$$\delta S_E = \frac{1}{\pi}\int\left(d^2z'\, b_{z'\cdots z'}\nabla'_{\bar z'} c^{z'\cdots z'} - d^2z\, b_{z\cdots z}\nabla_{\bar z} c^{z\cdots z}\right)$$

$$d^2z' = \frac{i}{2}\, dz'\wedge d\bar z' \Rightarrow \delta(d^2z) = \frac{i}{2}\left(d\,\delta z\wedge d\bar z + dz\wedge d\,\delta\bar z\right)$$

$$= d^2z\cdot\left(\partial_z\delta z + \partial_{\bar z}\delta\bar z\right)$$

$$\delta\Big(\underbrace{b_{z\cdots z}}_{(\lambda-1)+1}\nabla_{\bar z}\, c^{\overbrace{z\cdots z}^{\lambda-1}}\Big) =$$ Variation of indices ${}_{z\cdots z}\,{}_{\bar z}\,{}^{z\cdots z}$

and Variation of ∇ itself.

$(\lambda-1)$ Variations of "lower z's" and "upper z's" cancel out.

$$\delta\left(b_{z\cdots z}\nabla_{\bar z}\, c^{z\cdots z}\right) = -\partial_z \delta z\; b_{z\cdots z}\nabla_{\bar z}\, c^{z\cdots z} \quad \leftarrow \quad b_{z\cdots (\delta z)}\nabla_{\bar z}\, c^{z\cdots z}$$

$$-\partial_{\bar z}\delta z\; b_{z\cdots z}\nabla_z\, c^{z\cdots z} - \partial_{\bar z}\delta \bar z\; b_{z\cdots z}\nabla_{\bar z}\, c^{z\cdots z} \quad \leftarrow \quad b_{z\cdots z}\nabla_{(\partial \bar z)}\, c^{z\cdots z}$$

$$+\; b_{z\cdots z}(\delta\nabla)_{\bar z}\, c^{z\cdots z}$$

$$\delta\left(d^2z\cdot b_{z\cdots z}\nabla_{\bar z}\, c^{z\cdots z}\right) = d^2z\Big(\underbrace{-\partial_{\bar z}\delta z}_{=\;\frac{1}{2}g^{z\bar z}\delta g_{\bar z\bar z}}\; b_{z\cdots z}\nabla_z\, c^{z\cdots z} + b_{z\cdots z}\underbrace{(\delta\nabla)_{\bar z}\, c^{z\cdots z}}_{=\;(\lambda-1)\underbrace{(\delta\Gamma)^{\bar z}_{\bar z\bar z}}_{=\;\frac{1}{2}g^{z\bar z}\nabla_z\delta g_{\bar z\bar z}}\, c^{z\cdots z}}\Big)$$

$$\therefore\quad \delta S_E = \frac{1}{\pi}\int_\Sigma d^2z\left(-\frac{1}{2}g^{z\bar z}\delta g_{\bar z\bar z}\; b_{z\cdots z}\nabla_z\, c^{z\cdots z} + \frac{1}{2}(\lambda-1)\, g^{z\bar z}\nabla_z\delta g_{\bar z\bar z}\; b_{z\cdots z}\, c^{z\cdots z}\right)$$

use Stokes Thm to make it into

$$-\frac{1}{2}(\lambda-1)\, g^{z\bar z}\delta g_{\bar z\bar z}\nabla_z\left(b_{z\cdots z}\, c^{z\cdots z}\right)$$

$$= -\frac{1}{2\pi}\int_\Sigma \underbrace{d^2z\; g^{z\bar z}\delta g_{\bar z\bar z}}_{=\; g_{z\bar z}\, d^2z\; g^{z\bar z}\delta g_{\bar z\bar z}\, g^{z\bar z} \;=\; -\frac{1}{2}\sqrt{g}\, d^2\sigma\, (\delta g)^{zz}}\left(\lambda\, b_{z\cdots z}\nabla_z\, c^{z\cdots z} + (\lambda-1)\nabla_z b_{z\cdots z}\cdot c^{z\cdots z}\right)$$

$$= -\frac{1}{4\pi}\int \sqrt{g}\, d^2\sigma\, (\delta g)^{zz}\left(-\lambda\, b_{z\cdots z}\nabla_z\, c^{z\cdots z} + (1-\lambda)\nabla_z b_{z\cdots z}\cdot c^{z\cdots z}\right)$$

$$\boxed{T_{zz} = -\lambda\, b_{z\cdots z} \nabla_z c^{z\cdots z} + (1-\lambda) \nabla_z b_{z\cdots z}\, c^{z\cdots z}}$$

