The Kernel function for the Transfer Matrix

Change in convention : $a_x \leftrightarrow a_x^\dagger$,

Then,
$$T = (2\sinh(2K))^{\frac{N}{2}} \prod_x e^{\tilde{K}[a_x^\dagger, a_x]} \prod_x e^{K(a_x - a_x^\dagger)(a_{x+1} + a_{x+1}^\dagger)}$$

We'd like to find the Kernel function $T(\bar{\xi}, \eta)$

Outline

(I) We find equations of the form

$$a_p T = f_1 T a_p + f_2 T a_{-p}^\dagger$$
$$a_p^\dagger T = f_3 T a_p + f_4 T a_{-p}^\dagger$$

($a_p, a_p^\dagger$: Fourier components of $a_x, a_x^\dagger$)

(II) This is enough to fix

$$T(\bar{\xi}, \eta) \propto \exp\left(\sum_p \left(A_p \bar{\xi}_p \eta_p + B_p \bar{\xi}_{-p} \bar{\xi}_p + C_p \eta_{-p} \eta_p \right) \right)$$

(III) The proportionality constant can be fixed by computing

$$\langle 0 | T | 0 \rangle.$$

This way we find

$$\boxed{T(\bar{\xi}, \eta) = (\sinh(2K))^{N} \exp\left(\sum_p \frac{(1-t^2)t^{-1} \bar{\xi}_p \eta_p + i t^{-1} \sin p \, \bar{\xi}_{-p} \bar{\xi}_p + i t \sin p \, \eta_{-p} \eta_p}{1 + t^2 + 2t \cos p} \right)}$$

(I) Compute $T^{-1} a_x T$ & $T^{-1} a_x^{+} T$

$$T = (2\sinh(2K))^{\frac{N}{2}} \prod_x U_x \prod_x V_x \qquad U_x = e^{\tilde{K}[a_x^+,\, a_x]}$$

$$V_x = e^{K(a_x - a_x^+)(a_{x+1} + a_{x+1}^+)}$$

$U_{x'}$ commutes with a_x, a_x^+ if $x' \neq x$. $U_{x'}$'s mutually commute

$V_{x'}$ 〃 if $x' \neq x-1, x$. $V_{x'}$'s mutually commute.

$$\therefore\ T^{-1} a_x^{(+)} T_x = V_x^{-1} V_{x-1}^{-1} U_x^{-1} a_x^{(+)} U_x V_{x-1} V_x$$

- $U_x^{-1} a_x U_x = e^{\tilde{K}[a_x,\, a_x^+]} a_x\, e^{-\tilde{K}[a_x,\, a_x^+]} = ?$

$$[[a_x, a_x^+], a_x] = (a_x a_x^+ - \cancel{a_x^+ a_x}) a_x - a_x (\cancel{a_x a_x^+} - a_x^+ a_x)$$

$$= 2 a_x a_x^+ a_x = 2\{a_x, a_x^+\} a_x - 2 a_x^+ \cancel{a_x a_x} = 2 a_x$$

$$\therefore\ U_x^{-1} a_x U_x = e^{2\tilde{K}} a_x$$

- $V_{x-1}^{-1} a_x V_{x-1} = e^{K(a_{x-1}^+ - a_{x-1})(a_x + a_x^+)} a_x\, e^{-K(a_{x-1}^+ - a_{x-1})(a_x + a_x^+)} = ?$

$$[(a_{x-1}^+ - a_{x-1})(a_x + a_x^+), a_x] = (a_{x-1}^+ - a_{x-1})(a_x + a_x^+) a_x - a_x (a_{x-1}^+ - a_{x-1})(a_x + a_x^+)$$

$$= (a_{x-1}^+ - a_{x-1}) \{ \underbrace{(a_x + a_x^+) a_x + a_x (a_x + a_x^+)}_{1} \} = (a_{x-1}^+ - a_{x-1})$$

