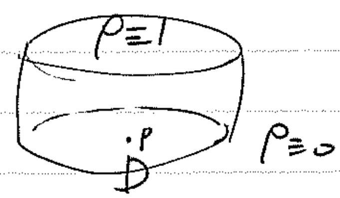
Ward identity

$$\langle \delta\mathcal{O}(p)\, \mathcal{O}_1' \cdots \mathcal{O}_s' \rangle = \frac{1}{2\pi}\int_\Sigma \sqrt{g}\, d^2\sigma\, \epsilon^\nu \nabla^\mu \langle T^c_{\mu\nu}\, \mathcal{O}(p)\, \mathcal{O}_1' \cdots \mathcal{O}_s' \rangle$$

if $\epsilon(p_1') = \cdots = \epsilon(p_s') = 0$.

Take $\epsilon = \rho\, \underset{\substack{\uparrow \\ \text{holomorphic}}}{\epsilon^z(z)} \dfrac{\partial}{\partial z}$



$$\rho = \begin{cases} 1 & \text{in } D \\ 0 & \text{outside} \end{cases}$$

operator identity

$$\Rightarrow \quad \delta_\epsilon \mathcal{O}(p) = \frac{1}{2\pi}\int_D \underbrace{\sqrt{g}\, d^2\sigma}_{= i g_{z\bar z}\, dz\wedge d\bar z}\, \epsilon^z \underbrace{\nabla^\mu T^c_{\mu z}}_{= g^{z\bar z}\partial_{\bar z} T^{hol}_{zz}}\, \mathcal{O}(p)$$

$T^{hol}_{zz} = T_{zz}$

$$= \frac{i}{2\pi}\int_D dz\wedge d\bar z\, \epsilon^z(z)\, \partial_{\bar z} T_{zz}\, \mathcal{O}(p)$$

$$= \frac{-i}{2\pi}\int_D d\left(dz\, \epsilon^z(z)\, T_{zz}(z)\, \mathcal{O}(p) \right)$$

$$= \frac{1}{2\pi i}\oint_{\partial D} dz\, \epsilon^z(z)\, T_{zz}(z)\, \mathcal{O}(p)$$

$$= \frac{1}{2\pi i}\oint_p dz\, \epsilon^z(z)\, T_{zz}(z)\, \mathcal{O}(p)$$

↑ small contour that encircles <u>p only</u>.

For $\epsilon = \rho\, \epsilon^{\bar z}(\bar z) \frac{\partial}{\partial \bar z}$

$$\delta_{\bar\epsilon} \mathcal{O}(p) = \frac{i}{2\pi} \int_D \underbrace{dz\, d\bar z\ \epsilon^{\bar z}(\bar z)\, \partial_z T_{\bar z \bar z}\, \mathcal{O}(p)}_{d\left(d\bar z\ \epsilon^{\bar z}(\bar z) T_{\bar z \bar z}\, \mathcal{O}(p)\right)}$$

$$= \frac{i}{2\pi} \oint_{\partial D} d\bar z\ \epsilon^{\bar z}(\bar z)\ T_{\bar z \bar z}\ \mathcal{O}(p)$$

$$= -\frac{1}{2\pi i} \oint_p d\bar z\ \epsilon^{\bar z}(\bar z)\ T_{\bar z \bar z}\ \mathcal{O}(p)$$

Thus Ward ids for holomorphic and antiholomorphic vectors are

$$\boxed{\begin{aligned} \delta_\epsilon \mathcal{O}(p) &= \frac{1}{2\pi i} \oint_p dz\ \epsilon^z(z)\ T_{zz}(z)\ \mathcal{O}(p) \\ \delta_{\bar\epsilon} \mathcal{O}(p) &= -\frac{1}{2\pi i} \oint_p d\bar z\ \epsilon^{\bar z}(\bar z)\, T_{\bar z \bar z}(z)\ \mathcal{O}(p) \end{aligned}}$$

