Properties of the free energy

$$f = \frac{1}{2}\int_0^{2\pi} \frac{d^2p}{(2\pi)^2} \log\left(\underbrace{\cosh^2(2K) - \sinh(2K)\left(\cos p_x + \cos p_y\right)}\right) + \log 2$$

$\exists$ possible singularity if the argument of log vanishes for some $p=(p_x,p_y)$.

$$\cosh^2(2K) - \sinh(2K)\left(\cos p_x + \cos p_y\right) \geq \cosh^2(2K) - 2\sinh(2K)$$

$\uparrow$ $=$ at $p_x = p_y = 0$

$$\text{RHS} = 1 + \sinh^2(2K) - 2\sinh(2K) = \left(\sinh(2K) - 1\right)^2 \geq 0$$

$$= 0 \quad \text{iff} \quad \sinh(2K) = 1 \quad \underline{\text{i.e. at } K = K_c}$$

$$\frac{df}{dK} = \frac{1}{2}\int_0^{2\pi} \frac{d^2p}{(2\pi)^2} \frac{2\cosh(2K)\,2\sinh(2K) - 2\cosh(2K)\left(\cos p_x + \cos p_y\right)}{\cosh^2(2K) - \sinh(2K)\left(\cos p_x + \cos p_y\right)}$$

$$= \frac{\cosh(2K)}{\sinh(2K)}\left[\int \frac{d^2p}{(2\pi)^2} \frac{\sinh(2K) - \sinh(2K)^{-1}}{\frac{\cosh^2(2K)}{\sinh(2K)} - \cos p_x - \cos p_y} + 1\right]$$

near $K=K_c$

$$= \left(\sqrt{2} + \cdots\right)\left(\int \frac{d^2p}{(2\pi)^2} \frac{4\sqrt{2}\,(K-K_c) + \cdots}{2 + 8(K-K_c)^2 + \cdots - \cos p_x - \cos p_y} + 1\right)$$

$$= \int \frac{d^2p}{(2\pi)^2} \frac{16\,(K-K_c)}{16\,(K-K_c)^2 + p_x^2 + p_y^2 + \cdots} + \text{reg}$$

$\uparrow$ higher power in $(K-K_c)$ & p_x, p_y

$$\sim -\frac{16(K-K_c)}{2\pi}\log|K-K_c| + \text{reg}$$

↑ ~~Singularity~~ from $p_x = p_y = 0$.

<u>Still continuous</u>

$$\frac{d^2 f}{dK^2} \sim -\frac{8}{\pi}\log|K-K_c| + \text{reg.}$$

<u>Singularity</u>

$\therefore$ $\exists$ second order phase transition at $K = K_c$.

<u>Meaning of $\frac{d^2f}{dK^2} \to \infty$ as $K \to K_c$</u>

$$f = \frac{1}{V}\log Z$$

$$\frac{df}{dK} = \frac{1}{V}\frac{\frac{dZ}{dK}}{Z} = \frac{1}{V}\frac{\sum_\sigma e^{K\sum_{\langle i,j\rangle}\sigma_i\sigma_j}\sum_{\langle i,j\rangle}\sigma_i\sigma_j}{Z}$$

$$= \frac{1}{V}\left\langle \sum_{\langle i,j\rangle}\sigma_i\sigma_j \right\rangle = \frac{E}{V}\langle \sigma_i\sigma_j \rangle = 2\langle \varepsilon_r \rangle$$

(E = # edges; i, j : nearest neighbor pair; $\sigma_i\sigma_j = \varepsilon_r$; $= 2$ ↑ Square lattice)

$$= \frac{2}{V}\left\langle \sum_r \varepsilon_r \right\rangle$$

$$\frac{d^2 f}{dK^2} = \frac{1}{V} \frac{\sum_\sigma e^{K\sum_{\langle ij\rangle}\sigma_i\sigma_j}\left(\sum_{\langle ij\rangle}\sigma_i\sigma_j\right)^2}{Z} - \frac{1}{V}\left(\frac{\sum_\sigma e^{K\sum_{\langle ij\rangle}\sigma_i\sigma_j}\sum_{\langle ij\rangle}\sigma_i\sigma_j}{Z}\right)^2$$

$$= \frac{4}{V}\left\langle\left(\sum_r \varepsilon_r\right)^2\right\rangle - \frac{4}{V}\left\langle\sum_r \varepsilon_r\right\rangle^2$$

$$= \frac{4}{V}\sum_{r,r'}\left\{\langle\varepsilon_r\varepsilon_{r'}\rangle - \langle\varepsilon_r\rangle\langle\varepsilon_{r'}\rangle\right\}$$