The same way, we find:

$$[(a_{x-1}^+ - a_{x-1})(a_x + a_x^+),\ a_{x-1}^+ - a_{x-1}] = 2(a_x + a_x^+)$$

$$[(a_{x-1}^+ - a_{x-1})(a_x + a_x^+),\ a_x + a_x^+] = 2(a_{x-1}^+ - a_{x-1})$$

Thus,

$$V_{x-1}^{-1} a_x V_{x-1} = a_x + K(a_{x-1}^\dagger - a_{x-1}) + \frac{K^2}{2!} 2(a_x + a_x^\dagger)$$
$$+ \frac{K^3}{3!} 2^2 (a_{x-1}^\dagger - a_{x-1}) + \frac{K^4}{4!} 2^3 (a_x + a_x^\dagger) + \cdots$$

$$= a_x + \frac{1}{2}\sinh(2K)(a_{x-1}^\dagger - a_{x-1}) + \frac{1}{2}(\cosh(2K) - 1)(a_x + a_x^\dagger),$$

$$= -\frac{1}{2}(a_x^\dagger - a_x) + \frac{1}{2}\cosh(2K)(a_x + a_x^\dagger) + \frac{1}{2}\sinh(2K)(a_{x-1}^\dagger - a_{x-1})$$

- $V_x^{-1}(V_{x-1}^{-1} a_x V_{x-1}) V_x = ?$

For this we need to know $V_x^{-1}(a_x^\dagger \mp a_x) V_x = e^{K(a_x^\dagger - a_x)(a_{x+1} + a_{x+1}^\dagger)}(a_x^\dagger \mp a_x) \times$
$\times\, e^{-K(a_x^\dagger - a_x)(a_{x+1} + a_{x+1}^\dagger)}$

$$[(a_x^\dagger - a_x)(a_{x+1} + a_{x+1}^\dagger), a_x^\dagger + a_x] = 0$$
$$[(a_x^\dagger - a_x)(a_{x+1} + a_{x+1}^\dagger), a_x^\dagger - a_x] = 2(a_{x+1} + a_{x+1}^\dagger)$$
$$[(a_x^\dagger - a_x)(a_{x+1} + a_{x+1}^\dagger), a_{x+1} + a_{x+1}^\dagger] = 2(a_x^\dagger - a_x)$$

$$\therefore\ V_x^{-1}(a_x^\dagger + a_x) V_x = a_x^\dagger + a_x$$

$$V_x^{-1}(a_x^\dagger - a_x) V_x = (a_x^\dagger - a_x) + K\, 2(a_{x+1} + a_{x+1}^\dagger) + \frac{K^2}{2!} 2^2 (a_x^\dagger - a_x)$$
$$+ \frac{K^3}{3!} 2^3 (a_{x+1} + a_{x+1}^\dagger) + \frac{K^4}{4!} 2^4 (a_x^\dagger - a_x) + \cdots$$

$$= \cosh(2K)(a_x^\dagger - a_x) + \sinh(2K)(a_{x+1} + a_{x+1}^\dagger)$$

$$\cdot\ V_x^{-1} V_{x-1}^{-1} a_x V_{x-1} V_x = -\frac{1}{2}\left(\cosh(2K)(a_x^\dagger - a_x) + \sinh(2K)(a_{x+1} + a_{x+1}^\dagger)\right) + \frac{1}{2}\cosh(2K)(a_x + a_x^\dagger)$$
$$+ \frac{1}{2}\sinh(2K)(a_{x-1}^\dagger - a_{x-1})$$

$$= \cosh(2K)\, a_x - \frac{1}{2}\sinh(2K)(a_{x+1} + a_{x+1}^\dagger) + \frac{1}{2}\sinh(2K)(a_{x-1}^\dagger - a_{x-1}).$$

$$\therefore\ T^{-1} a_x T = e^{2\tilde{K}} \left\{ \cosh(2K)\, a_x + \frac{1}{2}\sinh(2K)\left(-a_{x+1} - a^\dagger_{x+1} + a^\dagger_{x-1} - a_{x-1}\right) \right\}$$

Similarly

$$T^{-1} a^\dagger_x T = e^{-2\tilde{K}} \left\{ \cosh(2K)\, a^\dagger_x + \frac{1}{2}\sinh(2K)\left(a_{x+1} + a^\dagger_{x+1} + a^\dagger_{x-1} - a_{x-1}\right) \right\}$$