This defines an action of <u>hol / antihol vector fields</u> on the space of local operators.

infinitesmal conformal transf.

e.g.

$$\delta_\epsilon T_{ww}(w) = \frac{1}{2\pi i}\oint_w dz\, \overset{\epsilon(z)}{\epsilon^z(z)}\, \underbrace{T_{zz}(z)\, T_{ww}(w)}_{\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T_{ww} + \frac{1}{z-w}\partial_w T_{ww} + \text{reg}}$$

$$= \frac{1}{2\pi i}\oint_w dz \left\{ \frac{\epsilon(z)\, c/2}{(z-w)^4} + \frac{2\epsilon(z)}{(z-w)^2} T_{ww} + \frac{\epsilon(z)}{z-w}\partial_w T_{ww} \right\}$$

$$\epsilon(z) = \epsilon(w) + (z-w)\epsilon'(w) + \frac{1}{2}(z-w)^2\epsilon''(w) + \frac{1}{6}(z-w)^3\epsilon'''(w) + O((z-w)^4)$$

$$\epsilon(z) = \epsilon(w) + (z-w)\epsilon'(w) + O((z-w)^2)$$

$$= \frac{c}{12}\epsilon'''(w) + 2\epsilon'(w) T_{ww}(w) + \epsilon(w)\partial_w T_{ww}$$

Infinitesimal form of $T^{(w)}_{ww} = \left(z'(w)\right)^2 T_{zz}(z) + \frac{c}{12}\{z, w\}$

Similary

$$\delta_{\bar\epsilon} T_{\bar w\bar w}(\bar w) = \frac{c}{12}\bar\epsilon'''(\bar w) + 2\bar\epsilon'(\bar w) T_{\bar w\bar w} + \bar\epsilon(\bar w)\partial_{\bar w} T_{\bar w\bar w}$$

We denote the action of $z^{n+1}\frac{\partial}{\partial z}$, $\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$ on operators at $z=0$ as L_n, $\tilde{L}_n$:

$$\boxed{\begin{aligned} (L_n \mathcal{O})(0) &= \frac{1}{2\pi i}\oint_0 dz\, z^{n+1}\, T_{zz}(z)\, \mathcal{O}(0) \\ (\tilde{L}_n \mathcal{O})(0) &= -\frac{1}{2\pi i}\oint_0 d\bar{z}\, \bar{z}^{n+1}\, T_{\bar{z}\bar{z}}(\bar{z})\, \mathcal{O}(0) \end{aligned}}$$

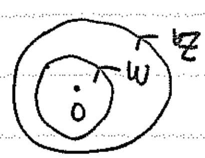
(L_n, $\tilde{L}_n$ can also be defined on operators at any point p :

$$(L_n \mathcal{O})(p) = \frac{1}{2\pi i}\oint_p dz\, (z - z(p))^{n+1}\, T_{zz}(z)\, \mathcal{O}(p)$$

$$(\tilde{L}_n \mathcal{O})(p) = -\frac{1}{2\pi i}\oint_p d\bar{z}\, (\bar{z} - \bar{z}(p))^{n+1}\, T_{\bar{z}\bar{z}}(\bar{z})\, \mathcal{O}(p) \quad)$$

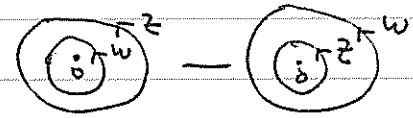
The algebra of L_n's & $\tilde{L}_n$'s

$$(L_n(L_m \mathcal{O}))(0) = \oint \frac{dz}{2\pi i} z^{n+1} T_{zz}(z) \oint \frac{dw}{2\pi i} w^{m+1} T_{ww}(w)\, \mathcal{O}(0)$$

