Operator Formalism $\to$ Path Integral in Fermionic System

A Fermionic System $\quad H = H(a_1, a_1^\dagger, \cdots, a_n, a_n^\dagger)$ Hamiltonian

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \qquad \text{Clifford algebra}$$

Irreducible representation (Clifford module)

$|0\rangle$ $\underbrace{\text{a vector}}_{\text{unit}}$ annihilated by a_i $\forall i$, $\quad a_i|0\rangle = 0$, $\langle 0|0\rangle = 1$

$$\mathcal{H} = \{ |0\rangle, a_i^\dagger|0\rangle, a_i^\dagger a_j^\dagger|0\rangle, \cdots, a_1^\dagger \cdots a_n^\dagger|0\rangle \}$$

$(-1)^F = 1 \qquad -1 \qquad 1 \qquad (-1)^n \qquad 2^n$ dim'l

A realization: $\quad \mathcal{H} = \mathbb{C}[\bar{\eta}_1, \cdots, \bar{\eta}_n]$ "functions" of n-anticommuting variables.

$$= \{ 1, \bar{\eta}_i, \bar{\eta}_i\bar{\eta}_j, \cdots, \bar{\eta}_1\cdots\bar{\eta}_n \}$$

$$(a_i f)(\bar{\eta}) = \frac{\partial}{\partial \bar{\eta}_i} f(\bar{\eta}), \quad (a_i^\dagger f)(\bar{\eta}) = \bar{\eta}_i f(\bar{\eta}).$$

inner product:

$$(f, g) = \int \prod_{i=1}^n \left(d\bar{\eta}_i d\eta_i\, e^{-\bar{\eta}_i \eta_i} \right) f(\bar{\eta})^* g(\bar{\eta})$$

in which $\bar{\eta}_i^* := \eta_i$

A Key to Path integral formulation : Kernel function

$$A \longmapsto A(\bar\eta,\eta) = A(\bar\eta_1,\dots,\bar\eta_n,\eta_1,\dots,\eta_n)$$

$\uparrow$ Some expression of $a_i, a_i^\dagger$ $\left(\mapsto f(\bar\eta) \mapsto (Af)(\bar\eta)\right)$

$\uparrow$ Kernel function for A defined by :

$$(Af)(\bar\xi) = \int \prod_{i=1}^{n}\left(d\bar\eta_i d\eta_i\, e^{-\bar\eta_i\eta_i}\right) \underbrace{A(\bar\xi,\eta)}\, f(\bar\eta)$$

[Notation: sometimes $\prod_{i=1}^{n} d\bar\eta_i d\eta_i\, e^{-\bar\eta_i\eta_i} = d^{2n}\eta\, e^{-\bar\eta\eta} \leftarrow \bar\eta\eta = \sum_{i=1}^{n}\bar\eta_i\eta_i$]

Examples

Useful formula : $\int d^{2n}\eta\, e^{-\bar\eta\eta + \bar\xi\eta} f(\bar\eta) = f(\bar\xi)$

(in this sense "$\int d^n\eta\, e^{-(\bar\eta-\bar\xi)\eta} = \delta^{(n)}(\bar\eta-\bar\xi)$")

Single variable case $f(\bar\eta) = f_0 + \bar\eta f_1$

$$\text{LHS} = \int d\bar\eta d\eta\,(1+\eta\bar\eta+\bar\xi\eta)(f_0+\bar\eta f_1)$$

$$= \int d\bar\eta d\eta\,(\cancel{f_0} + \cancel{\bar\eta f_1} + \eta\bar\eta f_0 + \cancel{\bar\xi\eta f_0} + \bar\xi\eta\bar\eta f_1)$$

$$= f_0 + \bar\xi f_1 = f(\bar\xi) = \text{RHS}$$

This means $\underline{(\mathrm{id})(\bar\xi,\eta) = e^{+\bar\xi\eta}}$ $\left(= e^{\sum_{i=1}^{n}\bar\xi_i\eta_i}\right)$

apply this to:

$$(a_i f)(\bar{\xi}) = \frac{\partial}{\partial \bar{\xi}_i} f(\bar{\xi}) = \frac{\partial}{\partial \bar{\xi}_i} \int d^{2n}\eta \; e^{-\bar{\eta}\eta + \bar{\xi}\eta} f(\bar{\eta})$$