<u>Fourier expansion</u>

$$a_x = \sum_p \frac{e^{ipx}}{\sqrt{N}} a_p \qquad p \in \begin{cases} \frac{2\pi}{N}\mathbb{Z} \text{ (periodic)} \\ \frac{2\pi}{N}\left(\mathbb{Z}+\frac{1}{2}\right) \text{ (antiperiodic)} \end{cases} \quad \begin{matrix} \text{mod } 2\pi\mathbb{Z} \\ \text{(N. values)} \end{matrix}$$

$$a^\dagger_x = \sum_p \frac{e^{-ipx}}{\sqrt{N}} a^\dagger_p$$

$$\{a_x, a^\dagger_{x'}\} = \delta_{x,x'}, \quad \{a_x, a_{x'}\} = 0 = \{a^\dagger_x, a^\dagger_{x'}\}$$

$$\Longleftrightarrow \quad \{a_p, a^\dagger_{p'}\} = \delta_{p,p'}, \quad \{a_p, a_{p'}\} = 0 = \{a^\dagger_p, a^\dagger_{p'}\}$$

Then

$$T^{-1} a_p T = f_1 a_p + f_2 a^\dagger_{-p}$$

$$T^{-1} a^\dagger_p T = f_3 a_p + f_4 a^\dagger_{-p}$$

where $f_1 = e^{2\tilde{K}}\left(\cosh(2K) - \sinh(2K)\cos p\right)$, $f_2 = e^{2\tilde{K}}\sinh(2K)(-i\sin p)$

$f_3 = e^{-2\tilde{K}}\sinh(2K)(i\sin p)$, $f_4 = e^{-2\tilde{K}}\left(\cos(2K) + \sinh(2K)\cos p\right)$

Note : $e^{\pm 2\tilde{K}} = t^{\mp 1}$ $\qquad t = \tanh(K) \iff e^{2K} = \frac{1+t}{1-t}$

$$\therefore \cosh(2K) = \frac{e^{2K}+e^{-2K}}{2} = \frac{\frac{1+t}{1-t}+\frac{1-t}{1+t}}{2} = \frac{(1+t)^2+(1-t)^2}{2(1-t^2)} = \frac{1+t^2}{1-t^2}$$

$$\sinh(2K) = \frac{e^{2K}-e^{-2K}}{2} = \frac{(1+t)^2-(1-t)^2}{2(1-t^2)} = \frac{2t}{1-t^2}$$

$$\Rightarrow \quad f_1 = \frac{1}{t(1-t^2)}\left(1+t^2-2t\cos p\right), \qquad f_2 = \frac{-2i\sin p}{1-t^2}$$

$$f_3 = \frac{2it^2\sin p}{1-t^2}, \qquad f_4 = \frac{t}{1-t^2}\left(1+t^2+2t\cos p\right)$$

II

$$a_p T = f_1 T a_p + f_2 T a_{-p}^\dagger$$

$$a_{-p}^\dagger T = f_3 T a_p + f_4 T a_{-p}^\dagger$$

means, for the Kernel function $T(\bar\eta, \eta) = T$

$$\left.\begin{aligned} \frac{\partial}{\partial \bar\eta_p} T &= f_1 T \eta_p + f_2 T \frac{\overleftarrow{\partial}}{\partial \eta_{-p}} \\ \bar\eta_{-p} T &= f_3 T \eta_p + f_4 T \frac{\overleftarrow{\partial}}{\partial \eta_{-p}} \end{aligned}\right\} (\star)$$

Assume the form $T = \underline{\text{const}} \cdot \exp\left(\sum_p A_p \bar\eta_p \eta_p + B_p \bar\eta_{-p} \bar\eta_p + C_p \eta_{-p} \eta_p \right)$.

