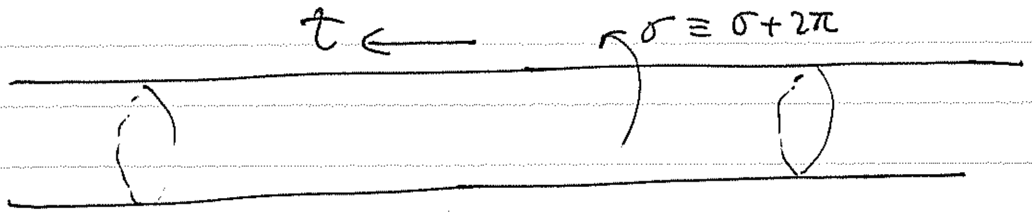
L_n's and $\tilde{L}_n$'s acting on states



Minkowski Cylinder

space of states

Define operators $L_n, \tilde{L}_n : \mathcal{H} \to \mathcal{H}$

$$\boxed{\begin{aligned} L_n &:= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \, e^{in(t-\sigma)} \frac{1}{2}\left(T^t{}_t - T^t{}_\sigma\right) \\ \tilde{L}_n &:= \frac{1}{2\pi}\int_0^{2\pi} d\sigma \, e^{in(t+\sigma)} \frac{1}{2}\left(T^t{}_t + T^t{}_\sigma\right) \end{aligned}}$$

c.f. $\left.\begin{aligned} L_0 &= H_R \\ \tilde{L}_0 &= H_L \end{aligned}\right\}$ so $H = L_0 + \tilde{L}_0$, $P = -L_0 + \tilde{L}_0$.

Hermiticity

$$\boxed{\begin{aligned} L_n^\dagger &= L_{-n} \\ \tilde{L}_n^\dagger &= \tilde{L}_{-n} \end{aligned}}$$

∵ $T^\mu{}_\nu$ real

Note reality or Hermiticity is defined on Minkowski signature!

Wick rotation $t \to -i\tau$

$$L_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{n(\tau - i\sigma)}\, \frac{1}{2}\left(T^\tau{}_\tau + i T^\tau{}_\sigma\right)$$

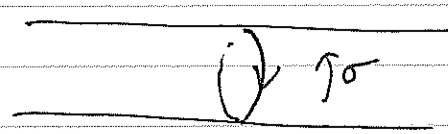
$$\tilde{L}_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{n(\tau + i\sigma)}\, \frac{1}{2}\left(T^\tau{}_\tau - i T^\tau{}_\sigma\right)$$

$$\zeta = \tau - i\sigma \quad T_{\zeta\zeta} = \frac{1}{4}\underbrace{\left(T_{\tau\tau} - T_{\sigma\sigma}\right)}_{\substack{\text{// traceless} \\ 2T_{\tau\tau}}} + \frac{i}{2} T_{\tau\sigma} = \frac{1}{2}\left(T_{\tau\tau} + i T_{\tau\sigma}\right)$$

$$= \frac{1}{2}\left(T^\tau{}_\tau + i T^\tau{}_\sigma\right)$$

$$T_{\bar\zeta\bar\zeta} = \frac{1}{2}\left(T^\tau{}_\tau - i T^\tau{}_\sigma\right)$$

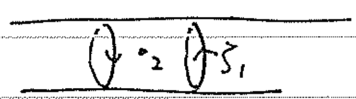
$$\boxed{L_n = \oint \frac{d\zeta}{2\pi i}\, e^{n\zeta}\, T_{\zeta\zeta}\,, \quad \tilde{L}_n = \oint \frac{d\bar\zeta}{-2\pi i}\, e^{n\bar\zeta}\, T_{\bar\zeta\bar\zeta}}$$

where the contour is 

Commutation relation

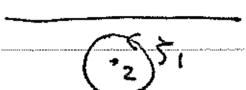
$$[L_n, L_m] = \oint \frac{d\zeta_2}{2\pi i}\, e^{m\zeta_2}\, [L_n, T_{\zeta\zeta}(2)]$$

$$[L_n, T_{\zeta\zeta}(2)] = \oint \frac{d\zeta_1}{2\pi i}\, e^{n\zeta_1}\, T\left(T_{\zeta\zeta}(1)\, T_{\zeta\zeta}(2)\right)$$