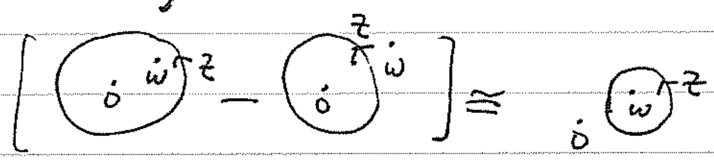


$$([L_n, L_m]\mathcal{O})(0) = (L_n(L_m\mathcal{O}))(0) - (L_m(L_n\mathcal{O}))(0)$$

$$= \int \frac{dz}{2\pi i}\frac{dw}{2\pi i} z^{n+1} w^{m+1} T_{zz}(z) T_{ww}(w) \mathcal{O}(0)$$



$$= \oint_0 \frac{dw}{2\pi i} w^{m+1} \int \frac{dz}{2\pi i} z^{n+1} T_{zz}(z) T_{ww}(w) \mathcal{O}(0)$$



$$= \oint_0 \frac{dw}{2\pi i} w^{m+1} \left\{ \frac{c}{12}(w^{n+1})''' + 2(w^{n+1})' T_{ww}(w) + w^{n+1}\partial_w T_{ww}(w) \right\} \mathcal{O}(0)$$

$$= \oint_0 \frac{dw}{2\pi i} \left\{ \underbrace{\frac{c}{12} w^{m+1}(w^{n+1})'''}_{= \frac{c}{12} w^{m+n-1}(n+1)n(n-1)} + \underbrace{2w^{m+1}(w^{n+1})' T_{ww} - (w^{m+n+2})' T_{ww}}_{= (2(n+1)-(m+n+2)) w^{m+n+1} T_{ww}} \right\} \mathcal{O}(0)$$

$$= \frac{c}{12}\delta_{m+n,0}(n^3-n)\mathcal{O}(0) + (n-m)\underbrace{\oint_0 \frac{dw}{2\pi i} w^{m+n+1} T_{ww} \mathcal{O}(0)}_{= (L_{n+m}\mathcal{O})(0)}$$

$$\therefore \boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3-n)}$$

This is called the Virasoro algebra of central charge C.

"Vir_c".

Similary we find

$$\boxed{\begin{aligned} [\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3-n). \\ [L_n, \tilde{L}_m] &= 0 \end{aligned}}$$

The infinitesimal conformal transformations of operators realized a representation of the Lie algebra

$$\begin{array}{ccc} \mathrm{Vir}_c & \oplus & \mathrm{Vir}_c \\ \uparrow & & \uparrow \\ \{L_n\} & & \{\tilde{L}_n\}. \end{array}$$

$L_{-1}, \tilde{L}_{-1}$

$T_{\mu\nu}$ can also be regarded as Noether current for translations. Thus,

$$L_{-1}\mathcal{O}(p) = \frac{1}{2\pi i}\oint_p dz\, T_{zz}(z)\,\mathcal{O}(p) = \frac{\partial}{\partial z}\mathcal{O}(p)$$

$$\tilde{L}_{-1}\mathcal{O}(p) = -\frac{1}{2\pi i}\oint_p d\bar{z}\, T_{\bar{z}\bar{z}}(z)\,\mathcal{O}(p) = \frac{\partial}{\partial \bar{z}}\mathcal{O}(p).$$

$L_0, \tilde{L}_0$

Suppose $L_0\mathcal{O} = \Delta\mathcal{O}$, $\tilde{L}_0\mathcal{O} = \tilde{\Delta}\mathcal{O}$

(i.e. $\mathcal{O}$ is an eigenoperator for $L_0, \tilde{L}_0$.)

recall $z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}$ ...... dilatation

$z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$ ..... rotation.

$\Delta + \tilde{\Delta}$ is the <u>dimension</u> of the operator $\mathcal{O}$

$\Delta - \tilde{\Delta}$ is the <u>spin</u> of the $\mathcal{O}$.