(may assume $B_{-p} = -B_p$, $C_{-p} = -C_p$)

Then the equs $(\star)$ meas

$$A_p \eta_p - B_p \bar\eta_{-p} + B_{-p} \bar\eta_{-p} = f_1 \eta_p + f_2 \left(A_{-p} \bar\eta_{-p} + C_{-p} \eta_p - C_p \eta_p \right)$$

$$\bar\eta_{-p} = f_3 \eta_p + f_4 \left(A_{-p} \bar\eta_{-p} + C_{-p} \eta_p - C_p \eta_p \right)$$

i.e.

$$A_p = f_1 - 2 f_2 C_p, \quad -2 B_p = f_2 A_{-p}$$

$$1 = f_4 A_{-p}, \quad 0 = f_3 - 2 f_4 C_p$$

$$\therefore \boxed{\begin{aligned} A_p &= \frac{1}{f_4} = \frac{1-t^2}{t} \frac{1}{1+t^2+2t\cos p} \\ B_p &= -\frac{f_2}{2f_4} = \frac{i t^{-1} \sin p}{1+t^2+2t\cos p}, \quad C_p = \frac{f_3}{2f_4} = \frac{i t \sin p}{1+t^2+2t\cos p} \end{aligned}}$$

III Fixing the proportionality constant in $T(\bar{\eta},\eta) = \underset{\uparrow}{\underline{\text{Const}}}\, e^{A\bar{\eta}\eta + B\bar{\eta}\bar{\eta} + C\eta\eta}$

Note that, for $f(\bar{\eta}) \equiv 1$ ($\Leftrightarrow$ state $|0\rangle$),

$$(Tf)(\bar{\xi}) = \int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} \underbrace{T(\bar{\xi},\eta)}_{\underline{\text{Const}} \cdot \exp\left(\sum_p A_p \bar{\xi}_p \eta_p + B_p \bar{\xi}_{-p}\bar{\xi}_p + C_p \eta_{-p}\eta_p\right)} \underbrace{f(\bar{\eta})}_{=1}$$

Non-zero answer for $\int d\bar{\eta}_p\, d\eta_p$ must come from $-\bar{\eta}_p\eta_p$ in $e^{-\bar{\eta}\eta}$.

$$\Rightarrow (Tf)(\bar{\xi}) = \underline{\text{Const}} \cdot \exp\left(\sum_p B_p \bar{\xi}_{-p}\bar{\xi}_p\right).$$

$$= \underline{\text{Const}} \cdot 1 + O(\bar{\xi}\bar{\xi})$$

Thus, to find $\underline{\text{Const}}$, we want to find

$$T|0\rangle = \underset{\uparrow}{(?)}\,|0\rangle + \text{Other components}$$

Remember $T = (2\sinh(2K))^{\frac{N}{2}} \prod_{x=1}^{N} \underbrace{U_x}_{e^{\tilde{K}[a_x^\dagger, a_x]}} \cdot \prod_{x=1}^{N} \underbrace{V_x}_{e^{K(a_x - a_x^\dagger)(a_{x+1} + a_{x+1}^\dagger)}}$ (assume periodic $\sigma_{x+N,y} = \sigma_{x,y}$)

$V_x|0\rangle = ?$

$$(a_x - a_x^\dagger)(a_{x+1} + a_{x+1}^\dagger)|0\rangle = (a_x - a_x^\dagger)\, a_{x+1}^\dagger |0\rangle = -a_x^\dagger a_{x+1}^\dagger |0\rangle.$$

$$(a_x - a_x^\dagger)(a_{x+1} + a_{x+1}^\dagger)\left(-a_x^\dagger a_{x+1}^\dagger|0\rangle\right) = (a_x - a_x^\dagger)\, a_x^\dagger|0\rangle = |0\rangle.$$

$$\therefore\ V_x|0\rangle = |0\rangle \underbrace{\cosh(K)}_{=:c} - a_x^\dagger a_{x+1}^\dagger|0\rangle \underbrace{\sinh(K)}_{=:s}$$

$$\underline{V_{x-1} V_x |0\rangle = ?}$$

$$\cdot\ V_{x-1}|0\rangle = |0\rangle c - a^\dagger_{x-1} a^\dagger_x |0\rangle s$$

$$\cdot\ V_{x-1}\, a^\dagger_x a^\dagger_{x+1}|0\rangle = ?$$

$$(a_{x-1} - a^\dagger_{x-1})(a_x + a^\dagger_x)(a^\dagger_x a^\dagger_{x+1}|0\rangle) = (a_{x-1} - a^\dagger_{x-1})\, a^\dagger_{x+1}|0\rangle = -a^\dagger_{x-1} a^\dagger_{x+1}|0\rangle$$

