1}{2} = g_{\bar{\zeta}\zeta}$$

We may also use $z = e^{\zeta} = e^{\tau + i\theta}$

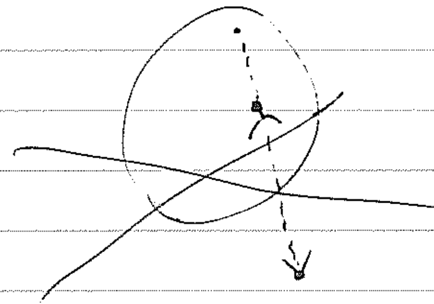
$\Rightarrow \quad dz = z\, d\zeta$

$$\therefore \ g = \frac{1}{2}\left(\frac{dz d\bar{z}}{z\bar{z}} + \frac{d\bar{z} dz}{\bar{z} z}\right) = \frac{1}{2}\frac{dz d\bar{z} + d\bar{z} dz}{|z|^2}$$

$$g_{z\bar{z}} = \frac{1}{2}\frac{1}{|z|^2} = g_{\bar{z}z}$$

$z = \infty$ i.e. $w = 0$

One can regard $\Sigma = \mathbb{R}^2 \cong \mathbb{C}$ as sphere $- \{\infty\}$



Cylinder as Sphere $- \{0, \infty\}$ (and as $\mathbb{C} - \{0\}$)

$$\zeta \longleftrightarrow z = e^{\zeta}$$

Metrics are all different but conformally equivalent.

$$g_{\text{sphere}} = \frac{1}{(1+|z|^2)^2}\, g_{\text{flat } \mathbb{R}^2}$$

$$g_{\text{cylinder}} = \frac{1}{|z|^2}\, g_{\text{flat } \mathbb{R}^2}$$

$$\left(g_{\text{sphere}} = \frac{|z|^2}{(1+|z|^2)^2}\, g_{\text{cylinder}} \right)$$

The Levi-Civita Connection

The L-C connection does not mix holomorphic and anti-holomorphic

$$\Gamma^{z}_{\mu\bar{z}} = \Gamma^{\bar{z}}_{\mu z} = 0$$

In fact all $\Gamma^{\mu}_{\nu\lambda} = 0$ except

$$\Gamma^{z}_{zz} = g^{z\bar{z}}\partial_z g_{z\bar{z}} \quad \text{and} \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = g^{z\bar{z}}\partial_{\bar{z}} g_{z\bar{z}}$$

e.g. use $g_{zz} = g_{\bar{z}\bar{z}} = 0$ in the def

$$\Gamma^{\mu}_{\lambda\nu} = \frac{1}{2} g^{\mu\rho}\left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu}\right)$$

The expression for Γ^{z}_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ can be found using these facts &

$$\partial_z g_{z\bar{z}} = \frac{\partial}{\partial z} g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = g\left(\underbrace{\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z}}_{=\,\Gamma^{z}_{zz}\frac{\partial}{\partial z}}, \frac{\partial}{\partial \bar{z}}\right) + g\left(\frac{\partial}{\partial z}, \underbrace{\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial \bar{z}}}_{=\,0}\right).$$

As a consequence

The curvature also does not mix holo and antiholo:

$$R^{z}{}_{\bar{z}\mu\nu} = R^{\bar{z}}{}_{z\mu\nu} = 0$$

$\therefore$ The only non-vanishing are $R^{z}{}_{zz\bar{z}} = -R^{z}{}_{z\bar{z}z}$ and $R^{\bar{z}}{}_{\bar{z}z\bar{z}} = -R^{\bar{z}}{}_{\bar{z}\bar{z}z}$

$$R^{z}{}_{z z \bar z} = \partial_z \cancel{\Gamma^{z}_{\bar z z}}^{\,0} - \partial_{\bar z} \Gamma^{z}_{z z} + \Gamma^{z}_{z \mu} \cancel{\Gamma^{\mu}_{\bar z z}}^{\,0} - \cancel{\Gamma^{z}_{\bar z \mu}}^{\,0} \Gamma^{\mu}_{z z}$$

$$= -\partial_{\bar z}\left(g^{z\bar z} \partial_z g_{z \bar z}\right) = -g^{z\bar z} \partial_z \partial_{\bar z} g_{z\bar z} + g^{z\bar z} \partial_{\bar z} g_{z\bar z}\, g^{z\bar z} \partial_z g_{z\bar z}$$

it is real, so $R^{\bar z}{}_{\bar z \bar z z} = R^{z}{}_{z z \bar z}$ (There is just one component)

<u>Ricci tensor</u> ($R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$)

$$R_{zz} = \cancel{R^{z}{}_{z z z}}^{\,0} + \cancel{R^{\bar z}{}_{z \bar z z}}^{\,0} = 0, \quad R_{\bar z \bar z} = 0$$

$$R_{z \bar z} = R^{z}{}_{z z \bar z} + \cancel{R^{\bar z}{}_{z \bar z \bar z}} = R^{z}{}_{z z \bar z}$$

<u>Scalar curvature</u> ($R = g^{\mu\nu} R_{\mu\nu}$)

$$R = g^{z\bar z} R_{z\bar z} + g^{\bar z z} R_{\bar z z} = 2 g^{z\bar z} R_{z\bar z}$$

Variation of metric

Consider varying the metric $g \to g + \delta g$.