If the correlation length $\xi < \infty$, then

$$\langle\varepsilon_r\varepsilon_{r'}\rangle - \langle\varepsilon_r\rangle\langle\varepsilon_{r'}\rangle \simeq c\, e^{-\frac{|r-r'|}{\xi}} \quad \text{as } |r-r'|\to\infty$$

$$\frac{d^2 f}{dK^2} = \text{finite} + \frac{4c}{V}\sum_{r,r'} e^{-\frac{|r-r'|}{\xi}} = \text{finite} + 4c\sum_r e^{-|r|/\xi} < \infty$$

But $\frac{d^2 f}{dK^2} \to \infty$ as $K \to K_c$

This means

$$\boxed{\xi \to \infty \quad \text{as} \quad K \to K_c}$$

long range interaction at critical temperature.

Rewriting the Partition function

$$Z_{\alpha,\beta} \overset{\sinh(2K)^{MN}}{=} \int \prod_p (d\bar{\eta}_p d\eta_p) \cdot \exp\left(\frac{1}{2}\sum_p (\bar{\eta}_p, \eta_{-p}) M_p \begin{pmatrix} \bar{\eta}_{-p} \\ \eta_p \end{pmatrix}\right)$$

where

$$M_p = \begin{pmatrix} 0 & -e^{+ip_y} \\ e^{-ip_y} & 0 \end{pmatrix} + \frac{1}{1+t^2+2t\cos p_x} \begin{pmatrix} -2it^{-1}\sin p_x & \frac{1-t^2}{t} \\ -\frac{1-t^2}{t} & 2it\sin p_x \end{pmatrix}$$

$$\tilde{M}_p = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} M_p \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -te^{+ip_y} \\ te^{-ip_y} & 0 \end{pmatrix} + \frac{1}{1+t^2+2t\cos p_x} \begin{pmatrix} -2it\sin p_x & 1-t^2 \\ -(1-t^2) & 2it\sin p_x \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -(1+te^{ip_y}) \\ 1+te^{-ip_y} & 0 \end{pmatrix} + \underbrace{\frac{1}{1+t^2+2t\cos p_x}}_{\frac{1}{(1+te^{ip_x})(1+te^{-ip_x})}} \underbrace{\begin{pmatrix} -2it\sin p_x & 2+2t\cos p_x \\ -2-2t\cos p_x & 2it\sin p_x \end{pmatrix}}_{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1+te^{ip_x} \\ -(1+te^{-ip_y}) & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & -(1+te^{ip_y}) \\ 1+te^{-ip_y} & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -(1+te^{ip_x}) \\ 1+te^{-ip_x} & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Denote $\bar{\xi}^v_p = t^{-1}\bar{\eta}_p$, $\xi^v_p = \eta_p \Rightarrow d\bar{\xi}^v_p = t\, d\bar{\eta}_p$, $d\xi^v_p = d\eta_p$

$$Z_{\alpha,\beta} = \underbrace{\sinh(2K)^{MN} t^{-MN}}_{(2\cosh^2(K))^{MN}} \int \prod_p (d\bar{\xi}^v_p d\xi^v_p) \exp\left(\frac{1}{2}\sum_p (\bar{\xi}^v_p, \xi^v_{-p}) \tilde{M}_p \begin{pmatrix} \bar{\xi}^v_{-p} \\ \xi^v_p \end{pmatrix}\right)$$

Claim

$$Z_{\alpha,\beta} = \left(2\cosh(K)\right)^{MN} \int \prod_p d\bar{\xi}_p^V d\xi_p^V \prod_p \frac{d\bar{\xi}_p^H d\xi_p^H}{(1+t^2+2t\cos p_x)^{\frac{1}{2}}} \exp\left(\hat{A}\right)$$

where

$$\hat{A} = \frac{1}{2}\sum_p \left(\bar{\xi}_p^H, \xi_{-p}^H, \bar{\xi}_p^V, \xi_{-p}^V\right) \begin{pmatrix} 0 & -(1+te^{ip_x}) & 1 & 1 \\ 1+te^{-ip_x} & 0 & -1 & 1 \\ -1 & 1 & 0 & -(1+te^{ip_y}) \\ -1 & -1 & 1+te^{-ip_y} & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^H \\ \xi_p^H \\ \bar{\xi}_{-p}^V \\ \xi_p^V \end{pmatrix}$$