$$(a_{x-1} - a^\dagger_{x-1})(a_x + a^\dagger_x)(-a^\dagger_{x-1} a^\dagger_{x+1}|0\rangle) = (a_{x-1} - a^\dagger_{x-1})\, a^\dagger_{x-1} a^\dagger_x a^\dagger_{x+1}|0\rangle = a^\dagger_x a^\dagger_{x+1}|0\rangle$$

$$\therefore\ V_{x-1}\, a^\dagger_x a^\dagger_{x+1}|0\rangle = a^\dagger_x a^\dagger_{x+1}|0\rangle c - a^\dagger_{x-1} a^\dagger_{x+1}|0\rangle s$$

$$\therefore\ V_{x-1} V_x |0\rangle = (|0\rangle c - a^\dagger_{x-1} a^\dagger_x |0\rangle s)c - (a^\dagger_x a^\dagger_{x+1}|0\rangle c - a^\dagger_{x-1} a^\dagger_{x+1}|0\rangle s)s$$

$$= |0\rangle c^2 - a^\dagger_{x-1} a^\dagger_x |0\rangle sc - a^\dagger_x a^\dagger_{x+1}|0\rangle cs + a^\dagger_{x-1} a^\dagger_{x+1}|0\rangle s^2$$

$$\underline{V_{x-2} V_{x-1} V_x |0\rangle = ?}$$

$$V_{x-2}|0\rangle = |0\rangle c - a^\dagger_{x-2} a^\dagger_{x-1}|0\rangle s$$

$$V_{x-2}\, a^\dagger_{x-1} a^\dagger_{\underset{x+1}{x}}|0\rangle = a^\dagger_{x-1} a^\dagger_{\underset{x+1}{x}}|0\rangle c - a^\dagger_{x-2} a^\dagger_{\underset{x+1}{x}}|0\rangle s$$

$$V_{x-2}\, a^\dagger_x a^\dagger_{x+1}|0\rangle = a^\dagger_x a^\dagger_{x+1}|0\rangle c - a^\dagger_{x-2} a^\dagger_{x-1} a^\dagger_x a^\dagger_{x+1}|0\rangle s$$

$$\therefore\ V_{x-2} V_{x-1} V_x |0\rangle = \cdots$$

$$= |0\rangle c^3 - a^\dagger_{x-2} a^\dagger_{x-1}|0\rangle sc^2 - a^\dagger_{x-1} a^\dagger_x |0\rangle csc + a^\dagger_{x-2} a^\dagger_x |0\rangle ssc$$

$$- a^\dagger_x a^\dagger_{x+1}|0\rangle c^2 s + a^\dagger_{x-2} a^\dagger_{x-1} a^\dagger_x a^\dagger_{x+1}|0\rangle scs$$

$$+ a^\dagger_{x-1} a^\dagger_{x+1}|0\rangle cs^2 - a^\dagger_{x-2} a^\dagger_{x+1}|0\rangle s^3$$

$$\therefore \prod_x V_x |0\rangle = \exp\Big(\sum_{x<x'} a_x^\dagger a_{x'}^\dagger \left(-\frac{S}{C}\right)^{x'-x} \Big) |0\rangle \times C^N$$

$$U_x = e^{\tilde{K}[a_x^\dagger, a_x]} = e^{\tilde{K}(a_x^\dagger a_x - a_x a_x^\dagger)} = e^{\tilde{K}(2a_x^\dagger a_x - 1)}$$

$$U = \prod_x U_x = e^{\sum_x \tilde{K}(2a_x^\dagger a_x - 1)} = e^{2\tilde{K} N} \cdot e^{-N\tilde{K}}$$

$$N = \sum_x a_x^\dagger a_x = \# \text{ operator}$$

$$\therefore \prod_x U_x \prod_x V_x |0\rangle = \exp\Big(\sum_{x<x'} a_x^\dagger a_{x'}^\dagger e^{4\tilde{K}} \left(-\frac{S}{C}\right)^{x'-x} \Big) |0\rangle \cdot e^{-N\tilde{K}} C^N$$