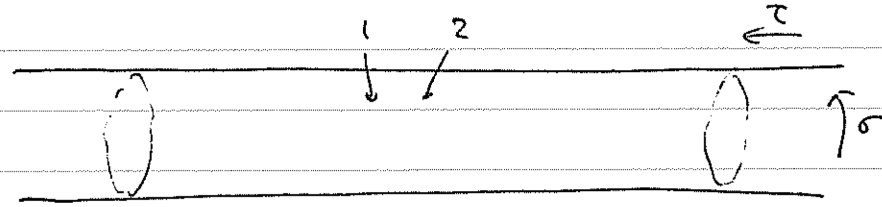


time ordered product.

$\Vert$



OPE of E-M tensors on Cylinder



$\xi = \tau - i\sigma$

recall $\langle X(1)\, X(2)\rangle = -\log\left|e^{\xi_1} - e^{\xi_2}\right| + \frac{1}{2}\log\left|e^{\xi_1} e^{\xi_2}\right| + \text{const}$

and

$$-\frac{2}{\pi}\,\partial_{\xi_1}\partial_{\bar\xi_1}\langle X(1)\, X(2)\rangle = \delta^{(2)}(1-2)$$

$$\| $$

$$\frac{1}{\pi}\,\partial_{\bar\xi_1}\left(\frac{e^{\xi_1}}{e^{\xi_1}-e^{\xi_2}}\right)$$

From

$$\frac{1}{2\pi}\,\partial_{\bar\xi_1} T_{\xi\xi}(1)\, T_{\xi\xi}(2) = -\frac{c}{24}\,\overbrace{\partial^3_{\xi_1}\delta^{(2)}(1-2)}^{-\partial^3_{\xi_2}\delta^{(2)}(1-2)} - \overbrace{\partial_{\xi_1}\delta^{(2)}(1-2)}^{-\partial_{\xi_2}\delta^{(2)}(1-2)}\, T_{\xi\xi}(2)$$

$$+\frac{1}{2}\,\delta^{(2)}(1-2)\,\partial_\xi T_{\xi\xi}(2) \quad + \text{reg}$$

we obtain

$$\boxed{\begin{aligned} T_{\xi\xi}(1)\, T_{\xi\xi}(2) &= \frac{c}{12}\,\partial^3_{\xi_2}\left(\frac{e^{\xi_1}}{e^{\xi_1}-e^{\xi_2}}\right) + 2\,\partial_{\xi_2}\left(\frac{e^{\xi_1}}{e^{\xi_1}-e^{\xi_2}}\right) T_{\xi\xi}(2) \\ &\quad + \frac{e^{\xi_1}}{e^{\xi_1}-e^{\xi_2}}\,\partial_\xi T_{\xi\xi}(2) \quad + \text{reg} \end{aligned}}$$

$$\therefore\ [L_n, T_{\zeta\zeta}(z)] = \oint \frac{d\zeta_1}{2\pi i} \left\{ \frac{c}{12} \partial_{\zeta_2}^3 \left(\frac{e^{(n+1)\zeta_1}}{e^{\zeta_1}-e^{\zeta_2}} \right) + 2\partial_{\zeta_2} \left(\frac{e^{(n+1)\zeta_1}}{e^{\zeta_1}-e^{\zeta_2}} \right) T_{\zeta\zeta}(z) + \frac{e^{(n+1)\zeta_1}}{e^{\zeta_1}-e^{\zeta_2}} \partial_\zeta T_{\zeta\zeta}(z) + \text{reg} \right\}$$

$$\oint \frac{e^{\zeta_1} d\zeta_1}{2\pi i} \frac{f(\zeta_1)}{e^{\zeta_1}-e^{\zeta_2}} = f(\zeta_2)$$

$$= \frac{c}{12} \partial_{\zeta_2}^3 (e^{n\zeta_2}) + 2\partial_{\zeta_2}(e^{n\zeta_2}) T_{\zeta\zeta}(z) + e^{n\zeta_2} \partial_\zeta T_{\zeta\zeta}(z)$$

$$= e^{n\zeta_2} \left(\frac{c}{12} n^3 + 2n T_{\zeta\zeta}(z) + \partial_\zeta T_{\zeta\zeta}(z) \right)$$

$$\therefore\ [L_n, L_m] = \oint \frac{d\zeta_2}{2\pi i} e^{(m+n)\zeta_2} \Big(\frac{c}{12} n^3 + 2n T_{\zeta\zeta}(z) + \underbrace{\partial_\zeta T_{\zeta\zeta}(z)}_{(-m-n)\, T_{\zeta\zeta}(z)} \Big)$$

$$= \delta_{m+n,0} \frac{c}{12} n^3 + (n-m) L_{m+n}.$$

Similarly

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{m+n} + \frac{c}{12} n^3 \delta_{m+n,0}$$