<u>or</u> angular momentum

$\Delta, \tilde{\Delta}$ play the analogous role as <u>energy</u> in ordinary QFT. (called <u>conformal weights</u>)

The algebra $\supset$

$$[L_0, L_{-n}] = n L_{-n} \quad , \quad [\tilde{L}_0, \tilde{L}_{-n}] = n \tilde{L}_{-n}$$

L_{-n} raises the (right) conformal weight by n

$\tilde{L}_{-n}$ raises the (left) 〃 by n.

An operator that cannot be lowered by $L_n, \tilde{L}_n$ $(n \geq 1)$

i.e. $$\boxed{L_n \mathcal{O} = 0 \quad \tilde{L}_n \mathcal{O} = 0 \quad \forall n \geq 1}$$

is called a <u>primary operator</u> (or <u>primary field</u>)

<u>The basic idea</u> (of representation theory of $\mathrm{Vir}_c \oplus \mathrm{Vir}_c$):

Classify all operators into primary operators

and their <u>descendants</u>.

[An operator of the form $L_{-n_1} \cdots L_{-n_k} \mathcal{O}$ $\quad n_1, \cdots, n_k \geq 1$

are called a <u>descendant</u> of $\mathcal{O}$.]

example

$$\mathcal{O} = \mathbb{1}$$

$$(L_n \mathbb{1})(0) = \frac{1}{2\pi i}\oint_0 dz\, \underbrace{z^{n+1} T_{zz}(z)\, \mathbb{1}(0)}_{\text{regular as } z\to 0 \text{ if } n\geq -1} = 0 \qquad n \geq -1$$

$$(\tilde{L}_n \mathbb{1})(0) = 0 \qquad n \geq -1.$$

$$\therefore \boxed{\mathcal{O} = \mathbb{1} \text{ is a primary operator of conformal weight } (\Delta, \tilde{\Delta}) = (0, 0).}$$

$$(L_{-2}\mathbb{1})(0) = \frac{1}{2\pi i}\oint_0 dz\, z^{-1} T_{zz}(z) = T_{zz}(0)$$

$$(\tilde{L}_{-2}\mathbb{1})(0) = -\frac{1}{2\pi i}\oint_0 d\bar{z}\, \bar{z}^{-1} T_{\bar{z}\bar{z}}(z) = T_{\bar{z}\bar{z}}(0)$$

$$\therefore \boxed{T_{zz},\ T_{\bar{z}\bar{z}} \text{ are descendants of } \mathcal{O} = \mathbb{1}.}$$

• **Note** : $L_n, \tilde{L}_n$ depends on the coordinate system.

Suppose z and w are holomorphic coordinates

~~both~~ $z(p) = w(p) = 0$. at a point p.

Then, they are related by $w = cz + O(z^2) = w(z)$

$$\left(L_n^{\{w\}} \mathcal{O}\right)(0) = \frac{1}{2\pi i} \oint_0 \underbrace{dw}_{= w'(z)\,dz} \; w^{n+1} \underbrace{T_{ww}(w)}_{= (w'(z))^{-2}\left(T_{zz}(z) - \frac{c}{12}\{w,z\}\right)} \mathcal{O}(0)$$

$$= \frac{1}{2\pi i} \oint_0 dz\, w(z)^{n+1} (w'(z))^{-1} \left(T_{zz}(z) - \frac{c}{12}\{w,z\} \right) \mathcal{O}(0)$$

$w'(z)^{-1}$ is analytic at $z=0$ $\therefore$ $w(z)^{n+1} w'(z)^{-1} \{w,z\}$ is analytic at $z=0$ ($\to$ no pole)

$$= \frac{1}{2\pi i} \oint_0 dz\, \underbrace{w(z)^{n+1} (w'(z))^{-1}} T_{zz}(z)\, \mathcal{O}(0)$$

$$z^{n+1} \cdot \left(\frac{w(z)}{z}\right)^{n+1} (w'(z))^{-1} = z^{n+1}\left(c^n + \sum_{m=1}^{\infty} c_m^n z^m \right)$$

$$= \left(\left(c^n L_n^{\{z\}} + \sum_{m=1}^{\infty} c_m^n L_{n+m}^{\{z\}}\right) \mathcal{O}\right)(0).$$

• The definition of primary does not depend on coordinate system you use.

