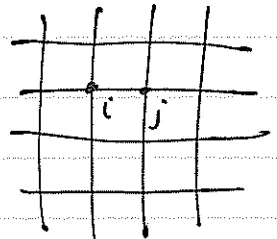
# 2d Ising Model




2d lattice

$$\sigma_i \in \{+1, -1\}$$

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \mu_B H \sum_i \sigma_i$$

$$Z = \sum_{\{\sigma_i\}} e^{-\mathcal{H}/kT} = \sum_{\{\sigma_i\}} e^{K \sum_{\langle i,j\rangle} \sigma_i \sigma_j + h \sum_i \sigma_i}$$

$$K = J/kT, \quad h = \mu_B H / kT$$

expectation

$T$

paramagnetic

$$\langle \sigma \rangle \to 0 \quad \text{as} \quad h \to \pm 0$$

or

$$\langle \sigma_i \sigma_j \rangle \Big|_{h=0} \to 0 \quad \text{as} \quad |i-j| \to \infty$$

$T_c$ ← Curie temperature (Critical)

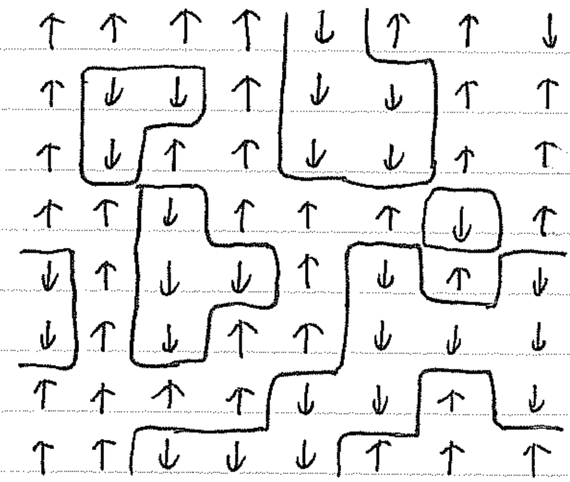
Ferromagnetic

$$\langle \sigma \rangle \to \pm M \quad \text{as} \quad h \to \pm 0$$

$$\langle \sigma_i \sigma_j \rangle \Big|_{h=0} \to M^2 \quad \text{as} \quad |i-j| \to \infty$$

$0$

Spin configuration $\{\sigma_i\}$ $\xrightarrow{2:1}$ Configuration of closed loops



Boltzmann weight $=$ $e^{K}$ for $\sigma_i = \sigma_j$ (blanck)

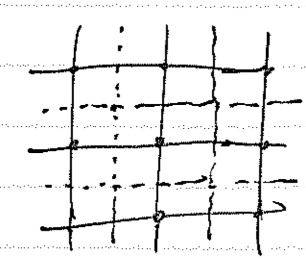
$e^{-K}$ for $\sigma_i = -\sigma_j$ (line)

Take out $e^{K}$ from each edge : $\begin{cases} 1 & \text{for blanck} \\ e^{-2K} & \text{for line} \end{cases}$

edge ↑ n.n. pair

$$Z = 2 \cdot e^{EK} \cdot \sum_{\text{loops}} \underbrace{e^{-2K}}_{= z}{}^{(\text{length})}$$

- $E = \#$ edges (n.n. pairs)
- loops are on the dual lattice



- Each loop is homotopically trivial (boundary of ↓ region)

High temperature expansion (strong coupling expansion)

$T$ large $\Leftrightarrow$ $K \ll 1$

$$e^{K\sigma_i\sigma_j} = 1 + K\sigma_i\sigma_j + \overset{K^2}{\overset{\parallel}{\frac{1}{2}(K\sigma_i\sigma_j)^2}} + \overset{K^3\sigma_i\sigma_j}{\overset{\parallel}{\frac{1}{3!}(K\sigma_i\sigma_j)^3}} + \cdots$$

$$= \cosh(K) + \sigma_i\sigma_j \sinh(K)$$

$$= \cosh(K)\left(1 + \sigma_i\sigma_j \tanh(K)\right)$$

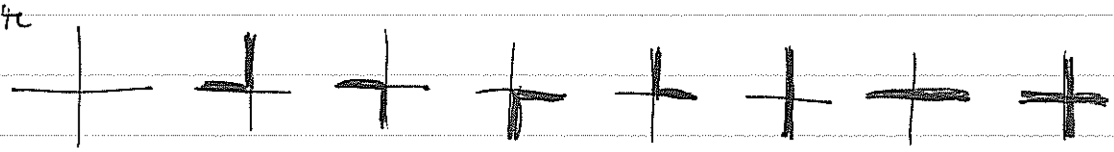
$$Z = (\cosh(K))^E \sum_{\{\sigma_i\}} \prod_{\langle i,j\rangle} (1 + \sigma_i\sigma_j \underbrace{\tanh(K)}_{=t})$$

$$\sum_{\{\sigma_i\}} \sigma_{i_1}\sigma_{j_1}\cdots\sigma_{i_s}\sigma_{j_s} t^s = \begin{cases} 0 & \text{some } \sigma_i \text{ appears odd \# of times} \\ 2^V t^s & \text{any } \sigma_i \text{ appears } \underline{\text{even}} \text{ \# of times} \end{cases}$$

include 0

$V =$ # vertices (# sites)

each site



$$Z = (\cosh(K))^E \sum_{\text{loops}} 2^V t^{\text{length}}$$

- on the lattice
- no topological constraint.