... (another version of) Virasoro algebra, central charge c.

We can develop the representation theory as in action on operators.

$$[L_0, L_{-n}] = n L_{-n}, \quad [\tilde{L}_0, \tilde{L}_{-n}] = n \tilde{L}_{-n}$$

Thus, L_{-n} raises the $L_0 = H_R$ eigenvalues by n.

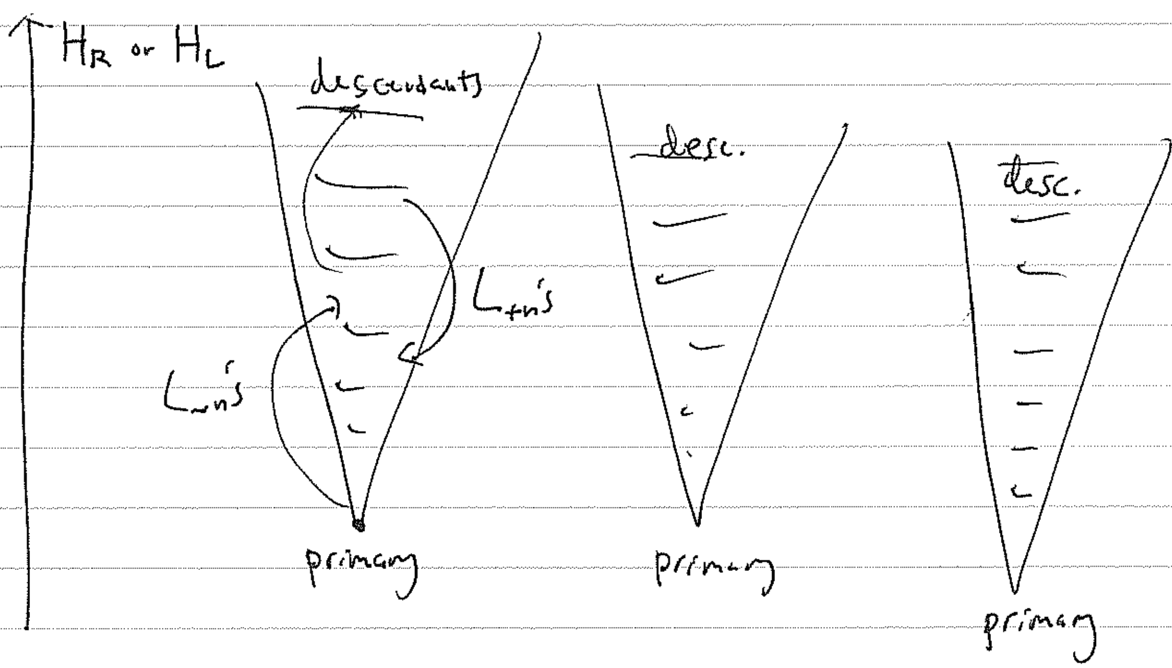
$\tilde{L}_{-n}$ " $\tilde{L}_0 = H_L$ " .

A <u>primary state</u> is a state annihilated by L_n's $\tilde{L}_n$'s with <u>positive n's</u>.

$$\boxed{L_n |\alpha\rangle = 0, \quad \tilde{L}_n |\alpha\rangle = 0 \qquad \forall n \geq 1}$$

(a state whose $H_{R,L}$ values <u>cannot be lowered</u> by L_n, $\tilde{L}_n$'s)

A <u>descendant</u> of a primary state $|\alpha\rangle$ is a state of the form $L_{-n_1} \cdots L_{-n_s} \tilde{L}_{-m_1} \cdots \tilde{L}_{-m_k} |\alpha\rangle \quad (n_i, m_j \geq 1)$.