Indeed

$$\hat{A} = \frac{1}{2}\sum_p \Bigg\{ \left(\bar{\xi}_p^V, \xi_{-p}^V\right) \begin{pmatrix} 0 & -(1+te^{ip_y}) \\ 1+te^{-ip_y} & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^V \\ \xi_p^V \end{pmatrix} + \left(\bar{\xi}_p^H, \xi_{-p}^H\right) \begin{pmatrix} 0 & -(1+te^{ip_x}) \\ 1+te^{-ip_x} & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^H \\ \xi_p^H \end{pmatrix}$$

$$+ \left(\bar{\xi}_p^H, \xi_{-p}^H\right) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^V \\ \xi_p^V \end{pmatrix} - \left(\bar{\xi}_p^V, \xi_{-p}^V\right) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^H \\ \xi_p^H \end{pmatrix} \Bigg\}$$

$$\Big\|$$

$$\left[\left(\bar{\xi}_p^H, \xi_{-p}^H\right) - \left(\bar{\xi}_p^V, \xi_{-p}^V\right) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} N_{p_x}^{-1}\right] N_{p_x} \left[\begin{pmatrix} \bar{\xi}_{-p}^H \\ \xi_p^H \end{pmatrix} + N_{p_x}^{-1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^V \\ \xi_p^V \end{pmatrix}\right]$$

$$+ \left(\bar{\xi}_p^V, \xi_{-p}^V\right) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} N_{p_x}^{-1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\xi}_{-p}^V \\ \xi_p^V \end{pmatrix}$$

where $N_{p_x} = \begin{pmatrix} 0 & -(1+te^{ip_x}) \\ 1+te^{-ip_x} & 0 \end{pmatrix}$.

Integrate out $\xi^H, \bar{\xi}^H \implies \prod_p (1+t^2+2t\cos p_x)^{\frac{1}{2}} \cdot \exp\left(\frac{1}{2}\sum_p \left(\bar{\xi}_p^V, \xi_{-p}^V\right) \widetilde{M}_p \begin{pmatrix} \bar{\xi}_{-p}^V \\ \xi_p^V \end{pmatrix}\right)$

//

Position space expression:
$$\xi_r^{V\text{ or }H} = \sum_p \frac{e^{ip\cdot r}}{\sqrt{MN}}\, \xi_p^{V\text{ or }H} \qquad (p\cdot r = p_x x + p_y y)$$
$$\bar{\xi}_r^{V\text{ or }H} = \sum_p \frac{e^{-ip\cdot r}}{\sqrt{MN}}\, \bar{\xi}_p^{V\text{ or }H}$$

<u>Note</u>. $\prod_p d\bar{\xi}_p^H d\xi_p^H d\bar{\xi}_p^V d\xi_p^V = \prod_r d\bar{\xi}_r^H d\xi_r^H d\bar{\xi}_r^V d\xi_r^V$

- $\prod_p (1+t^2+2t\cos p_x)^{\frac{1}{2}} = (1 \pm t^N)^M$

$$Z_{\alpha,\beta} = \frac{(2\cosh^2(K))^{MN}}{(1\pm t^N)^M} \int \prod_r d\bar{\xi}_r^H d\xi_r^H d\bar{\xi}_r^V d\xi_r^V \cdot \exp(\hat{A})$$

where
$$\hat{A} = \sum_r \Big\{ -\bar{\xi}_r^H \xi_r^H - t\,\bar{\xi}_r^H \xi_{r+\hat{x}}^H - \bar{\xi}_r^V \xi_r^V - t\,\bar{\xi}_r^V \xi_{r+\hat{y}}^V + \bar{\xi}_r^H \bar{\xi}_r^V + \bar{\xi}_r^H \xi_r^V + \bar{\xi}_r^V \xi_r^H + \xi_r^H \xi_r^V \Big\}$$

where for $r=(x,y)$, $r+\hat{x} = (x+1,y)$ & $r+\hat{y} = (x, y+1)$.