$$= e^{-N\tilde{K}} (\cosh(K))^N |0\rangle + \text{other components.}$$

$$\therefore T|0\rangle = \underbrace{(2\sinh(2K))^{\frac{N}{2}} (e^{-\tilde{K}} \cosh(K))^N}_{} |0\rangle + \text{other components.}$$

$$\Big(2\sinh(2K)\, \underbrace{e^{-2\tilde{K}} \cosh^2(K)}_{\frac{1}{2}\sinh(2K)}\Big)^{\frac{N}{2}} = (\sinh(2K))^N$$

$$= (\sinh(2K))^N |0\rangle + \text{other components.}$$

$$\therefore \underline{\text{const}} = (\sinh(2K))^N$$

$$\therefore \underline{T(\bar{\xi}, \eta) = (\sinh(2K))^N \exp\Big(\sum_p A_p \bar{\xi}_p \eta_p + B_p \bar{\xi}_{-p} \bar{\xi}_p + C_p \eta_{-p} \eta_p \Big)}$$

where A_p, B_p, C_p are as given before.

The Partition function

Now that we know $T(\bar{\eta}, \eta)$, we can compute

$$\mathrm{Tr}_{\alpha} (\pm 1)^F T^M = \int \prod_{y=1}^{M} \prod_{p_x} \left(d\bar{\eta}_{p_x, y}\, d\eta_{p_x, y+1}\, e^{-\bar{\eta}_{p_x, y} \eta_{p_x, y+1}} \right)$$

AP or P in x

$$T(\bar{\eta}_{\cdot, M}, \eta_{\cdot, M})\, T(\bar{\eta}_{\cdot, M-1}, \eta_{\cdot, M-1}) \cdots T(\bar{\eta}_{\cdot, 1}, \bar{\eta}_{\cdot, 1})$$

where $\eta_{p_x, M+1} = \mp \eta_{p_x, 1}$ $\begin{cases} AP \\ P \end{cases}$ in y

$$= (\sinh(2K))^{MN} \int \prod_{y, p_x} d\bar{\eta}_{p_x, y}\, d\eta_{p_x, y+1}\, e^{A},$$

$$A := -\sum_{y=1}^{M} \sum_{p_x} \bar{\eta}_{p_x, y} \eta_{p_x, y+1} + \sum_{y=1}^{M} \sum_{p_x} \left(A_{p_x} \bar{\eta}_{p_x, y} \eta_{p_x, y} + B_{p_x} \bar{\eta}_{-p_x, y} \bar{\eta}_{p_x, y} + C_{p_x} \eta_{-p_x, y} \eta_{p_x, y} \right)$$

Fourier expansion

$$\eta_{p_x, y} = \sum_{p_y} \frac{e^{i p_y y}}{\sqrt{M}} \eta_{p_x, p_y} \qquad p_y \in \left\{ \begin{array}{l} \frac{2\pi}{M} \mathbb{Z} \ (P) \\ \frac{2\pi}{M} (\mathbb{Z} + \frac{1}{2}) \ (AP) \end{array} \right\} \mod 2\pi\mathbb{Z}$$

$$\bar{\eta}_{p_x, y} = \sum_{p_y} \frac{e^{-i p_y y}}{\sqrt{M}} \bar{\eta}_{p_x, p_y} \qquad (M\text{-values})$$

<u>exercise</u> $\displaystyle \prod_{y, p_x} d\bar{\eta}_{p_x, y}\, d\eta_{p_x, y+1} = \prod_{p_x, p_y} d\bar{\eta}_{p_x, p_y}\, d\eta_{p_x, p_y}$