• For a primary operator, conformal weights do not depend on coordinate system you use.

· Note: $(L_n \mathcal{O})(0) = \oint_0 \frac{dz}{2\pi i} z^{n+1} T_{zz}(z) \mathcal{O}(0)$

$$\Leftrightarrow \quad T_{zz}(z)\mathcal{O}(0) = \sum_{n\in\mathbb{Z}} z^{-n-2} (L_n \mathcal{O})(0)$$

If $\mathcal{O}$ is a primary operator, $n \geq 1$ terms vanish:

$$T_{zz}(z)\mathcal{O}(0) = \sum_{n\leq 0} z^{-n-2}(L_n\mathcal{O})(0)$$

$$= z^{-2}\underbrace{(L_0\mathcal{O})(0)}_{\Delta \mathcal{O}(0)} + z^{-1}\underbrace{(L_{-1}\mathcal{O})(0)}_{\partial_z \mathcal{O}(0)} + z^0 (L_{-2}\mathcal{O})(0) + \cdots$$

if conformal weight Δ

∴ For a primary operator $\mathcal{O}$ of conformal weight $\Delta, \tilde{\Delta}$:

$$\boxed{\begin{aligned} T_{zz}(z)\mathcal{O}(w) &= \frac{\Delta}{(z-w)^2}\mathcal{O}(w) + \frac{1}{z-w}\partial_w \mathcal{O}(w) + \text{regular as } z\to w \\ T_{\bar{z}\bar{z}}(\bar{z})\mathcal{O}(w) &= \frac{\tilde{\Delta}}{(\bar{z}-\bar{w})^2}\mathcal{O}(w) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\mathcal{O}(w) + \text{regular} \end{aligned}}$$

Note: $\mathcal{O}(w) = T_{ww}(w)$ is indeed <u>NOT</u> a primary operator.

Consequence :

$$(\delta_\epsilon \mathcal{O})(z) = \oint_z \frac{dw}{2\pi i} \epsilon(w) \underbrace{T_{ww}(w)\,\mathcal{O}(z)}$$

$\Leftarrow$ primary of weight Δ

$$\frac{\Delta}{(w-z)^2}\mathcal{O}(z) + \frac{1}{w-z}\partial_z \mathcal{O}(z) + \text{reg}$$

$$= \Delta\,\epsilon'(z)\,\mathcal{O}(z) + \epsilon(z)\,\partial_z \mathcal{O}(z)$$

$$(\delta_{\bar\epsilon}\mathcal{O})(z) = \tilde\Delta\,\bar\epsilon'(\bar z)\,\mathcal{O}(z) + \bar\epsilon(\bar z)\,\partial_{\bar z}\mathcal{O}(z)$$

$\rightsquigarrow$ Finite version : under $z \to w(z)$

$$\boxed{\mathcal{O}(z) \longrightarrow \left(w'(z)\right)^{\Delta}\left(\bar w'(\bar z)\right)^{\tilde\Delta}\mathcal{O}(w(z))}$$

$$\mathcal{O} = \mathcal{O}(z)(dz)^{\Delta}(d\bar z)^{\tilde\Delta} \longrightarrow \mathcal{O}(w(z))(dw)^{\Delta}(d\bar w)^{\tilde\Delta} = w^*\mathcal{O}.$$

$\therefore$ primary operator of conformal weight $(\Delta, \tilde\Delta)$

defines a "$(\Delta,\tilde\Delta)$-differential".