Since the complex structure is defined by g, it will deform the complex structure. In particular, a "holomorphic coordinate z" will no longer be holomorphic (in general).

If we want z to continued to be holomophic, z must also change: $z \to z + \delta z$. (not unique, of course)

$$g = g_{z\bar z}(dz\,d\bar z + d\bar z\,dz)$$

$$\delta g = \delta(g_{z\bar z})(dz\,d\bar z + d\bar z\,dz) + g_{z\bar z}\big(d\delta z\,d\bar z + dz\,d\delta\bar z + d\delta\bar z\,dz + d\bar z\,d\delta z\big)$$

$$d\delta z = (\partial_{\bar z}\delta z)d\bar z + (\partial_z \delta z)dz$$

$$d\delta\bar z = (\partial_{\bar z}\delta\bar z)d\bar z + (\partial_z\delta\bar z)dz$$

$$= \underbrace{\left(\delta(g_{z\bar z}) + \partial_z\delta z + \partial_{\bar z}\delta\bar z\right)}_{(\delta g)_{z\bar z}}(dz\,d\bar z + d\bar z\,dz) + \underbrace{2g_{z\bar z}\,\partial_z\delta\bar z}_{(\delta g)_{zz}}\,dz\,dz + \underbrace{2g_{z\bar z}\,\partial_{\bar z}\delta z}_{(\delta g)_{\bar z\bar z}}\,d\bar z\,d\bar z$$

$$\therefore \boxed{\begin{aligned} \partial_{\bar z}\delta z &= \frac{1}{2}g^{z\bar z}(\delta g)_{\bar z\bar z} \\ \partial_z\delta\bar z &= \frac{1}{2}g^{z\bar z}(\delta g)_{zz} \end{aligned}}$$

$\delta(\partial_{\bar z}) = ?$, $\delta(\partial_z) = ?$

We want $\partial_z z = 1$, $\partial_z \bar z = 0$
$\partial_{\bar z} z = 0$, $\partial_{\bar z} \bar z = 1$ to continue to hold.

$\Rightarrow$ $0 = \delta(\partial_z) z + \partial_z \delta z$, $0 = \delta(\partial_z)\bar z + \partial_z \delta \bar z$

$0 = \delta(\partial_{\bar z}) z + \partial_{\bar z}\delta z$, $0 = \delta(\partial_{\bar z})\bar z + \partial_{\bar z}\delta\bar z$

$$\therefore \quad \boxed{\begin{aligned} \delta(\partial_z) &= -(\partial_z \delta z)\,\partial_z - (\partial_z \delta \bar z)\,\partial_{\bar z} \\ \delta(\partial_{\bar z}) &= -(\partial_{\bar z}\delta z)\,\partial_z - (\partial_{\bar z}\delta\bar z)\,\partial_{\bar z} \end{aligned}}$$

If we have a vector field $V = V^z \frac{\partial}{\partial z} + V^{\bar z}\frac{\partial}{\partial \bar z}$, even if it does not vary $\delta V = 0$, its components vary as

$$\delta(V^z) = V^z \partial_z \delta z + V^{\bar z}\partial_{\bar z}\delta z$$
$$\delta(V^{\bar z}) = V^z \partial_z \delta \bar z + V^{\bar z}\partial_{\bar z}\delta\bar z .$$

while $(\delta V)^z = (\delta V)^{\bar z} = 0$

Same for 1-form $\omega = \omega_z dz + \omega_{\bar z} d\bar z$. i.e. if $\delta\omega = 0$, then

$$\delta(\omega_z) = -(\partial_z \delta z)\,\omega_z - (\partial_z \delta\bar z)\,\omega_{\bar z}$$
$$\delta(\omega_{\bar z}) = -(\partial_{\bar z}\delta z)\,\omega_z - (\partial_{\bar z}\delta\bar z)\,\omega_{\bar z} .$$

while $(\delta\omega)_z = (\delta\omega)_{\bar z} = 0$

Similarly for higher rank symmetric (or antisym) tensors, e.g.

while $(\delta t)_{\mu\nu} = 0$

$$\begin{cases} \delta(t_{zz}) = -(\partial_z \delta z)\, t_{zz} \times 2 - \partial_z \delta\bar z\,(t_{z\bar z} + t_{\bar z z}) \\ \delta(t_{z\bar z}) = -(\partial_z\delta z)\, t_{z\bar z} - (\partial_z \delta\bar z)\, t_{\bar z\bar z} - (\partial_{\bar z}\delta z)\, t_{zz} - (\partial_{\bar z}\delta\bar z)\, t_{z\bar z} \\ \text{etc} \end{cases}$$

Variation of Levi-Civita Connection $\delta\Gamma$

under $g \to g + \delta g$.

It no longer has holomorphic/antiholomorphic separation.