(Check it for $M = 2, 3, 4$)

$$A = \sum_{p=(p_x,p_y)} \left(-e^{ip_y} \bar{\eta}_p \eta_p + A_{p_x} \bar{\eta}_p \eta_p + B_{p_x} \bar{\eta}_{-p} \bar{\eta}_p + C_{p_x} \eta_{-p} \eta_p \right)$$

$$= \frac{1}{2} \sum_p (\bar{\eta}_p , \eta_{-p}) \underbrace{\begin{pmatrix} -2B_{p_x} & -e^{ip_y}+A_{p_x} \\ e^{-ip_y}-A_{p_x} & 2C_{p_x} \end{pmatrix}}_{M_p} \begin{pmatrix} \bar{\eta}_{-p} \\ \eta_p \end{pmatrix}$$

$$M_p = \begin{pmatrix} \frac{f_2}{f_4} & -e^{ip_y} + \frac{1}{f_4} \\ e^{-ip_y} - \frac{1}{f_4} & \frac{f_3}{f_4} \end{pmatrix} = \begin{pmatrix} 0 & -e^{ip_y} \\ e^{-ip_y} & 0 \end{pmatrix} + \frac{1}{1+t^2+2t\cos p_x} \begin{pmatrix} -2it^{-1}\sin p_x & \frac{1-t^2}{t} \\ -\frac{1-t^2}{t} & 2it\sin p_x \end{pmatrix}$$

$$\det M_p = \frac{f_2 f_3}{f_4^2} - \left(e^{-ip_y} - \frac{1}{f_4}\right)\left(e^{ip_y} + \frac{1}{f_4}\right) = \frac{\overbrace{f_2 f_3 + 1}^{= f_1 f_4}}{f_4^2} + 1 - \frac{2\cos p_y}{f_4}$$

$$= \frac{1}{f_4} \left(f_1 + f_4 - 2\cos p_y \right)$$

$$= \frac{1-t^2}{t(1+t^2+2t\cos p_x)} \left\{ \frac{1+t^2-2t\cos p_x}{t(1-t^2)} + \frac{t(1+t^2+2t\cos p_x)}{1-t^2} - 2\cos p_y \right\}$$

$$= \frac{1}{t^2(1+t^2+2t\cos p_x)} \left\{ 1+t^2-2t\cos p_x + t^2(1+t^2+2t\cos p_x) - 2t(1-t^2)\cos p_y \right\}$$

$$= \frac{1}{t^2(1+t^2+2t\cos p_x)} \left\{ (1+t^2)^2 - 2t(1-t^2)(\cos p_x + \cos p_y) \right\}$$

$$\therefore \; \mathrm{Tr}_\alpha (\pm 1)^F T^M = (\sinh(2K))^{MN} \cdot \mathrm{Pf}\, M$$

$$= (\sinh(2K))^{MN} \cdot \prod_p (\det M_p)^{\frac{1}{2}} \quad \text{up to sign}$$

$$= (\sinh(2K))^{MN} \cdot \prod_p \frac{1}{t\,(1+t^2+2t\cos p_x)^{\frac{1}{2}}} \left\{ (1+t^2)^2 - 2t(1-t^2)(\cos p_x + \cos p_y) \right\}^{\frac{1}{2}}$$

$$= (\sinh(2K))^{MN} t^{-MN} \prod_p \frac{1}{(1+t^2+2t\cos p_x)^{\frac{1}{2}}} \left\{ (1+t^2)^2 - 2t(1-t^2)(\cos p_x + \cos p_y) \right\}^{\frac{1}{2}}$$

<u>Note</u>

$$\left.\begin{array}{l} \alpha = (\text{AP or P in } x) \\ (\pm 1)^F \leftrightarrow \text{AP or P in } y \end{array}\right\} \text{ enters in the range of } p = (p_x, p_y)$$

$$p_x \in \frac{2\pi}{N} \times \begin{cases} \mathbb{Z} & \text{P in } x \\ \mathbb{Z} + \frac{1}{2} & \text{AP in } x \end{cases} \pmod{2\pi}$$

$$p_y \in \frac{2\pi}{M} \times \begin{cases} \mathbb{Z} & \text{P in } y \\ \mathbb{Z} + \frac{1}{2} & \text{AP in } y \end{cases} \pmod{2\pi}$$

