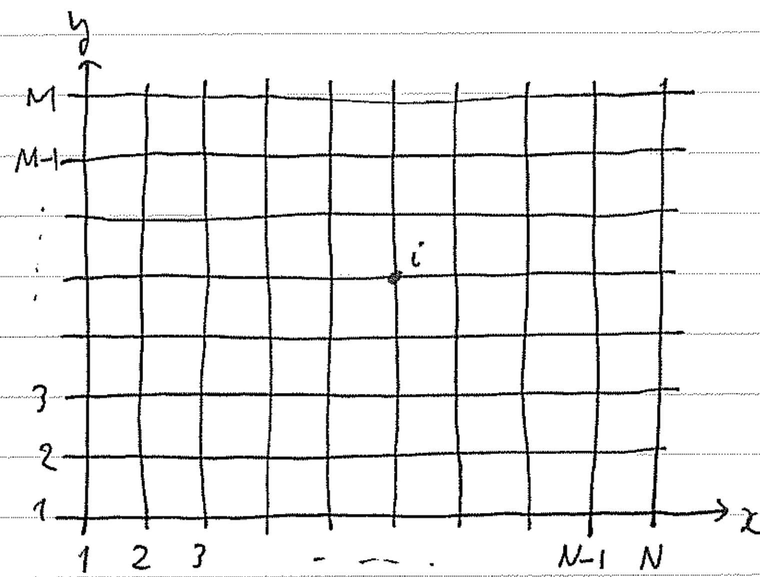
Transfer Matrix

$N \times M$ Square lattice



$$Z = \sum_{\sigma} e^{\sum_{\langle i,j \rangle} K\sigma_i\sigma_j} \qquad i,j = (x, y) \quad \begin{array}{l} x \in \{1, \cdots, N\} \\ y \in \{1, \cdots, M\} \end{array}$$

$$\sum_{\langle i,j \rangle} K\sigma_i\sigma_j = \sum_{x,y} K\sigma_{x,y}\sigma_{x+1,y} + \sum_{x,y} K\sigma_{x,y}\sigma_{x,y+1}$$

Various cases

- Open $\quad \begin{array}{l} x \in \{1, \cdots, N-1\} \\ y \in \{1, \cdots, M\} \end{array} \qquad \begin{array}{l} x \in \{1, \cdots, N\} \\ y \in \{1, \cdots, M-1\} \end{array}$

- y-periodic $\quad y = M+1$ is identified with $y = 1$.

- x-periodic $\quad x = N+1$ is identified with $x = 1$

- both-periodic $\quad (x, y) \equiv (x+N, y) \equiv (x, y+M)$

We consider these cases

$$e^{\sum_{\langle i,j\rangle} K\sigma_i\sigma_j} = \prod_{x,y} e^{K\sigma_{x,y}\sigma_{x+1,y} + K\sigma_{x,y}\sigma_{x,y+1}}$$

$$\begin{aligned}
&= \prod_{x} e^{K\sigma_{x,M}\sigma_{x+1,M} + K\sigma_{x,M}\sigma_{x,\overset{\equiv 1}{M+1}}} && \Big] = T_{\sigma_{\cdot,1},\sigma_{\cdot,M}} \\
&\times \prod_{x} e^{K\sigma_{x,M-1}\sigma_{x+1,M-1} + K\sigma_{x,M-1}\sigma_{x,M}} && \Big] = T_{\sigma_{\cdot,M},\sigma_{\cdot,M-1}} \\
&\times \cdots && \vdots \\
&\times \prod_{x} e^{K\sigma_{x,1}\sigma_{x+1,1} + K\sigma_{x,1}\sigma_{x,2}} && \Big] = T_{\sigma_{\cdot,2},\sigma_{\cdot,1}}
\end{aligned}$$

$$Z = \sum_{\sigma_{\cdot,M},\sigma_{\cdot,M-1}\cdots\sigma_{\cdot,1}} T_{\sigma_{\cdot,1},\sigma_{\cdot,M}} T_{\sigma_{\cdot,M},\sigma_{\cdot,M-1}} \cdots T_{\sigma_{\cdot,2},\sigma_{\cdot,1}}$$

$$= \mathrm{Tr}\, T^M$$

T : transfer matrix.