$$= \int d^{2n}\eta \; e^{-\bar{\eta}\eta + \bar{\xi}\eta} \eta_i f(\bar{\eta})$$

$$= \int d^{2n}\eta \; e^{-\bar{\eta}\eta} \left(e^{\bar{\xi}\eta} \eta_i \right) f(\bar{\eta})$$

$$\therefore \underline{a_i(\bar{\xi}, \eta) = e^{\bar{\xi}\eta} \eta_i}$$

$$(a_i^\dagger f)(\bar{\xi}) = \bar{\xi}_i f(\bar{\xi}) = \int d^{2n}\eta \; e^{-\bar{\eta}\eta} \left(\bar{\xi}_i e^{\bar{\xi}\eta} \right) f(\bar{\eta})$$

$$\therefore \underline{a_i^\dagger(\bar{\xi}, \eta) = \bar{\xi}_i e^{\bar{\xi}\eta}}$$

$$(a_i^\dagger a_j f)(\bar{\xi}) = \bar{\xi}_i \frac{\partial}{\partial \bar{\xi}_j} \int d^{2n}\eta \; e^{-\bar{\eta}\eta + \bar{\xi}\eta} f(\bar{\eta})$$

$$= \int d^{2n}\eta \; e^{-\bar{\eta}\eta} \left(\bar{\xi}_i e^{\bar{\xi}\eta} \eta_j \right) f(\bar{\eta})$$

$$\therefore \underline{(a_i^\dagger a_j)(\bar{\xi}, \eta) = \bar{\xi}_i e^{\bar{\xi}\eta} \eta_j}$$

in general $\underline{(a_{i_1}^\dagger \cdots a_{i_\ell}^\dagger a_{j_1} \cdots a_{j_m})(\bar{\xi}, \eta) = \bar{\xi}_{i_1} \cdots \bar{\xi}_{i_\ell} e^{\bar{\xi}\eta} \eta_{j_1} \cdots \eta_{j_m}}$

$$(a_i A f)(\bar{\xi}) = \frac{\partial}{\partial \bar{\xi}_i} \int d^{2n}\eta \, e^{-\bar{\eta}\eta} A(\bar{\xi}, \eta) f(\bar{\eta})$$

$$= \int d^{2n}\eta \, e^{-\bar{\eta}\eta} \underbrace{\frac{\partial}{\partial \bar{\xi}_i} A(\bar{\xi}, \eta)} f(\bar{\eta})$$

$$(a_i^\dagger A f)(\bar{\xi}) = \bar{\xi}_i \int d^{2n}\eta \, e^{-\bar{\eta}\eta} A(\bar{\xi}, \eta) f(\bar{\eta}) = \int d^{2n}\eta \, e^{-\bar{\eta}\eta} \underbrace{\bar{\xi}_i A(\bar{\xi}, \eta)} f(\bar{\eta})$$

$$(A a_i f)(\bar{\xi}) = \int d^{2n}\eta \, e^{-\bar{\eta}\eta} A(\bar{\xi}, \eta) \frac{\partial}{\partial \bar{\eta}_i} f(\bar{\eta})$$

partial integration

$$= \int d^{2n}\eta \left[e^{-\bar{\eta}\eta} A(\bar{\xi}, \eta) \right] \left(+ \frac{\overleftarrow{\partial}}{\partial \bar{\eta}_i} \right) f(\bar{\eta})$$

$$= \int d^{2n}\eta \, e^{-\bar{\eta}\eta} \underbrace{A(\bar{\xi}, \eta) \eta_i} f(\bar{\eta})$$

$$(A a_i^\dagger f)(\bar{\xi}) = \int d^{2n}\eta \, e^{-\bar{\eta}\eta} A(\bar{\xi}, \eta) \bar{\eta}_i f(\bar{\eta}) = \int d^{2n}\eta \, A(\bar{\xi}, \eta) \left(+ \frac{\partial}{\partial \eta_i} \right) \left(e^{-\bar{\eta}\eta} f(\bar{\eta}) \right)$$

part. int.