Recall Dirac fermion

$$S_E = \frac{1}{2\pi}\int d^2\sigma \left(\overline{\psi_-} \underbrace{(\partial_\tau - i\partial_\sigma)}_{= 2\partial_{\bar z}} \psi_- + \overline{\psi_+} \underbrace{(\partial_\tau + i\partial_\sigma)}_{= 2\partial_z} \psi_+ \right) \qquad (z = \tau - i\sigma)$$

$$= \frac{1}{\pi}\int d^2\sigma \left(\overline{\psi_-}\, \partial_{\bar z} \psi_- + \overline{\psi_+}\, \partial_z \psi_+ \right)$$

$\Longleftrightarrow$ $\lambda = \frac{1}{2}$ system $b = \overline{\psi_-}$, $c = \psi_-$

$$\boxed{\begin{aligned} \therefore\ T_{zz} &= -\frac{1}{2} \overline{\psi_-} D_z \psi_- + \frac{1}{2} D_z \overline{\psi_-} \cdot \psi_- \\ T_{\bar z \bar z} &= -\frac{1}{2} \overline{\psi_+} D_{\bar z} \psi_+ + \frac{1}{2} D_{\bar z} \overline{\psi_+} \cdot \psi_+ \end{aligned}}$$

… Differ from the ones defined as Noether Currents,

but the conserved charges are equivalent

H & P

Consequence of the Ward id $\langle \nabla^\mu T_{\mu\nu} \cdots \rangle = 0$

Note: $\underbrace{T^\mu{}_\mu}_{= -\frac{c}{12} R} = g^{\mu\nu} T_{\mu\nu} = g^{z\bar{z}} T_{z\bar{z}} + g^{\bar{z}z} T_{\bar{z}z} = 2 g^{z\bar{z}} T_{z\bar{z}}$

$$\boxed{\therefore \ T_{z\bar{z}} = -\frac{c}{24} g_{z\bar{z}} R}$$

$$\therefore \ \left\langle \left(\nabla^z T_{zz} + \underbrace{\nabla^{\bar{z}} T_{\bar{z}z}}_{= -\frac{c}{24} \nabla^{\bar{z}} (g_{z\bar{z}} R) = -\frac{c}{24} g_{z\bar{z}} \nabla^{\bar{z}} R = -\frac{c}{24} \partial_z R} \right) \cdots \right\rangle_{\Sigma, g} = 0$$

$$\therefore \ \left\langle \left(\nabla^z T_{zz} - \frac{c}{24} \partial_z R \right) \cdots \right\rangle_{\Sigma, g} = 0$$

Take the variation $g \to g + \delta g$ of this equation.

$$0 = \left\langle \left(\delta(\nabla^z T_{zz}) - \frac{c}{24} \delta(\partial_z R) \right) \cdots \right\rangle$$

$$+ \left\langle \left(\nabla^z T_{zz} - \frac{c}{24} \partial_z R \right) \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma\, \delta g^{\mu\nu} T_{\mu\nu} \cdots \right\rangle$$

This does not have to vanish because of possible singularity in $\langle \nabla^z T_{zz}(\rho) T_{\mu\nu}(\sigma) \rangle$ as $\sigma \to \rho$.