T can be regarded as an operator acting on

$$(\mathbb{C}^2)^{\otimes N} = \left\{ |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots \otimes |\sigma_N\rangle \;\middle|\; \sigma_i \in \{+1,-1\} \right\}$$

$$T_{\sigma'_{\cdot},\sigma_{\cdot}} = \prod_x e^{K\sigma_x\sigma_{x+1}} \underbrace{e^{K\sigma_x\sigma'_x}}$$

$$\left(e^{K\sigma\sigma'}\right)_{\sigma,\sigma'=\pm 1} = \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix} = e^{K}\begin{pmatrix} 1 & e^{-2K} \\ e^{-2K} & 1 \end{pmatrix}$$

$$= e^{K}\left(\mathbb{1}_2 + \sigma^1 \underbrace{e^{-2K}}_{\substack{\parallel \\ \tanh\tilde{K}}}\right) \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Pauli matrices)

$$= \frac{e^{K}}{\cosh\tilde{K}}\left(\mathbb{1}_2 \cosh\tilde{K} + \sigma^1 \sinh\tilde{K}\right)$$

$$= \frac{e^{K}}{\cosh\tilde{K}}\, e^{\tilde{K}\sigma^1} = \left(2\sinh(2K)\right)^{\frac{1}{2}} e^{\tilde{K}\sigma^1}$$

$$\therefore\ T = \left(2\sinh(2K)\right)^{\frac{N}{2}} \prod_x e^{\tilde{K}\sigma^1_x} \circ \prod_x e^{K\sigma^3_x \sigma^3_{x+1}}$$

$$\text{where}\quad \sigma^i_x = \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 \otimes \underset{\widehat{x}}{\sigma^i} \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2$$

Note For $V = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ we have $V^{-1}\sigma^1 V = -\sigma^3$, $V^{-1}\sigma^2 V = \sigma^2$, $V^{-1}\sigma^3 V = \sigma^1$

$$\therefore\ T = \prod_x V_x^{-1} \circ \tilde{T} \circ \prod_x V_x$$

$$\text{where}\quad \boxed{\tilde{T} = \left(2\sinh(2K)\right)^{\frac{N}{2}} \prod_x e^{\tilde{K}\sigma^3_x} \circ \prod_x e^{K\sigma^1_x \sigma^1_{x+1}}}$$

In what follows, we denote this $\tilde{T}$ by T.

$$\sigma^{\pm} := \frac{1}{2}(\sigma^1 \pm i\sigma^2) \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} &\{\sigma^+, \sigma^-\} = 1_2 \\ &\{\sigma^+, \sigma^+\} = \{\sigma^-, \sigma^-\} = 0 \\ &\sigma^3\sigma^{\pm} = -\sigma^{\pm}\sigma^3 \end{aligned}\right\}$$

Like a fermion system

with $\sigma^3 = (-1)^F$.

$$\sigma_x^i(v_1 \otimes \cdots \otimes v_N) = v_1 \otimes \cdots \otimes v_{x-1} \otimes \sigma^i v_x \otimes \cdots \otimes v_N \qquad i = \pm, 3$$

σ_x^i and $\sigma_{x'}^j$ commute ... unlike fermion system.

Jordan-Wigner transform

define $$a_x(v_1 \otimes v_2 \otimes \cdots \otimes v_N) = (-1)^{|v_1| + \cdots + |v_{x-1}|} v_1 \otimes \cdots \otimes v_{x-1} \otimes \sigma^+ v_x \otimes v_{x+1} \otimes \cdots \otimes v_N$$

$$a_x^\dagger(v_1 \otimes v_2 \otimes \cdots \otimes v_N) = (-1)^{|v_1| + \cdots + |v_{x-1}|} v_1 \otimes \cdots \otimes v_{x-1} \otimes \sigma^- v_x \otimes v_{x+1} \otimes \cdots \otimes v_N$$

$$(-1)^F(v_1 \otimes v_2 \otimes \cdots \otimes v_N) = (-1)^{|v_1| + \cdots + |v_N|} v_1 \otimes v_2 \otimes \cdots \otimes v_N$$

Where $(-1)^{\left|\binom{1}{0}\right|} = 1$, $(-1)^{\left|\binom{0}{1}\right|} = -1$. i.e. $\sigma_3 v = (-1)^{|v|} v$

Then
$$\left.\begin{aligned} &\{a_x, a_{x'}^\dagger\} = \delta_{x,x'} \\ &\{a_x, a_{x'}\} = \{a_x^\dagger, a_{x'}^\dagger\} = 0 \\ &(-1)^F a_x = -a_x(-1)^F \ \ \& \ \ (-1)^F a_x^\dagger = -a_x^\dagger(-1)^F \end{aligned}\right\}$$

a Fermion System!

- $\sigma^3 = [\sigma^+, \sigma^-]$

- $\sigma^1 = \sigma^+ + \sigma^-$, $+i\sigma^2 = \sigma^+ - \sigma^-$, $\sigma^2\sigma^3 = i\sigma^1$

$$\sigma_x^3 = [\sigma_x^+, \sigma_x^-] = [a_x, a_x^+]$$

$$(\sigma_x^1 \circ \sigma_{x+1}^1)(v_1 \otimes \cdots \otimes v_N) = v_1 \cdots v_{x-1} \otimes \underbrace{\sigma^1 v_x}_{\substack{\| \\ -i\sigma^2\sigma^3 v_x = -i\sigma^2 v_x (-1)^{|v_x|}}} \otimes \sigma^1 v_{x+1} \otimes \cdots \otimes v_N$$

$$= -i(\sigma_x^2 \circ \sigma_{x+1}^1)(v_1 \otimes \cdots \otimes v_N)(-1)^{|v_x|}$$

$$= -(a_x - a_x^+) \circ (a_{x+1} + a_{x+1}^+)(v_1 \otimes \cdots \otimes v_N)$$

Thus $$T = (2\sinh(2K))^{\frac{N}{2}} \cdot \prod_x e^{\tilde{K}[a_x, a_x^+]} \circ \prod_x e^{K(a_x^+ - a_x)\circ(a_{x+1}^+ + a_{x+1})}$$