$$\boxed{\mathcal{O} = \mathcal{O}(z)(dz)^{\Delta}(d\bar z)^{\tilde\Delta}}$$

Example massless scalar

$$T_{zz} = - :\partial_z X \partial_z X: \quad , \quad T_{\bar{z}\bar{z}} = - :\partial_{\bar{z}} X \partial_{\bar{z}} X:$$

$$T_{zz}(z)\, \partial_w X(w) = - :\partial_z X(z) \underbrace{\partial_z X(z)\, \partial_w X(w)}_{\underset{-\frac{1}{2}\frac{1}{(z-w)^2}}{\parallel}}: \times 2 \; - \underset{\underset{\text{reg}}{\uparrow}}{:(\partial_z X)^2 \partial_w X:}$$

$$= \frac{1}{(z-w)^2}\, \partial_z X(z) \quad + \text{reg}$$

$$= \frac{1}{(z-w)^2}\, \partial_w X(w) + \frac{1}{z-w}\, \partial_w(\partial_w X) \; + \text{reg.}$$

$$T_{\bar{z}\bar{z}}(z)\, \partial_w X(w) = - :\partial_{\bar{z}} X(z) \underbrace{\partial_{\bar{z}} X(z)\, \partial_w X(w)}_{\text{const } \delta(z-w)}: \times 2 \; - \underset{\underset{\text{reg}}{\uparrow}}{:(\partial_{\bar{z}} X)^2 \partial_w X:}$$

$$\left(\Rightarrow (\bar{L}_n \partial_w X) = 0 \quad n \geq 0 \right)$$

$\therefore$ $\partial_z X$ is a primary operator of conformal weight $(\Delta, \tilde{\Delta}) = (1, 0)$.

$(\Rightarrow$ dimension 1, spin $= 1)$

similarly

$\partial_{\bar{z}} X$ is a 〃 〃 $(\Delta, \tilde{\Delta}) = (0, 1)$

$(\Rightarrow$ dimension 1, spin $= -1)$

example massless Dirac

$$T_{zz} = -\frac{1}{2} :\bar{\psi}_- \partial_z \psi_-: + \frac{1}{2} :\partial_z \bar{\psi}_- \psi_-:$$

$$T_{\bar{z}\bar{z}} = -\frac{1}{2} :\bar{\psi}_+ \partial_{\bar{z}} \psi_+: + \frac{1}{2} :\partial_{\bar{z}} \bar{\psi}_+ \psi_+:$$

$$T_{zz}(z)\,\psi_-(w) = -\frac{1}{2} :\bar{\psi}_-(z)\,\partial_z \psi_-(z)\,\psi_-(w): + \frac{1}{2} :\partial_z \bar{\psi}_-(z)\,\psi_-(z)\,\psi_-(w): + \text{reg.}$$

$$= -\frac{1}{2}\,\frac{1}{z-w}\,\partial_z \psi_-(z) + \frac{1}{2}\,(-1)\,\frac{-1}{(z-w)^2}\,\psi_-(z) + \text{reg}$$

(-1) < statistics

$$= \frac{\frac{1}{2}}{(z-w)^2}\,\psi_-(z) + \frac{1}{z-w}\,\partial_z \psi_-(z) + \text{reg.}$$

$$T_{\bar{z}\bar{z}}(z)\,\psi_-(w) = \text{reg.}$$

$\therefore$ ψ_- is a primary operator of weight $(\Delta, \tilde{\Delta}) = (\frac{1}{2}, 0)$

$\Rightarrow$ dimension $= \frac{1}{2}$, spin $= \frac{1}{2}$

$\bar{\psi}_-$ is also. $(\Delta, \tilde{\Delta}) = (\frac{1}{2}, 0)$, dimension $\frac{1}{2}$, spin $\frac{1}{2}$.

ψ_+ is a primary op of $(\Delta, \tilde{\Delta}) = (0, \frac{1}{2})$ $\left(\text{dim} = \frac{1}{2}, \text{spin} = -\frac{1}{2}\right)$

$\bar{\psi}_+$ 〃

exercises (HW)

$$T_{zz}(z) :\bar{\psi}_-(w)\psi_-(w): = ?$$

$$T_{zz}(z) :e^{ikX(w)}: = ?$$