$$= \int d^{2n}\eta \, \underbrace{[A(\bar{\xi}, \eta)] \frac{\overleftarrow{\partial}}{\partial \eta_i}} e^{-\bar{\eta}\eta} f(\bar{\eta})$$

$$\Rightarrow \boxed{\begin{aligned} &(a_i A)(\bar{\xi}, \eta) = \frac{\partial}{\partial \bar{\xi}_i} A(\bar{\xi}, \eta), \quad (a_i^\dagger A)(\bar{\xi}, \eta) = \bar{\xi}_i A(\bar{\xi}, \eta) \\ &(A a_i)(\bar{\xi}, \eta) = A(\bar{\xi}, \eta) \eta_i, \quad (A a_i^\dagger)(\bar{\xi}, \eta) = A(\bar{\xi}, \eta) \frac{\overleftarrow{\partial}}{\partial \eta_i} \end{aligned}}$$

Hermitian Conjugation

$$(f, A^\dagger g) = \int d^{2n}\xi\, e^{-\bar\xi\xi} f(\bar\xi)^* \int d^{2n}\eta\, e^{-\bar\eta\eta} A^\dagger(\bar\xi,\eta)\, g(\bar\eta)$$

$$\parallel$$

$$(Af, g) = \int d^{2n}\eta\, e^{-\bar\eta\eta} \underbrace{(Af)(\bar\eta)^*}\, g(\bar\eta)$$

$$= \left(\int d^{2n}\xi\, e^{-\bar\xi\xi} A(\bar\eta,\xi) f(\bar\xi)\right)^*$$

$$= \int d^{2n}\xi\, e^{-\bar\xi\xi} A(\bar\eta,\xi)^* f(\bar\xi)^*$$

$$\Rightarrow \boxed{A^\dagger(\bar\xi,\eta) = A(\bar\eta,\xi)^*}$$

Composition

$$(ABf)(\bar\xi) = \int d^{2n}\eta_1\, e^{-\bar\eta_1\eta_1} A(\bar\xi,\eta_1)(Bf)(\bar\eta_1)$$

$$= \int d^{2n}\eta_1\, e^{-\bar\eta_1\eta_1} A(\bar\xi,\eta_1) \int d^{2n}\eta\, e^{-\bar\eta\eta} B(\bar\eta_1,\eta) f(\bar\eta)$$

$$= \int d^{2n}\eta\, e^{-\bar\eta\eta} \underbrace{\int d^{2n}\eta_1\, e^{-\bar\eta_1\eta_1} A(\bar\xi,\eta_1) B(\bar\eta_1,\eta)} f(\bar\eta)$$

$$\Rightarrow \boxed{(AB)(\bar\xi,\eta) = \int d^{2n}\eta_1\, e^{-\bar\eta_1\eta_1} A(\bar\xi,\eta_1) B(\bar\eta_1,\eta)}$$

Traces

$$\mathrm{Tr}\,A = \int d^{2n}\eta \; e^{-\bar\eta\eta} A(-\bar\eta, \eta)$$

$$\mathrm{Tr}\,(-1)^F A = \int d^{2n}\eta \; e^{-\bar\eta\eta} A(\bar\eta, \eta)$$

<u>proof</u> $\mathrm{Tr}\,A = \sum_\alpha (f_\alpha, A f_\alpha)$ $\qquad f_\alpha$ .. orthonormal basis

$$= \sum_\alpha \int d^{2n}\xi \; e^{-\bar\xi\xi} f_\alpha(\bar\xi)^* \int d^{2n}\eta \; e^{-\bar\eta\eta} A(\bar\xi, \eta) f_\alpha(\bar\eta)$$

$$= \int d^{2n}\eta \; e^{-\bar\eta\eta} \int d^{2n}\xi \; e^{-\bar\xi\xi} \sum_\alpha f_\alpha(\bar\xi)^* f_\alpha(\bar\eta) A(\bar\xi, \eta)$$