Focus on the Variation $\delta g_{\bar{z}\bar{z}} = -g_{z\bar{z}}{}^2 \delta g^{zz}$

$$\delta(\nabla^z T_{zz}) = -\frac{1}{2}(g^{z\bar{z}})^2 \delta g_{\bar{z}\bar{z}} \nabla_z T_{zz} - (g^{z\bar{z}})^2 \nabla_z \delta g_{\bar{z}\bar{z}} T_{zz}$$

$$\delta(\partial_z R) = (g^{z\bar{z}})^2 \nabla_z \nabla_z \nabla_z \delta g_{\bar{z}\bar{z}}$$

Thus, we find that

$$0 = \left[-\frac{1}{2}(g^{z\bar{z}})^2 \delta g_{\bar{z}\bar{z}} \nabla_z T_{zz} - (g^{z\bar{z}})^2 \nabla_z \delta g_{\bar{z}\bar{z}} T_{zz} - \frac{c}{24}(g^{z\bar{z}})^2 \nabla_z^3 \delta g_{\bar{z}\bar{z}} \right](p)$$

$$+ \nabla^z T_{zz}(p) \cdot \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, d^2\sigma \underbrace{\delta g^{zz}}_{= -(g^{z\bar{z}})^2 \delta g_{\bar{z}\bar{z}}} T_{zz}$$

holds inside correlators.

This meas the following operator relation:

$$\boxed{\begin{aligned} \frac{1}{4\pi} \nabla^z T_{zz}(z) T_{ww}(w) = & -\frac{c}{24} \nabla_z^3 \delta^{(2)}(z-w) - \nabla_z \delta^{(2)}(z-w) T_{ww}(w) \\ & + \frac{1}{2} \delta^{(2)}(z-w) \nabla_w T_{ww}(w) \quad + \text{regular as } z \to w. \end{aligned}}$$

For simplicity, let us consider flat (region of (Σ, g))

and focus only on the singular behaviour as $z \to w$

- $\delta^{(2)}(z-w) = \frac{1}{\pi} \partial_{\bar{z}} \left(\frac{1}{z-w} \right)$
- $\nabla^{z} T_{zz} = g^{z\bar{z}} \partial_{\bar{z}} T_{zz} = 2 \partial_{\bar{z}} T_{zz}$

$\partial_{\bar{z}}$ appears everywhere. Removing it, we have

$$T_{zz}(z) T_{ww}(w) = -\frac{c}{12} \partial_z^3 \left(\frac{1}{z-w} \right) - 2\partial_z \left(\frac{1}{z-w} \right) T_{ww}(w)$$

$$+ \frac{1}{z-w} \partial_w T_{ww}(w) \quad + \text{reg.}$$

- $\partial_z^3 \left(\frac{1}{z-w} \right) = \partial_z^2 \frac{-1}{(z-w)^2} = \partial_z \frac{2}{(z-w)^3} = \frac{-6}{(z-w)^4}$
- $\partial_z \left(\frac{1}{z-w} \right) = \frac{-1}{(z-w)^2}$

We find an operator product relation

$$\boxed{T_{zz}(z) T_{ww}(w) = \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2}{(z-w)^2} T_{ww}(w) + \frac{1}{z-w} \partial_w T_{ww}(w) + \text{regular}}$$

"AFOO."

Note : $\nabla^z T_{zz} + \nabla^{\bar z} T_{\bar z z} = 0$ (away from other operators)

$$\nabla^z T_{zz} = g^{z\bar z}\partial_{\bar z} T_{zz} - \frac{c}{24}\partial_z R$$

Thus, on a flat region of (Σ, g) , $\partial_{\bar z} T_{zz} = 0$ (AFOO)

i.e. T_{zz} is holomorphic in flat region.

But, on a curved region , T_{zz} is not holomorphic.

In fact, it is straightforward to show

$$\partial_z R = -g^{z\bar z}\partial_{\bar z}\left(2\,\partial_z\partial_z \log g_{z\bar z} - (\partial_z \log g_{z\bar z})^2\right).$$

Thus

$$0 = \nabla^z T_{zz} + \nabla^{\bar z} T_{\bar z z} = g^{z\bar z}\partial_{\bar z} T^{hol}_{zz}$$

where $\boxed{T^{hol}_{zz} := T_{zz} + \frac{c}{24}\left(2\partial_z\partial_z \log g_{z\bar z} - (\partial_z \log g_{z\bar z})^2\right)}$

- T^{hol}_{zz} is holomorphic (AFOO) even if $R \neq 0$
- Also, $T^{hol}_{zz}(z)\,T^{hol}_{ww}(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T^{hol}_{ww}(w) + \frac{1}{z-w}\partial_w T^{hol}_{ww}(w)$

  $+$ reg.

  even if $R \neq 0$.

