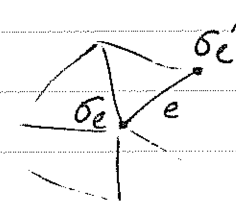
Further generalization

$$K = (K_e)_{e \in E(\Lambda)}$$

K for different edges may be different.

$$-\mathcal{H}/kT = \sum_{e \in E(\Lambda)} K_e \sigma_e \sigma_e'$$



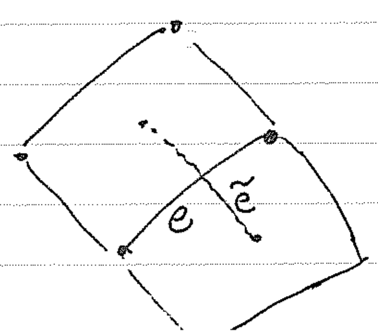
$$Z_\Lambda(K) = Z_\Lambda\left((K_e)_{e \in E(\Lambda)}\right) = \sum_{(\sigma_v)_{v \in V(\Lambda)}} e^{-\mathcal{H}/kT}$$

$$Y_\Lambda(K) = 2^{-\frac{V(\Lambda)}{2}} \prod_{e \in E(\Lambda)} \left(\cosh(2K_e)\right)^{-\frac{1}{2}} \cdot Z_\Lambda(K)$$

Then, Kramers-Wannier duality also holds

$$Y_\Lambda(K) = Y_{\tilde{\Lambda}}(\tilde{K})$$

Here $\tilde{K}$ is defined by



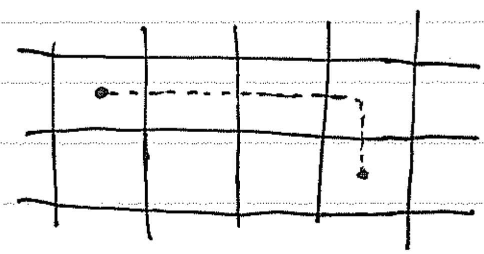
$$\tilde{K}_{\tilde{e}} = \widetilde{(K_e)}$$ i.e. $e^{-2\tilde{K}_{\tilde{e}}} = \tanh K_e$

$e \leftrightarrow \tilde{e}$ under $E(\Lambda) \cong E(\tilde{\Lambda})$

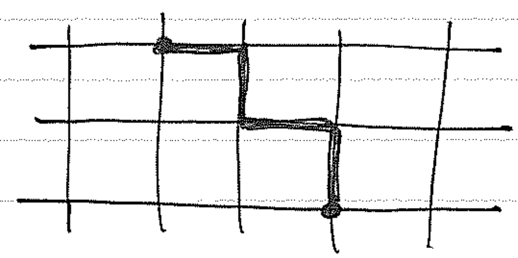
Can we consider/describe the dual variables within the original model?

$\longrightarrow$ disorder operator.

$\tilde{\Gamma}$: path $\subset \tilde{\Lambda}$



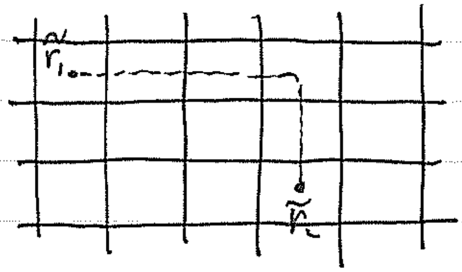
Γ : path $\subset \Lambda$



Operatoons on $K = (K_e)_{e \in E(\Lambda)}$: $K \to K_{\tilde{\Gamma}}$, $K \to K^{\Gamma}$

$$(K_{\tilde{\Gamma}})_e = \begin{cases} K_e & e \text{ does not meet } \tilde{\Gamma} \\ -K_e & e \text{ intersects with } \tilde{\Gamma} \end{cases}$$

$$(K^{\Gamma})_e = \begin{cases} K_e & e \text{ is off the path } \Gamma \\ K_e + \frac{\pi i}{2} & e \text{ is on the path } \Gamma \end{cases}$$