# Kramers - Wannier duality

two sums for $Z$ : $\sum_{\text{loops}} z^{\text{length}}$ , $\sum_{\text{loops}} t^{\text{length}}$

$$z = e^{-2K} \qquad t = \tanh(K)$$

define $\tilde{K}$ by $\boxed{\tanh(K) = e^{-2\tilde{K}}}$

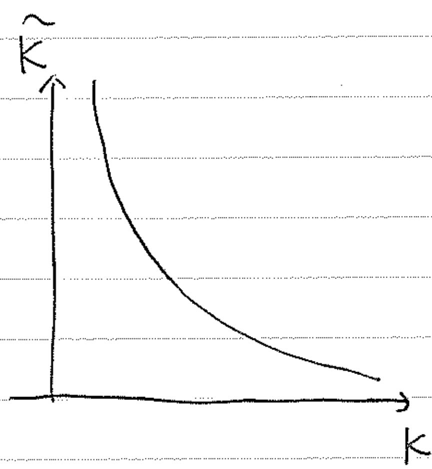
Then $Z(K) = 2^V (\cosh(K))^E \sum_{\text{loop}} \underbrace{t}_{e^{-2\tilde{K}}}{}^{\text{length}} = \dfrac{2^V (\cosh(K))^E}{2\, e^{E\tilde{K}}} Z(\tilde{K})$ !

Note

$$\sinh(2K)\sinh(2\tilde{K}) = 2\sinh(K)\cosh(K)\,\frac{e^{2\tilde{K}} - e^{-2\tilde{K}}}{2}$$

$$= \cancel{2}\sinh(K)\cosh(K)\left(\frac{\cosh(K)}{\sinh(K)} - \frac{\sinh(K)}{\cosh(K)}\right)$$

$$= \cosh^2(K) - \sinh^2(K) = 1$$




$\Rightarrow \tanh(\tilde{K}) = e^{-2K}$ also!

When $K = \tilde{K}$ ? $\quad \sinh^2(2K) = 1$

$$\frac{e^{2K} - e^{-2K}}{2} = 1 \Rightarrow e^{4K} - 2e^{2K} - 1 = 0$$

$$\Rightarrow (e^{2K} - 1)^2 = 2$$

$$\therefore \underline{e^{2K} = 1 + \sqrt{2}} \quad \text{or} \quad \underline{K = \frac{1}{2}\log(\sqrt{2}+1)}$$

Note also

$$2\cosh^2(K)\, e^{-2\tilde{K}} = 2\cosh^2(K)\,\frac{\sinh(K)}{\cosh(K)} = \sinh(2K)$$

<u>For square lattice</u>: $2V = E$

$$Z(K) = \frac{1}{2}\left(\sinh(2K)\right)^V \cdot Z(\tilde{K})$$

$$\begin{array}{ccc} K \text{ large} & \longleftrightarrow & K \text{ small} \\ \Updownarrow & & \Updownarrow \\ \text{low temperature} & & \text{high temperature} \end{array}$$

Free energy per site : $Z = e^{-VF}$ ($V \to \infty$ thermodynamic limit)

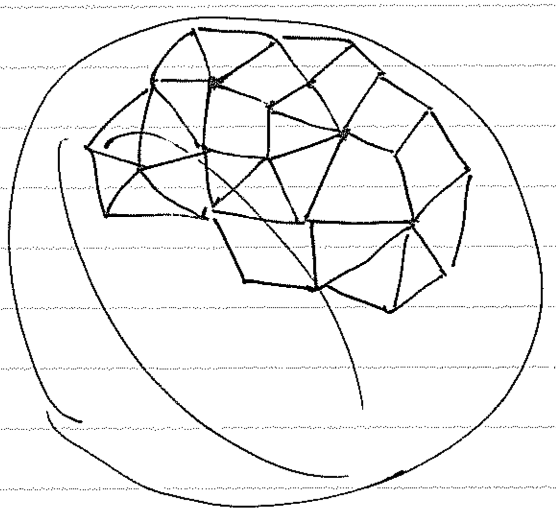
$$-F(K) = \underbrace{\log \sinh(2K)}_{= -\log\sinh(2\tilde{K})} - F(\tilde{K})$$

Consistent!