Change of notation

From now on

$$T_{\mu\nu} \longrightarrow T^c_{\mu\nu} \leftarrow \text{covariant}$$

$$T^{hol}_{zz} \longrightarrow T_{zz}$$

$$T^{antihol}_{\bar z\bar z} \longrightarrow T_{\bar z\bar z}$$

i.e. $$T_{zz} = T^c_{zz} + \frac{c}{24}\left(2\partial_z\partial_z \log g_{z\bar z} - (\partial_z \log g_{z\bar z})^2\right)$$

$$T_{\bar z\bar z} = T^c_{\bar z\bar z} + \frac{c}{24}\left(2\partial_{\bar z}\partial_{\bar z} \log g_{z\bar z} - (\partial_{\bar z} \log g_{z\bar z})^2\right)$$

so that $\partial_{\bar z} T_{zz} = \partial_z T_{\bar z\bar z} = 0$ (AFOO) even if $R \neq 0$.

Note ① On a flat region with $g_{z\bar z} = $ constant

$$T_{zz} = T^c_{zz}, \quad T_{\bar z\bar z} = T^c_{\bar z\bar z}$$

e.g. $\Sigma = \mathbb{R}^2$ $\quad g = \frac{1}{2}(dz\,d\bar z + d\bar z\,dz)$ $\quad (g_{z\bar z} = \frac{1}{2})$.

However, if $g_{z\bar z}$ is not constant (even if $R=0$)

They can be different: e.g. $\Sigma = $ Cylinder $g = \dfrac{dz\,d\bar z + d\bar z\,dz}{2|z|^2}$

$g_{\bar z z} = \frac{1}{2|z|^2}$: $\quad T_{zz} = T^c_{zz} + \frac{c}{24}\frac{1}{z^2}$

(But for ζ, $z = e^{\zeta}$ $\quad T_{\zeta\zeta} = T^c_{\zeta\zeta}$ $\quad \left(g = \dfrac{d\zeta\,d\bar\zeta + d\bar\zeta\,d\zeta}{2}\right)$)

Note ② The last example shows that T_{zz}, $T_{\bar{z}\bar{z}}$ are not covariant!

($T^c_{\mu\nu}$ is covariant by definition!)

For a holomorphic coordinate change $z \to w(z)$

$$T_{zz} = (w'(z))^2 T_{ww} + \frac{c}{12}\{w, z\}$$

$$T_{\bar{z}\bar{z}} = \overline{(w'(z))^2}\, T_{\bar{w}\bar{w}} + \frac{c}{12}\overline{\{w, z\}}.$$

$$\{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 \quad \text{Schwarzian derivative}$$

e.g. Cylinder $z = e^{\zeta}$ $(\zeta \equiv \zeta + 2\pi i)$

$$T_{\zeta\zeta} = (e^{\zeta})^2 T_{zz} + \frac{c}{12}\left[\frac{e^{\zeta}}{e^{\zeta}} - \frac{3}{2}\left(\frac{e^{\zeta}}{e^{\zeta}}\right)^2\right] = z^2 T_{zz} - \frac{c}{24}$$

indeed it is compatible with

$$T^c_{\zeta\zeta} = T^c_{zz}\left(\frac{\partial z}{\partial \zeta}\right)^2 = z^2 T^c_{zz}$$

$$\begin{cases} T_{zz} = T^c_{zz} + \dfrac{c}{24}\dfrac{1}{z^2} \\ T_{\zeta\zeta} = T^c_{\zeta\zeta} \end{cases}$$

We will next look at the Operator product relations of T_{zz} ($, T_{\bar{z}\bar{z}}$) in

- massless scalar theory (or σ-model on $S^1_{2\pi R}$)
- Dirac fermion. massless

Before doing it, let us formulate these theories on $\Sigma = \mathbb{R}^2$, $g = \dfrac{dz d\bar{z} + d\bar{z} dz}{2}$.