$\tilde{r}_1, \tilde{r}_2 \in V(\tilde{\Lambda})$

pick a path $\tilde{\Gamma} \subset \tilde{\Lambda}$

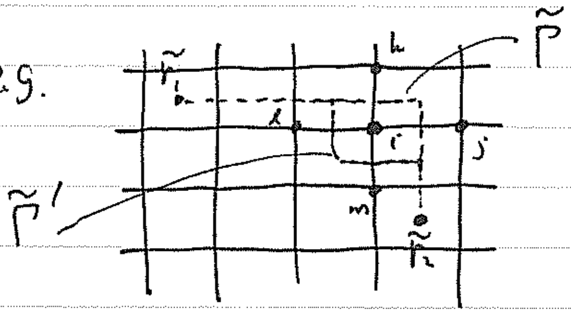
that connects them.

define

$$\langle \mu_{\tilde{r}_1} \mu_{\tilde{r}_2} \rangle := \frac{Y_\Lambda(K_{\tilde{\Gamma}})}{Y_\Lambda(K)} \underset{\substack{\uparrow \\ \cosh(-2K) = \cosh(2K)}}{=} \frac{Z_\Lambda(K_{\tilde{\Gamma}})}{Z_\Lambda(K)}$$

<u>Note</u> The result does not depend on the choice of path $\tilde{\Gamma}$.

e.g.



$$\mathcal{H}_{\text{for } K} \supset K_{ij}\sigma_i\sigma_j + K_{ih}\sigma_i\sigma_h + K_{il}\sigma_i\sigma_l + K_{im}\sigma_i\sigma_m$$

$$\Rightarrow \quad \mathcal{H}_{\text{for } K_{\tilde{\Gamma}}} \supset -K_{ij}\sigma_i\sigma_j - K_{ih}\sigma_i\sigma_h + K_{il}\sigma_i\sigma_l + K_{im}\sigma_i\sigma_m$$

$$\mathcal{H}_{\text{for } K_{\tilde{\Gamma}'}} \supset K_{ij}\sigma_i\sigma_j + K_{ih}\sigma_i\sigma_h - K_{il}\sigma_i\sigma_l - K_{im}\sigma_i\sigma_m$$

$$\therefore \quad \mathcal{H}_{\text{for } K_{\tilde{\Gamma}'}} = \mathcal{H}_{\text{for } K_{\tilde{\Gamma}}} \Big|_{\sigma_i \to -\sigma_i, \text{ others intact.}}$$

$$\sum_{\cdots\sigma_i\cdots} e^{-\mathcal{H}_{\text{for } K_{\tilde{p}'}}/kT} = \sum_{\cdots -\sigma_i \cdots} e^{-\mathcal{H}_{\text{for } K_{\tilde{p}}}/kT} \Rightarrow Y_\Lambda(K_{\tilde{p}'}) = Y_\Lambda(K_{\tilde{p}})$$

//.

<u>NB</u> The proof fails if there was an insertion of σ_i.

Of course, we are interested in the case

$$K_e \equiv K \quad \forall e \in E(\Lambda)$$

<u>constant coupling</u>.

From now on, we assume such K in the definition

$$\langle \mu_{\tilde{r}_1} \mu_{\tilde{r}_2} \rangle := \frac{Y_\Lambda(K_{\tilde{p}})}{Y_\Lambda(K)}$$

$\mu_{\tilde{r}}$ measures disorder

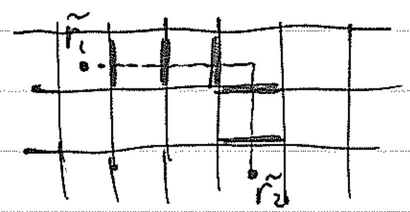
- $T \to \infty \Rightarrow K \to 0$ then $Z(K_{\tilde{\Gamma}}) \to Z(K)$

$$\therefore \langle \mu_{\tilde{r}_1} \mu_{\tilde{r}_2} \rangle = \frac{Z(K_{\tilde{\Gamma}})}{Z(K)} \longrightarrow 1 \quad \forall \tilde{r}_1, \tilde{r}_2$$

If we define $\langle \mu \rangle$ by $\langle \mu \rangle^2 = \lim_{|\tilde{r}_1 - \tilde{r}_2| \to \infty} \langle \mu_{\tilde{r}_1} \mu_{\tilde{r}_2} \rangle$

then $\langle \mu \rangle \to 1$ as $T \to \infty$.