$$[L_n, L_{-n}] = 2n L_0 + \frac{c}{12} n^3$$

For a primary state $|\alpha\rangle$ $L_n|\alpha\rangle = 0$ primary

$$\underset{n\geq 1}{0 \leq} \| L_{-n}|\alpha\rangle \|^2 = \langle\alpha| \underset{\overset{\|}{L_n}}{L_{-n}^{\dagger}} L_n |\alpha\rangle = \langle\alpha|[L_n, L_{-n}]|\alpha\rangle$$

$$= \langle\alpha| \left(2n L_0 + \frac{c}{12} n^3\right) |\alpha\rangle$$

$\therefore$ L_0 eigenvalue of a primary state $\geq -\frac{c}{24} n^2$ $\forall_{n\geq 1}$.

Impossible if $c < 0$!

Thus, if $\exists$ positive definite inner product with $L_n^{\dagger} = L_{-n}$

(such a theory is called a unitary CFT)

then $\boxed{c \geq 0}$ $\cdots$ "unitarity bound" (on central charge)

If $c > 0$, $n=1$ gives the lowerbound:

L_0 eigenvalue of a primary state $|\alpha\rangle$ $\geq -\frac{c}{24}$

$= -\frac{c}{24}$ if and only if $L_{-1}|\alpha\rangle = 0$

Similarly for $\tilde{L}_0$ e.v. $\left(\geq -\frac{c}{24}\right.$ and $= -\frac{c}{24}$ iff $\left.\tilde{L}_{-1}|\alpha\rangle = 0\right)$

Note $H = L_0 + \tilde{L}_0$ $\quad (P = -L_0 + \tilde{L}_0)$

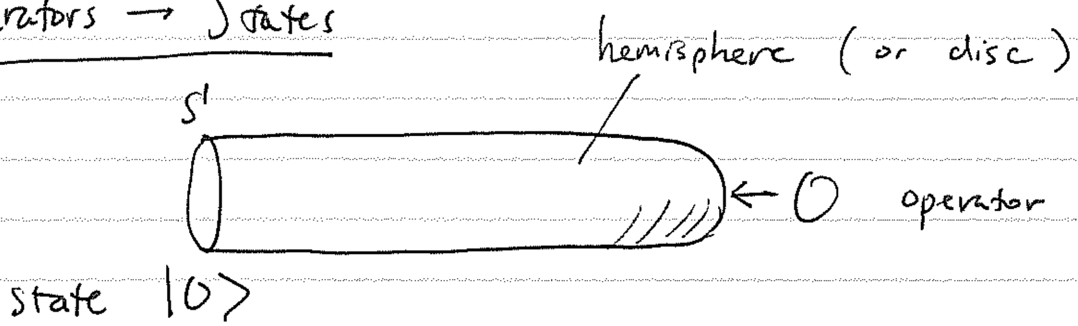
The ground state is a primary state that is annihilated by L_{-1} and $\tilde{L}_{-1}$. The ground state energy is $-\frac{c}{12}$ (momentum is 0)

But, is there such a state?

State-Operator Correspondence

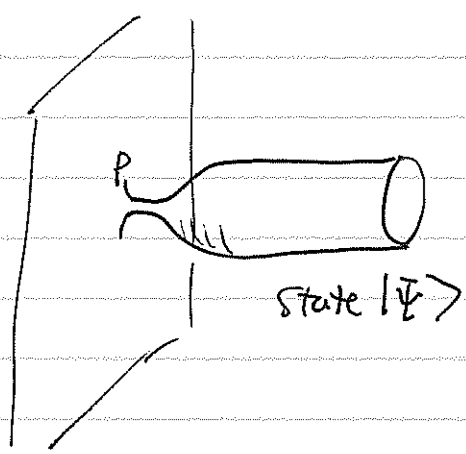
There is a one to one correspondence between states and local operators.