[We have done that on Σ = Cylinder $\quad \varsigma \equiv \varsigma + 2\pi i$

$$g = \frac{d\varsigma d\bar{\varsigma} + d\bar{\varsigma} d\varsigma}{2} = \frac{dz d\bar{z} + d\bar{z} dz}{2|z|^2} \qquad z = e^{\varsigma}$$
]

- massless scalar

$d^2\sigma = d^2 z = \frac{i}{2} dz \wedge d\bar{z}$

$$S_E = \frac{1}{4\pi}\int_{\mathbb{R}^2} d^2\sigma\, \partial_\mu X \partial_\mu X = \frac{1}{\pi}\int_{\mathbb{R}^2} d^2 z\, \partial_z X \partial_{\bar{z}} X$$

$$= \frac{1}{2}\int d^2 z\, X\left(-\frac{2}{\pi}\partial_z \partial_{\bar{z}}\right) X$$

$$\therefore \quad -\frac{2}{\pi}\partial_z \partial_{\bar{z}} \langle X(z) X(w) \rangle = \delta^{(2)}(z-w)$$

$$\Rightarrow \quad \langle X(z)\, X(w) \rangle = -\log|z-w| \quad + \text{const}$$

$$\therefore \quad \langle \partial_z X(z)\, X(w) \rangle = -\frac{1}{2}\frac{1}{z-w}$$

$$\langle \partial_z X(z)\, \partial_w X(w) \rangle = -\frac{1}{2}\frac{1}{(z-w)^2}$$

cf. $$\langle \partial_{\bar{z}} X(z)\, \partial_{\bar{w}} X(w) \rangle = -\frac{1}{2}\frac{1}{(\bar{z}-\bar{w})^2}$$

c.f. On the cylinder, we obtained

$$\langle X(1)\, X(2) \rangle = -\log\left|e^{\tau_1 - i\sigma_1} - e^{\tau_2 - i\sigma_2}\right| + \frac{\tau_1+\tau_2}{2} + \text{const}$$

$$= -\log|z_1 - z_2| \quad + \frac{1}{2}\log|z_1 z_2| \quad + \text{const.}$$

The results agree as long as we consider

$$\langle \partial_{\mu_1} \cdots \partial_{\mu_n} X(1)\, \partial_{\nu_1} \cdots \partial_{\nu_m} X(w) \rangle \qquad \text{with} \quad n, m \geq 1$$

Correlators with at least one derivative

· massless Dirac

$$S_E = \frac{1}{\pi} \int d^2 z \left(\bar{\psi}_- \partial_{\bar z} \psi_- + \bar{\psi}_+ \partial_z \psi_+ \right)$$

$$\therefore \quad \frac{1}{\pi} \partial_{\bar z} \langle \psi_-(z) \bar{\psi}_-(w) \rangle = \delta^{(2)}(z-w)$$

$$\frac{1}{\pi} \partial_z \langle \psi_+(z) \bar{\psi}_+(w) \rangle = \delta^{(2)}(z-w)$$

$$\Rightarrow \quad \langle \psi_-(z) \bar{\psi}_-(w) \rangle = \frac{1}{z-w}$$

$$\langle \psi_+(z) \bar{\psi}_+(w) \rangle = \frac{1}{\bar z - \bar w}$$

c.f. On the cylinder, we had $\langle \psi_-^{cyl}(1) \bar{\psi}_-^{cyl}(2) \rangle = \dfrac{z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}}{z_1 - z_2}$

Note $\psi_-^{cyl} (d\zeta)^{\frac{1}{2}} = \psi_-^{plane} (dz)^{\frac{1}{2}}$ $\qquad (dz)^{\frac{1}{2}} = (e^{\zeta} d\zeta)^{\frac{1}{2}} = z^{\frac{1}{2}} (d\zeta)^{\frac{1}{2}}$

$\therefore \psi_-^{cyl} = z^{\frac{1}{2}} \psi_-^{plane}$. Similarly $\bar{\psi}_-^{cyl} = z^{\frac{1}{2}} \bar{\psi}_-^{plane}$