As the basis, take $1, \bar\eta_i, \bar\eta_i\bar\eta_j, \cdots, \bar\eta_1 \cdots \bar\eta_n$

$$\sum_\alpha f_\alpha(\bar\xi)^* f_\alpha(\bar\eta) = 1 + \sum_i \xi_i \bar\eta_i + \sum_{i<j} \xi_j \xi_i \bar\eta_i \bar\eta_j + \sum_{i<j<k} \xi_k \xi_j \xi_i \bar\eta_i \bar\eta_j \bar\eta_k + \cdots$$

$$= 1 + \sum_i \xi_i \bar\eta_i + \sum_{i<j} \xi_i \bar\eta_i \xi_j \bar\eta_j + \sum_{i<j<k} \xi_i \bar\eta_i \xi_j \bar\eta_j \xi_k \bar\eta_k + \cdots$$

$$= (1 + \xi_1 \bar\eta_1)(1 + \xi_2 \bar\eta_2) \cdots (1 + \xi_n \bar\eta_n) = e^{\sum_i \xi_i \bar\eta_i} = e^{\xi\bar\eta} = e^{-\bar\eta\xi}$$

$$\therefore \mathrm{Tr}\,A = \int d^{2n}\eta \; e^{-\bar\eta\eta} \underbrace{\int d^{2n}\xi \; e^{-\bar\xi\xi - \bar\eta\xi} A(\bar\xi, \eta)}_{} = A(-\bar\eta, \eta)$$

$$= \int d^{2n}\eta \; e^{-\bar\eta\eta} A(-\bar\eta, \eta) \quad //$$

$$\mathrm{Tr}\,(-1)^F A = \mathrm{Tr}\,A\,(-1)^F$$

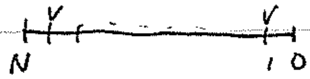
$$\vdots$$

$$= \int d^4\eta\, e^{-\bar\eta\eta} \int d^4\xi\, e^{-\bar\xi\xi}\, \underbrace{\sum_\alpha f_\alpha(\bar\xi)^* \left((-1)^F f_\alpha\right)(\bar\eta)}\, A(\bar\xi,\eta)$$

$$\begin{aligned} &= 1 - \sum_i \xi_i\bar\eta_i + \sum_{i<j} \xi_i\bar\eta_i\xi_j\bar\eta_j - \sum_{i<j<k} \xi_i\bar\eta_i\xi_j\bar\eta_j\xi_k\bar\eta_k + \cdots \\ &= e^{-\xi\bar\eta} = e^{\bar\eta\xi} \end{aligned}$$

$$= \int d^4\eta\, e^{-\bar\eta\eta}\, \underbrace{\int d^4\xi\, e^{-\bar\xi\xi + \bar\eta\xi} A(\bar\xi,\eta)}_{= A(\bar\eta,\eta)} \qquad //.$$

$$U_{t_f,t_i} = P\exp\left[-i\int_{t_i}^{t_f} dt'\, H(a,a^\dagger,t')\right]$$



$$= \lim_{\substack{N\to\infty \\ \epsilon\to 0}} e^{-i\epsilon H(a,a^\dagger,t_N)}\, e^{-i\epsilon H(a,a^\dagger,t_{N-1})} \cdots e^{-i\epsilon H(a,a^\dagger,t_1)}$$

$$U_{t_f,t_i}(\bar\xi,\eta) = \int \prod_{i=1}^{N-1}\left(d\bar\eta_i\, d\eta_i\, e^{-\bar\eta_i\eta_i}\right)\left(e^{-i\epsilon H(a,a^\dagger,t_N)}\right)(\bar\xi,\eta_{N-1})\, e^{-i\epsilon H(a,a^\dagger,t_{N-1})}(\bar\eta_{N-1},\eta_{N-2})$$