## Careful look

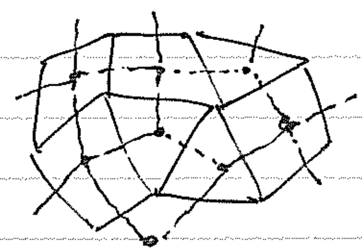
$\sum_{\text{loops}} z^{\text{length}}$ must be Only for homotopically trivial loops.

this condition is vacuous if the (dual) lattice is Spherical



lattice $\Lambda$

$\tilde{\Lambda}$ -- dual lattice



$$Z_\Lambda(K) = 2 \cdot e^{E(\Lambda)K} \sum_{\text{loops } C \tilde{\Lambda}} \left(e^{-2K}\right)^{\text{length}}$$

$$Z_\Lambda(K) = \left(\cosh(K)\right)^{E(\Lambda)} \cdot 2^{V(\Lambda)} \sum_{\text{loops } C \Lambda} \left(\tanh(K)\right)^{\text{length}}$$

$V(\Lambda)$ = (# of) vertices of $\Lambda$

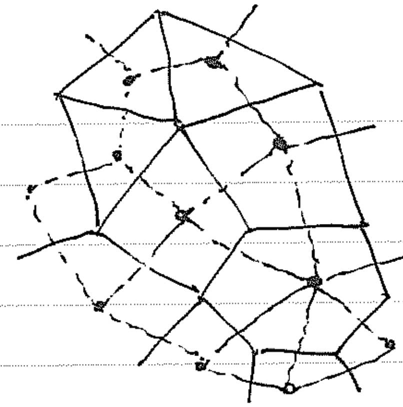
$E(\Lambda)$ = (# of) edges of $\Lambda$

$F(\Lambda)$ = (# of) faces of $\Lambda$

**Note:** $V(\tilde{\Lambda}) \cong F(\Lambda)$

$E(\tilde{\Lambda}) \cong E(\Lambda)$

$F(\tilde{\Lambda}) \cong V(\Lambda)$



Also $V - E + F = \chi$ Euler # $= 2$ for spherical lattice.

$\underline{\tanh(K) = e^{-2\tilde{K}}}$

$$Z_\Lambda(K) = \left(\cosh(K)\right)^{E(\Lambda)} \cdot 2^{V(\Lambda)} \cdot \underbrace{\sum_{\text{loops } C(\Lambda) = \tilde{\Lambda}} \left(e^{-2\tilde{K}}\right)^{\text{length}}}_{= \frac{1}{2 \cdot e^{E(\tilde{\Lambda})\tilde{K}}} Z_{\tilde{\Lambda}}(\tilde{K})}$$

$E(\tilde{\Lambda}) = E(\Lambda)$

$$= 2^{V(\Lambda)-1}\left(\cosh(K)\, e^{-\tilde{K}}\right)^{E(\Lambda)} Z_{\tilde{\Lambda}}(\tilde{K})$$

$$= \underbrace{2^{V(\Lambda)-1-\frac{E(\Lambda)}{2}}}_{= 2^{V(\Lambda) - \frac{1}{2}(V(\Lambda) - E(\Lambda) + F(\Lambda)) - \frac{E(\Lambda)}{2}}} \left(\sinh(2K)\right)^{\frac{E(\Lambda)}{2}} Z_{\tilde{\Lambda}}(\tilde{K})$$

$2 = V(\Lambda) - E(\Lambda) + F(\Lambda)$

$F(\Lambda) = V(\tilde{\Lambda})$

$$= 2^{\frac{V(\Lambda)}{2} - \frac{V(\tilde{\Lambda})}{2}}$$

$$\therefore\ 2^{-\frac{V(\Lambda)}{2}} Z_\Lambda(K) = \left(\sinh(2K)\right)^{\frac{E(\Lambda)}{2}} 2^{-\frac{V(\tilde{\Lambda})}{2}} Z_{\tilde{\Lambda}}(\tilde{K})$$

Completely symmetric

duality

Furthermore

$$\frac{\cosh 2K}{\cosh 2\tilde{K}} = \frac{\cosh 2K}{\frac{e^{2\tilde{K}} + e^{-2\tilde{K}}}{2}} = \frac{2\cosh(2K)}{\frac{\sinh K}{\cosh K} + \frac{\cosh K}{\sinh K}} = \frac{2\cosh K \sinh K \cancel{\cosh 2K}}{\cancel{\cosh^2 K + \sinh^2 K}}$$

$$= \sinh(2K).$$

$\therefore$ If we define

$$Y_\Lambda(K) = 2^{-\frac{V(\Lambda)}{2}} \left(\cosh(2K)\right)^{-\frac{E(\Lambda)}{2}} \cdot Z_\Lambda(K)$$

Then Kramers-Wannier duality takes the form

$$Y_\Lambda(K) = Y_{\tilde{\Lambda}}(\tilde{K}).$$