By a direct computation, we find

$$(\delta\Gamma)^z_{zz} = \partial_z \delta\phi - \frac{1}{2} g^{z\bar z} \nabla_{\bar z} \delta g_{zz} \qquad \left(\delta\phi := g^{z\bar z} (\delta g)_{z\bar z} \right)$$

$$(\delta\Gamma)^z_{\bar z z} = \frac{1}{2} g^{z\bar z} \nabla_z \delta g_{\bar z\bar z} = (\delta\Gamma)^z_{z\bar z}$$

$$(\delta\Gamma)^z_{\bar z\bar z} = \frac{1}{2} g^{z\bar z} \nabla_{\bar z} \delta g_{\bar z\bar z}$$

$$\left(\begin{aligned} &(\delta\Gamma)^{\bar z}_{\bar z\bar z} = \text{c.c. of } (\delta\Gamma)^z_{zz} \\ &(\delta\Gamma)^{\bar z}_{z\bar z} = (\delta\Gamma)^{\bar z}_{\bar z z} = \text{c.c. of } (\delta\Gamma)^z_{\bar z z} = (\delta\Gamma)^z_{z\bar z} \\ &(\delta\Gamma)^{\bar z}_{zz} = \text{c.c. of } (\delta\Gamma)^z_{\bar z\bar z}. \end{aligned} \right)$$

Scalar curvature:

$$\delta R = -g^{\mu\nu} \nabla_\mu \partial_\nu \delta\phi - R\,\delta\phi + g^{z\bar z} g^{z\bar z} \nabla_z \nabla_z \delta g_{\bar z\bar z} + g^{z\bar z} g^{z\bar z} \nabla_{\bar z} \nabla_{\bar z} \delta g_{zz}.$$

A General 1st Order System

$$S_E = \frac{1}{\pi}\int d^2z\ \underbrace{b_{z\cdots z}}_{\lambda}\ \nabla_{\bar z}\ C^{\overbrace{z\cdots z}^{\lambda-1}}$$

$$z = x^1 + i x^2 \qquad d^2z = \frac{i}{2}\, dz\wedge d\bar z = dx^1 dx^2$$

If we vary the metric $g \to g + \delta g$ (that induces $z \to z + \delta z$), how does S_E vary?

$$\boxed{\text{First, how should we vary } b_{z\cdots z} \text{ and } C^{z\cdots z}\ ?}$$

We want $b = b_{z\cdots z}(dz)^\lambda$, $C = C^{z\cdots z}(dz)^{1-\lambda}$ to continue to be of the form $b_{z'\cdots z'}(dz')^\lambda$, $C^{z'\cdots z'}(dz')^{1-\lambda}$, where $z' = z + \delta z$ is the new holomorphic coordinate <u>after</u> deformation $g \to g + \delta g$.

$$b_{z'\cdots z'} = b_{z\cdots z} - \lambda(\partial_z \delta z)\, b_{z\cdots z} - (\partial_z \delta \bar z)(\underset{0}{\underset{\parallel}{b_{\bar z z\cdots z}}} + \underset{0}{\underset{\parallel}{b_{z\bar z z\cdots z}}} + \underset{0}{\underset{\parallel}{b_{\cdots}}} \cdots + \underset{0}{\underset{\parallel}{b_{z\cdots z\bar z}}})$$

$$= (1 - \lambda(\partial_z \delta z))\, b_{z\cdots z}$$

$$(dz')^\lambda = (dz + d\delta z)^\lambda = (dz + (\partial_z \delta z)\, dz + (\partial_{\bar z}\delta z)\, d\bar z)^\lambda$$

$$= (dz)^\lambda + \lambda(\partial_z \delta z)(dz)^\lambda + (\partial_{\bar z}\delta z)(d\bar z\, dz\cdots dz + dz\, d\bar z\, dz\cdots dz + \cdots + dz\cdots dz\, d\bar z)$$

$$\therefore\ b' = b_{z'\cdots z'}(dz')^{\lambda}$$

$$= \left(1-\lambda(\partial_z\delta z)\right) b_{z\cdots z}\left\{ (dz)^{\lambda} + \lambda(\partial_z\delta z)(dz)^{\lambda} + (\partial_{\bar z}\delta \bar z)\left(d\bar z(dz)^{\lambda-1} + \cdots + (dz)^{\lambda-1}d\bar z\right)\right\}$$

$$= b_{z\cdots z}(dz)^{\lambda} - \cancel{\lambda(\partial_z\delta z)\, b_{z\cdots z}(dz)^{\lambda}} + \cancel{b_{z\cdots z}\lambda(\partial_z\delta z)(dz)^{\lambda}}$$

$$+\ b_{z\cdots z}\underbrace{(\partial_{\bar z}\delta z)}_{=\frac{1}{2}g^{z\bar z}(\delta g)_{\bar z\bar z}}\left(d\bar z(dz)^{\lambda-1} + \cdots + (dz)^{\lambda-1}d\bar z\right)$$

$$= b + \frac{1}{2}g^{z\bar z}(\delta g)_{\bar z\bar z}\, b_{z\cdots z}\left(d\bar z(dz)^{\lambda-1} + \cdots + (dz)^{\lambda-1}d\bar z\right)$$

$\cdots$ unambiguously determed by δg.

<u>Same for c.</u>