$\psi_+^{cyl} = \bar{z}^{\frac{1}{2}} \psi_+^{plane}$, $\bar{\psi}_+^{cyl} = \bar{z}^{\frac{1}{2}} \bar{\psi}_+^{plane}$

$$\therefore \langle \psi_-^{plane}(1) \bar{\psi}_-^{plane}(2) \rangle = z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \langle \psi_-^{cyl}(1) \bar{\psi}_-^{cyl}(2) \rangle = \frac{1}{z_1 - z_2}$$

$$\langle \psi_+^{plane}(1) \bar{\psi}_+^{plane}(2) \rangle = \bar{z}_1^{-\frac{1}{2}} \bar{z}_2^{-\frac{1}{2}} \langle \psi_+^{cyl}(1) \bar{\psi}_+^{cyl}(2) \rangle = \frac{1}{\bar z_1 - \bar z_2}$$

perfect match!

OPE of T_{zz}'s

• massless scalar $\quad T_{zz} = - :\partial_z X \partial_z X:$

$$T_{zz}(z)\, T_{ww}(w) = (-1)^2 :\partial_z X \partial_z X \partial_w X \partial_w X: \times 2$$

$$+ (-1)^2 :\partial_z X \partial_z X \partial_w X \partial_w X: \times 4 \quad + (-1)^2 :\partial_z X \partial_z X \partial_w X \partial_w X:$$

$$= 2\left(-\frac{1}{2}\frac{1}{(z-w)^2}\right)^2 + 4\times\left(-\frac{1}{2}\frac{1}{(z-w)^2}\right)\underbrace{:\partial_z X \partial_w X:}_{:\partial_w X \partial_w X: + (z-w):\partial_w^2 X \partial_w X: + \cdots} + :(\partial_z X)^2 (\partial_w X)^2:$$

$$= \frac{1}{2}\frac{1}{(z-w)^4} - \frac{2}{(z-w)^2}\underbrace{:\partial_w X \partial_w X:}_{-T_{ww}} - \frac{2}{z-w}\underbrace{:\partial_w^2 X \partial_w X:}_{-\frac{1}{2}\partial_w T_{ww}}$$

$+$ regular

$$= \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}T_{ww} + \frac{1}{z-w}\partial_w T_{ww} \quad + \text{reg.}$$

∴ $\boxed{\text{massless scalar theory is a } c=1 \text{ C.F.T.}}$

· <u>massless Dirac</u> $\quad T_{zz} = -\frac{1}{2} :\overline{\psi}_- \partial_z \psi_-: + \frac{1}{2} :\partial_z \overline{\psi}_- \psi_-:$

$$T_{zz}(z)\, T_{ww}(w) = \cdots\cdots$$

$$= \frac{1}{2} \frac{1}{(z-w)^4} + \frac{2}{(z-w)^2} T_{ww} + \frac{1}{z-w} \partial_w T_{ww} + \text{reg.}$$

<u>exercise</u>

$\therefore$ massless Dirac fermion is also a $c=1$ C.F.T.

Just to find the central charge, compute

$$\langle T_{zz}(z)\, T_{ww}(w) \rangle = \frac{1}{4} \Big[\overline{\psi}_-(z) \partial_z \psi_-(z) \overline{\psi}_-(w) \partial_w \psi_-(w) - \overline{\psi}_-(z) \partial_z \psi_-(z) \partial_w \overline{\psi}_-(w) \psi_-(w)$$

$$- \partial_z \overline{\psi}_-(z) \psi_-(z) \overline{\psi}_-(w) \partial_w \psi_-(w) + \partial_z \overline{\psi}_-(z) \psi_-(z) \partial_w \overline{\psi}_-(w) \psi_-(w) \Big]$$

$$= \frac{1}{4} \left[\frac{1}{(z-w)^2} \frac{-1}{(z-w)^2} - \frac{1}{z-w} \cdot \frac{-2}{(z-w)^3} - \frac{-2}{(z-w)^3} \frac{1}{z-w} + \frac{-1}{(z-w)^2} \cdot \frac{1}{(z-w)^2} \right]$$

$$= \frac{1}{2} \frac{1}{(z-w)^4} \qquad \boxed{\therefore\ c = 1}$$