- If $T < T_c$, nearly all spins align



has extra energy $\Delta E = 2J\,\ell(\tilde{\Gamma})$

$\uparrow$ length of $\tilde{\Gamma}$

$$\therefore Z(K_{\tilde{\Gamma}}) \sim e^{-2K\ell(\tilde{\Gamma})} Z(K)$$

$$\therefore \langle \mu_{\tilde{r}_1} \mu_{\tilde{r}_2} \rangle \sim e^{-2K\ell(\tilde{\Gamma})} \sim e^{-2Kc|\tilde{r}_1 - \tilde{r}_2|} \longrightarrow 0$$

as $|\tilde{r}_1 - \tilde{r}_2| \to \infty$

$$\therefore \langle \mu \rangle = 0 \quad \text{if } T < T_c$$

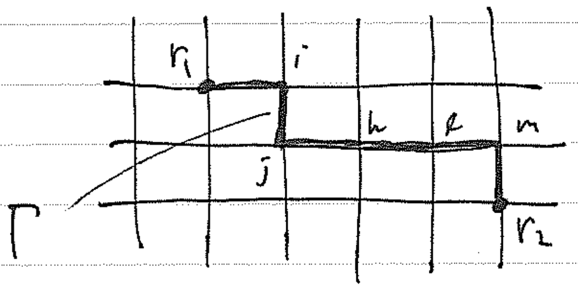
$\langle \mu \rangle$ vanishes in the ordered phase, and is non-vanishing in the disordered phase.

$\longrightarrow$ $\mu_{\tilde{r}}$ disorder operator

What is $Y_\Lambda(K^\Gamma)/Y_\Lambda(K)$?

$$e^{K\sigma_j\sigma_k} \xrightarrow[K\to K+\frac{\pi i}{2}]{} e^{(K+\frac{\pi i}{2})(\sigma_j\sigma_k)} = e^{K\sigma_j\sigma_k}\, e^{\frac{\pi i}{2}\sigma_j\sigma_k}$$

$$e^{\frac{\pi i}{2}\sigma_j\sigma_k} = i\sigma_j\sigma_k$$



$$(i\sigma_{r_1}\sigma_i)(i\sigma_i\sigma_j)(i\sigma_j\sigma_k)(i\sigma_k\sigma_\ell)(i\sigma_\ell\sigma_m)(i\sigma_m\sigma_{r_2})$$
$$= i^{\ell(\Gamma)}\sigma_{r_1}\sigma_{r_2}$$

$$\frac{Z_\Lambda(K^\Gamma)}{Z_\Lambda(K)} = \frac{\sum_{\{\sigma\}} e^{-\mathcal{H}/kT}\, i^{\ell(\Gamma)}\sigma_{r_1}\sigma_{r_2}}{\sum_{\{\sigma_i\}} e^{-\mathcal{H}/kT}} = i^{\ell(\Gamma)}\langle \sigma_{r_1}\sigma_{r_2}\rangle$$

Note $\cosh(2K) \longrightarrow \cosh 2(K+\frac{\pi i}{2}) = -\cosh(2K)$

$$(\cosh(2K))^{\frac{1}{2}} \longrightarrow i(\cosh(2K))^{\frac{1}{2}}$$

$$\therefore \frac{Y_\Lambda(K^\Gamma)}{Y_\Lambda(K)} = \langle \sigma_{r_1}\sigma_{r_2}\rangle$$

Note : $\sinh 2\widetilde{K} = \dfrac{1}{\sinh 2K}$

$$\left.\begin{array}{l} \widetilde{K} \xrightarrow[K \to K + \frac{\pi i}{2}]{} -\widetilde{K} \\[2ex] \widetilde{K} \xrightarrow[K \to -K]{} \widetilde{K} + \dfrac{\pi i}{2} \end{array}\right\} \star$$

Recall $\widetilde{K} = (\widetilde{K}_{\tilde{e}})_{\tilde{e} \in E(\tilde{\Lambda})}$; $\widetilde{K}_{\tilde{e}} = \widetilde{(K_e)}$ for $\tilde{e} \leftrightarrow e$

Then, we find from $\star$

$$\left.\begin{array}{l} \widetilde{K}^{\tilde{P}} = \widetilde{K_{\tilde{P}}} \\[2ex] \text{and} \quad \widetilde{K}_P = \widetilde{K^P} \end{array}\right\} \star$$