Single variable case

$$\cdots\cdots \left(e^{-i\epsilon H(a,a^\dagger,t_1)}\right)(\bar\eta_1,\eta)$$

$$e^{-i\epsilon H} \sim \mathrm{id} - i\epsilon H$$

$$\therefore \left(e^{-i\epsilon H}\right)(\bar\xi,\eta) = e^{\bar\xi\eta} - i\epsilon H(\bar\xi,\eta;t)\, e^{\bar\xi\eta}$$

$$\cong e^{\bar\xi\eta - i\epsilon H(\bar\xi,\eta;t)}$$

$$\therefore U_{t_f,t_i}(\bar\xi,\eta) = \int \prod_{i=1}^{N-1}\left(d\bar\eta_i\, d\eta_i\, e^{-\bar\eta_i\eta_i}\right) e^{\bar\xi\eta_{N-1} - i\epsilon H(\bar\xi,\eta_{N-1};t_N)}$$

$$e^{\bar\eta_{N-1}\eta_{N-2} - i\epsilon H(\bar\eta_{N-1},\eta_{N-2};t_{N-1}) + \cdots + \bar\eta_1\eta - i\epsilon H(\bar\eta_1,\eta,t_1)}$$

rename $\eta_i \to \eta_{i+1}$

$$= \int \prod_{i=1}^{N-1} d\bar\eta_i\, d\eta_{i+1}\, e^{-\sum_{i=1}^{N-1}\bar\eta_i\eta_{i+1} + \sum_{i=1}^{N}\left(\bar\eta_i\eta_i - i\epsilon H(\bar\eta_i,\eta_i,t_i)\right)}$$

$$\bar\eta_N := \bar\xi \qquad \eta_1 := \eta$$

$$\mathrm{Tr}\, U_{t_f, t_i} = \int d\bar{\xi}\, d\eta\, e^{-\bar{\xi}\eta}\, U_{t_f, t_i}(-\bar{\xi}, \eta)$$

$$= \int \underbrace{d\bar{\xi}\, d\eta}_{\mathbin{/\!/}\ d\bar{\eta}_N d\eta_{N+1}} \prod_{i=1}^{N-1} d\bar{\eta}_i d\eta_{i+1}\ e^{\underbrace{-\bar{\xi}\eta}_{\mathbin{/\!/}\ \bar{\eta}_N \eta_{N+1}} - \sum_{i=1}^{N-1} \bar{\eta}_i \eta_{i+1} + \sum_{i=1}^{N} \left(\bar{\eta}_i \eta_i - i\epsilon H(\bar{\eta}_i, \eta_i; t_i)\right)}$$

$$\bar{\eta}_N = -\bar{\xi}, \quad \eta_1 = \eta \underset{!!}{} = -\eta_{N+1}$$

$$= \int \prod_{i=1}^{N} d\bar{\eta}_i d\eta_{i+1}\ e^{-\sum_{i=1}^{N} \bar{\eta}_i \eta_{i+1} + \sum_{i=1}^{N} \left(\bar{\eta}_i \eta_i - i\epsilon H(\bar{\eta}_i, \eta_i, t_i)\right)}$$

$$= \int \prod_{i=1}^{N} d\bar{\eta}_i d\eta_{i+1}\ e^{i \sum_{i=1}^{N} \epsilon \left[i \bar{\eta}_i \frac{\eta_{i+1} - \eta_i}{\epsilon} - H(\bar{\eta}_i, \eta_i, t_i) \right]}$$

$$\xrightarrow{N\to\infty} \int \mathcal{D}\bar{\eta}\mathcal{D}\eta\ e^{i\int_{t_i}^{t_f} \left(i\bar{\eta}\dot{\eta} - H(\bar{\eta}, \eta; t)\right) dt}$$

$$\left.\begin{aligned} \eta(t_i) &= -\eta(t_f) \\ \bar{\eta}(t_i) &= -\bar{\eta}(t_f) \end{aligned}\right\} \text{Anti Periodic B.C.}$$

$$\mathrm{Tr}(-1)^F U_{t_f, U_i} = \int d\bar{\xi}\, d\eta\, e^{-\bar{\xi}\eta}\, U_{t_f, t_i}(\bar{\xi}, \eta)$$

$$= \int \mathcal{D}\bar{\eta}\mathcal{D}\eta\ e^{i\int_{t_i}^{t_f} \left(i\bar{\eta}\dot{\eta} - H(\bar{\eta}, \eta; t)\right) dt}$$

$$\left.\begin{aligned} \eta(t_i) &= \eta(t_f) \\ \bar{\eta}(t_i) &= \bar{\eta}(t_f) \end{aligned}\right\} \text{Periodic B.C.}$$