Operators → States



$$\langle x|\mathcal{O}\rangle = \int_{X|_{S^1}=x} \mathcal{D}X \, e^{-S_E(X)} \mathcal{O}(X) \quad \text{is the wavefunction.}$$

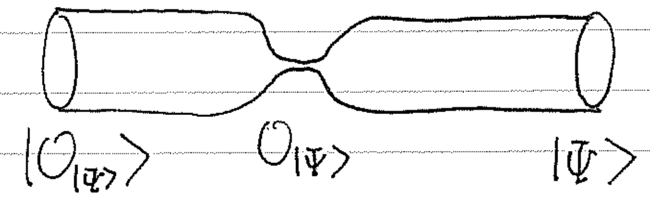
States → Operators



gives an operator $\mathcal{O}_{|\Psi\rangle}(P)$

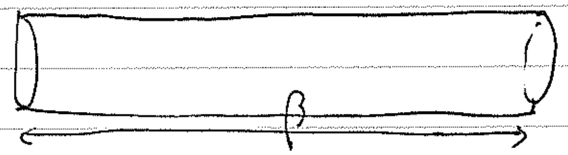
They are inverse to each other (up to $L_0 + \tilde{L}_0$ operations).

$\underline{(\text{State} \leftarrow \text{Op}) \circ (\text{Op} \leftarrow \text{State})}$



$$g = g_{\zeta\bar\zeta}\, d\zeta\, d\bar\zeta + \text{c.c.}$$

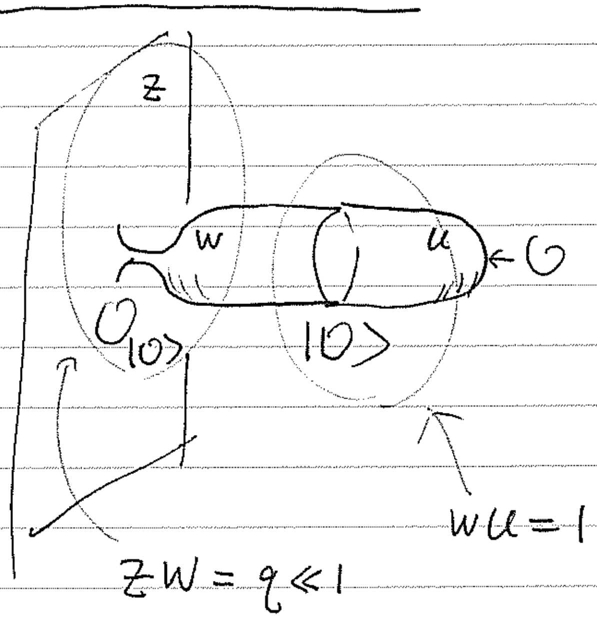
$\Vert\wr$ conformal



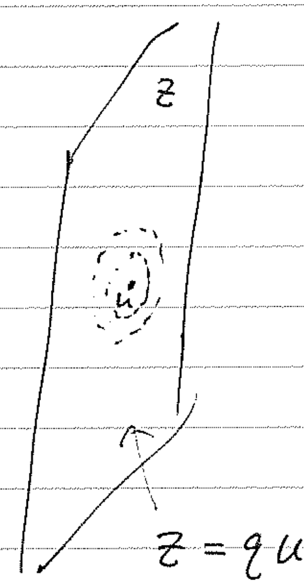
$$g = \tfrac{1}{2}\, d\zeta\, d\bar\zeta + \text{c.c.}$$

$$\therefore\ |\mathcal{O}_\Psi\rangle = e^{C(g,\phi)} \cdot e^{-\beta(L_0+\tilde{L}_0)} |\Psi\rangle$$

$\underline{(\text{Op} \leftarrow \text{State}) \circ (\text{State} \leftarrow \text{Op})}$



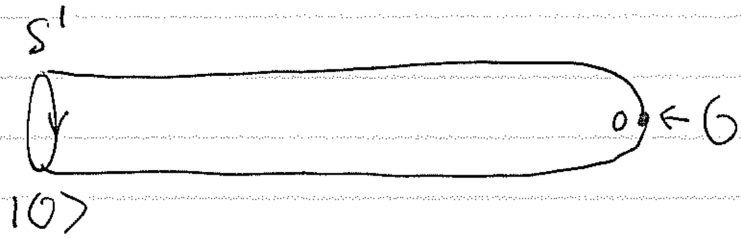
conformal $\simeq$



$$\mathcal{O}_{|0\rangle} = e^{C(g,\phi')}\, q^{L_0+\tilde{L}_0}\, \mathcal{O}$$

[One can also show $q = e^{-\beta}$]