Kramers-Wannier : $Y_\Lambda(K) = Y_{\tilde{\Lambda}}(\widetilde{K}) \quad \forall K.$

$$\langle \mu_{\tilde{r}_1} \mu_{\tilde{r}_2} \rangle_{\Lambda, K} = \frac{Y_\Lambda(K_{\tilde{P}})}{Y_\Lambda(K)} = \frac{Y_{\tilde{\Lambda}}(\widetilde{K_{\tilde{P}}})}{Y_{\tilde{\Lambda}}(\widetilde{K})} \overset{\star}{=} \frac{Y_{\tilde{\Lambda}}(\widetilde{K}^{\tilde{P}})}{Y_{\tilde{\Lambda}}(\widetilde{K})}$$

$$= \langle \sigma_{\tilde{r}_1} \sigma_{\tilde{r}_2} \rangle_{\tilde{\Lambda}, \widetilde{K}}$$

$$\langle \sigma_{r_1} \sigma_{r_2} \rangle_{\Lambda, K} = \frac{Y_\Lambda(K^\Gamma)}{Y_\Lambda(K)} = \frac{Y_{\tilde\Lambda}(\widetilde{K^\Gamma})}{Y_{\tilde\Lambda}(\tilde K)} \overset{!}{=} \frac{Y_{\tilde\Lambda}(\tilde K_\Gamma)}{Y_{\tilde\Lambda}(\tilde K)}$$

$$= \langle \mu_{r_1} \mu_{r_2} \rangle_{\tilde\Lambda, \tilde K}$$

Thus, $\mu_{\tilde r_1}$ is indeed the dual variable

($\sigma_{\tilde r}$ for the dual theory)

Order/disorder in K, Λ $\Longleftrightarrow$ disorder/order in $\tilde K, \tilde \Lambda$

$\therefore$ Curie temperature T_c must be s.t. $K = \tilde K$

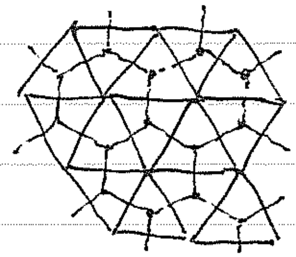
if $\Lambda \cong \tilde\Lambda$ (recall, then $K = \frac{1}{2}\log(\sqrt{2}+1)$)

e.g. square lattice.

$$\therefore \quad \frac{J}{kT_c} = \frac{1}{2}\log(\sqrt{2}+1)$$

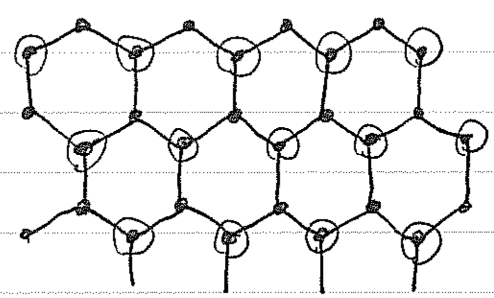
$$\therefore \quad T_c = \frac{2}{\log(\sqrt{2}+1)} \frac{J}{k}$$

Λ : triangular lattice $\Longleftrightarrow$ $\tilde{\Lambda}$: hexagonal lattice

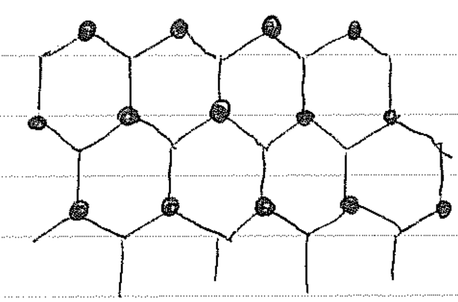


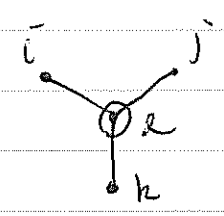
Order/disorder in K, Λ $\Leftrightarrow$ disorder/order in $\tilde{K}, \tilde{\Lambda}$

$$\text{hexagonal} \xrightarrow{\text{RG-transform}} \text{triangular}$$



sum over σ_i at $\odot$ sites only $\longrightarrow$





$$\sum_{\sigma_\ell} e^{\tilde{K}(\sigma_i\sigma_\ell+\sigma_j\sigma_\ell+\sigma_k\sigma_\ell)} = Z\, e^{\tilde{K}'(\sigma_i\sigma_j+\sigma_j\sigma_k+\sigma_k\sigma_i)}$$

for some $\tilde{K}'$ (find it!)

$\therefore$ disorder/order in $\tilde{K}, \tilde{\Lambda}$ $\Leftrightarrow$ disorder/order in $\tilde{K}', \Lambda$

Curie temperature is s.t. $\boxed{K = \tilde{K}'}$

for triangular lattice $\quad \dfrac{J}{kT_c} = \dfrac{1}{2}\log\sqrt{3}$