- $$\sinh(2K)\, t^{-1} = 2\sinh(K)\cosh(K) \cdot \underbrace{\frac{\cosh(K)}{\sinh(K)}}_{t^{-1}} = 2\cosh^2(K)$$

  $$\stackrel{\text{or}}{=} \frac{2t}{1-t^2} \cdot t^{-1} = \frac{2}{1-t^2}$$

- $$1+t^2+2t\cos p = (1+te^{ip})(1+te^{-ip})$$

$$\prod_p (1+te^{ip}) = 1 + \underbrace{\sum_p t e^{ip}}_{=0} + \underbrace{\sum_{p_1<p_2} t^2 e^{ip_1} e^{ip_2}}_{=0} + \underbrace{\cdots}_{=0} + t^N e^{i\sum_p p}$$

in some order

$$= 1 + t^N e^{i\sum_p p}$$

$$\therefore \prod_{p_x}(1+t^2+2t\cos p_x) = \prod_{p_x}(1+te^{ip_x})(1+te^{-ip_x}) = \left(1+t^N e^{i\sum_{p_x} p_x}\right)\left(1+t^N e^{-i\sum_{p_x} p_x}\right)$$

$$\left(\sum_{p_x} p_x \equiv \begin{cases} \pi \pmod{2\pi} & x\text{-periodic \& } N \text{ even} \\ 0 \pmod{2\pi} & \text{otherwise} \end{cases}\right)$$

$$= \left.\begin{cases} (1-(-1)^N t^N)^2 & x\text{-periodic} \\ (1+t^N)^2 & x\text{-antiperiodic} \end{cases}\right\} \xrightarrow{N\to\infty} 1$$

$$\boxed{\begin{aligned} Z_{\alpha,\beta} &= \mathrm{Tr}_\alpha\left((-1)^{\nu_\beta F} T^M\right) \qquad \begin{pmatrix} \alpha,\beta = AP \text{ or } P \\ \nu_{AP}=0,\ \nu_P = 1 \end{pmatrix} \\ &= \frac{2^{MN}}{(1 \underset{\smile}{\pm} t^N)^M} \prod_p \left\{ \left(\frac{1+t^2}{1-t^2}\right)^2 - \frac{2t}{1-t^2}(\cos p_x + \cos p_y) \right\}^{\frac{1}{2}} \end{aligned}}$$

$\left\{\begin{array}{ll} - & \alpha = \text{Periodic}, N \text{ even} \\ + & \text{otherwise} \end{array}\right.$

$p_x \in \frac{2\pi}{N}(\mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2})$ for $\alpha = P$ or AP

$p_y \in \frac{2\pi}{M}(\mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2})$ for $\beta = P$ or AP

$$Z = \frac{1}{2}\left(Z_{AP,AP} + Z_{AP,P} + Z_{P,AP} \underbrace{- (-1)^N Z_{P,P}}\right\}$$

$-Z_{P,P}$ before the convention change

$a_x \leftrightarrow a_x^\dagger$

$|0\rangle$ even, $a_1^\dagger \cdots a_N^\dagger|0\rangle$ $(-1)^N$

Thermodynamic limit

$$f = \lim_{M,N\to\infty} \frac{1}{MN} \log Z \qquad \left(\text{free energy per site}/kT \times (-1)\right)$$

$$\frac{1}{MN}\log Z = \log 2 - \underbrace{\frac{1}{N}\log(1\pm t^N)}_{\xrightarrow{N\to\infty}\ 0} + \frac{1}{2}\underbrace{\sum_p \frac{1}{MN}}_{\to \int_0^{2\pi}\int_0^{2\pi}\frac{dp_x dp_y}{(2\pi)(2\pi)}} \log\left\{\left(\frac{1+t^2}{1-t^2}\right)^2 - \frac{2t}{1-t^2}(\cos p_x + \cos p_y)\right\}$$

$$\boxed{\begin{aligned} \therefore\ f &= \frac{1}{2}\iint_0^{2\pi} \frac{d^2p}{(2\pi)^2} \log\left\{\left(\frac{1+t^2}{1-t^2}\right)^2 - \frac{2t}{1-t^2}(\cos p_x + \cos p_y)\right\} + \log 2 \\ &= \frac{1}{2}\int_{[0,2\pi]^2} \frac{d^2p}{(2\pi)^2} \log\left(\cosh^2(2K) - \sinh(2K)(\cos p_x + \cos p_y)\right) + \log 2 \end{aligned}}$$

This is the celebrated result by Onsager!