L_n on states vs L_n on operators



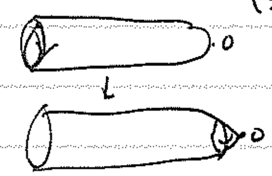
$$z = e^{\zeta}$$

$$L_n^{\text{cylinder}} = \oint_{S^1} \frac{d\zeta}{2\pi i}\, e^{n\zeta}\, \underbrace{T_{\zeta\zeta}(\zeta)}_{\parallel}$$

$$\underbrace{T_{\zeta\zeta}(\zeta)} = z^2 T_{zz}(z) - \frac{c}{24}$$

$$= \oint_{S^1} \frac{dz/z}{2\pi i}\, z^n \left(z^2 T_{zz} - \frac{c}{24} \right)$$

T_{zz} holo



$$= \oint_0 \frac{dz}{2\pi i} \left(z^{n+1} T_{zz} - \frac{c}{24} z^{n-1} \right)$$

$$= L_n^{\text{plane}} - \frac{c}{24}\delta_{n,0}$$

$$\therefore\ |L_n O\rangle = |L_n^{\text{plane}} O\rangle$$

$$= \left(L_n^{\text{cylinder}} + \frac{c}{24}\delta_{n,0} \right) |O\rangle$$

$$= \left(L_n + \frac{c}{24}\delta_{n,0} \right) |O\rangle$$

similarly $|\tilde{L}_n O\rangle = \left(\tilde{L}_n + \frac{c}{24}\delta_{n,0} \right) |O\rangle$

- $\mathcal{O}$ is a primary operator $\Longleftrightarrow$ $|\mathcal{O}\rangle$ is a primary state

$$L_n \mathcal{O} = \tilde{L}_n \mathcal{O} = 0 \quad \forall n \geq 1 \quad \Longleftrightarrow \quad L_n |\mathcal{O}\rangle = \tilde{L}_n |\mathcal{O}\rangle = 0 \quad \forall n \geq 1.$$

- $\mathcal{O}$ has conformal weights $\Delta, \tilde{\Delta}$ $\Longleftrightarrow$ $\mathcal{O}$ has $H_R = \Delta - \frac{c}{24}$, $H_L = \tilde{\Delta} - \frac{c}{24}$

<u>Note</u> $\mathcal{O} = \text{id}$ is a primary operator of $\Delta = \tilde{\Delta} = 0$

$\Rightarrow$ $|0\rangle = |\text{id}\rangle$ is a primary state of $H_R = -\frac{c}{24}$, $H_L = -\frac{c}{24}$.

i.e. that is the ground state we were looking for!

<u>Note</u> $H_R \geq -\frac{c}{24}$, $H_L \geq -\frac{c}{24}$ in a unitary CFT

$\Rightarrow$ $\Delta \geq 0$, $\tilde{\Delta} \geq 0$ in a unitary CFT

$\boxed{= 0}$ iff $\mathcal{O} = \text{id}$

Note added after the class

Is such a state (i.e. primary, ann by $L_{-1}, \tilde{L}_{-1}$) unique?

Suppose $|\Psi\rangle$ is such a state.

Then $\mathcal{O}_{|\Psi\rangle}$ is a primary operator s.t. $L_{-1}\mathcal{O}_{|\Psi\rangle} = \tilde{L}_{-1}\mathcal{O}_{|\Psi\rangle} = 0$

i.e. $$\frac{\partial \mathcal{O}_{|\Psi\rangle}}{\partial z} = \frac{\partial \mathcal{O}_{|\Psi\rangle}}{\partial \bar{z}} = 0$$

$$\Longrightarrow \quad \mathcal{O}_{|\Psi\rangle} = \text{const} \cdot id$$

$\therefore \ |\Psi\rangle \propto |id\rangle$ unique!

The free massless scalar theory & free massless Dirac fermion are $c=1$ CFT, and unitary.

(find the expressions for L_n, $\tilde{L}_n$'s and check $L_n^\dagger = L_{-n}$, $\tilde{L}_n^\dagger = \tilde{L}_{-n}$)

Thus, the ground state energy is

$$E_{|0\rangle} = -\frac{c}{12} = -\frac{1}{12}.$$

This is one justification of "$\sum_{n=1}^{\infty} n = -\frac{1}{12}$".