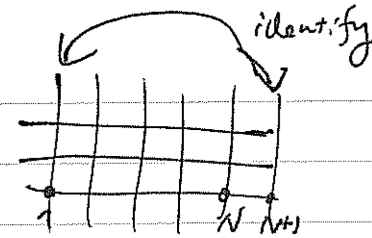
$$=: e^{-H}$$

H = some expression in a_x's and a_x^+'s.

$$Z = \mathrm{Tr}_{(\mathbb{C}^2)^{\otimes N}} e^{-MH}$$

∴ Partition function of some fermionic system!

Some Subtlety in x-periodic System



$$(N+1, y) \equiv (1, y)$$

$$T = (2\sinh(2K))^{\frac{N}{2}} \prod_{x=1}^{N} e^{\tilde{K}\sigma_x^3} \circ \underbrace{\prod_{x=1}^{N} e^{K\sigma_x^1 \sigma_{x+1}^1}}$$

The last factor $= e^{K\sigma_N^1 \underbrace{\sigma_{N+1}^1}_{\sigma_1^1}}$

$$\sigma_x^3 = [a_x, a_x^\dagger] \quad \text{OK} \quad \forall x = 1, \cdots, N$$

$$\sigma_x^1 \circ \sigma_{x+1}^1 = (a_x^\dagger - a_x)\circ(a_{x+1}^\dagger + a_{x+1}) \quad \text{OK for } x = 1, \cdots, N-1$$

Is this OK also for $\underline{x=N}$? where $\sigma_{N+1}^1 = \sigma_1^1$

$a_{N+1} \overset{?}{=} a_1$, $a_{N+1}^\dagger \overset{?}{=} a_1^\dagger$

$$(a_N^\dagger - a_N)\circ(a_1^\dagger + a_1)(v_1 \otimes \cdots \otimes v_N)$$

$$= -(a_1^\dagger + a_1)\circ(a_N^\dagger - a_N)(v_1 \otimes \cdots \otimes v_N)$$

$$= -(\sigma^1 v_1)\otimes v_2 \otimes \cdots \otimes \underbrace{(-i\sigma_2 v_N)}_{=\sigma^1\sigma^3 v_N = \sigma^1 v_N (-1)^{|v_N|}} \times (-1)^{|v_1| + \cdots + |v_{N-1}|}$$

$$= -(-1)^{|v_1| + \cdots + |v_N|}(\sigma^1 v_1)\otimes \cdots \otimes (\sigma^1 v_N)$$

$$= -(-1)^F(\sigma_1^1 \circ \sigma_N^1)(v_1 \otimes \cdots \otimes v_N)$$

$$\sigma_N^1 \circ \sigma_1^1 = -(-1)^F (a_N^\dagger - a_N) \circ (a_1^\dagger + a_1)$$

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N} = \underset{(-1)^F = 1}{\mathcal{H}_+} \oplus \underset{(-1)^F = -1}{\mathcal{H}_-}$$

On $\mathcal{H}_\pm$: $\sigma_N^1 \circ \sigma_1^1 = \mp (a_N^\dagger - a_N) \circ (a_1^\dagger + a_1)$

Define $\boxed{a_{N+1} := -a_1 , \quad a_{N+1}^\dagger := -a_1^\dagger \quad \text{on } \mathcal{H}_+}$

$\boxed{a_{N+1} := a_1 , \quad a_{N+1}^\dagger := a_1^\dagger \quad \text{on } \mathcal{H}_-}$

antiperiodic

periodic.

Then, $T = (2\sinh(2K))^{\frac{N}{2}} \prod_{x=1}^{N} e^{\tilde{K}[a_x, a_x^\dagger]} \cdot \prod_{x=1}^{N} e^{K(a_x^\dagger - a_x)\circ(a_{x+1}^\dagger + a_{x+1})}$

holds always!

$$Z = \mathrm{Tr}_{\mathcal{H}_+} T^M + \mathrm{Tr}_{\mathcal{H}_-} T^M$$

antiperiodic periodic

$$= \mathrm{Tr}_{\mathcal{H}} \frac{1 + (-1)^F}{2} T^M + \mathrm{Tr}_{\mathcal{H}} \frac{1 - (-1)^F}{2} T^M$$

antiperiodic (NS-NS) periodic (R-R)

like a GSO projection!