<u>Remark</u> (Up to the factor $\frac{1}{(1\pm t^N)^M}$), it produces the high temperature expansion
$$Z = 2^V (\cosh(K))^E \sum_{\text{loops}} t^{\text{length}}$$

Indeed, with the identification

$\bar{\xi}^H_r = R_r$ right of the vertex r

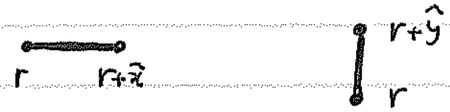
$\xi^H_r = L_r$ left of   "

$\bar{\xi}^V_r = U_r$ above   "   (up)

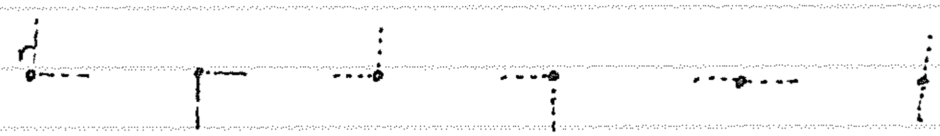
$\xi^V_r = D_r$ below   "   (down)

$$\hat{A} = \hat{A}_{line} + \hat{A}_{empty}$$

- $\hat{A}_{line} = \sum_r \left(t\, L_{r+\hat{x}} R_r + t\, D_{r+\hat{y}} U_r \right)$



- $\hat{A}_{empty} = \sum_r \left(RU + RD + UL + LD + LR + DU \right)_r$



The sign works all right!

Further rewriting

$$\bar{\xi}^H = \frac{1}{2}\sum_{s=\pm1,\pm3} e^{\pi i s/4}\,\psi_s , \qquad \xi^H = \frac{1}{2}\sum_{s=\pm1,\pm3} e^{3\pi i s/4}\,\psi_s$$

$$\bar{\xi}^V = \frac{1}{2}\sum_{s=\pm1,\pm3} e^{2\pi i s/4}\,\psi_s , \qquad \xi^V = \frac{1}{2}\sum_{s=\pm1,\pm3} e^{4\pi i s/4}\,\psi_s$$

at each r.

Then

$$A = \sum_r \Big\{ \frac{t}{4}\big(\psi_1(d_x+id_y)\psi_1 + \psi_{-1}(d_x-id_y)\psi_{-1}\big) + i\big(t-(\sqrt{2}-1)\big)\psi_1\psi_{-1}$$

$$+\frac{t}{4}\big(\psi_3(d_x-id_y)\psi_3 + \psi_{-3}(d_x+id_y)\psi_3 + i(t+\sqrt{2}+1)\psi_{-3}\psi_3$$

$$-\frac{t}{4}\big(\psi_1(d_x-id_y)\psi_{-3} + \psi_{-3}(d_x-id_y)\psi_1 + \psi_{-1}(d_x+id_y)\psi_3 + \psi_3(d_x+id_y)\psi_{-1}\big)\Big\}_r$$

where

$$\psi_{r+\hat{x}} = \psi_r + (d_x\psi)_r$$

$$\psi_{r+\hat{y}} = \psi_r + (d_y\psi)_r$$

d_i = difference operators

and ignored all "total derivatives"

like $\sum_r \{(d_x\psi)\psi' + \psi\, d_x\psi'\}_r$

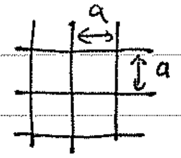
Note The mass of $\psi_{\pm 1}$ is $m_1 = \frac{2}{t}(t - \sqrt{2} + 1) = \frac{2}{t}(t - t_c)$

$\longrightarrow 0$ at the critical temperature.

The mass of $\psi_{\pm 3}$ is $m_3 = \frac{2}{t}(t + \sqrt{2} + 1)$: always non-zero

$\longrightarrow$ integrate them out!

In the continuum limit, we set $a =$ lattice spacing $\longrightarrow 0$



$$\psi_{r+\hat{x}} = \psi_r + a\,\partial_x \psi + \mathcal{O}(a^2)$$
$$\psi_{r+\hat{y}} = \psi_r + a\,\partial_y \psi + \mathcal{O}(a^2)$$

$m_1 = a^{-1}\frac{2}{t}(t - t_c)$ can still be finite or zero

$m_3 = a^{-1}\frac{2}{t}(t + \sqrt{2} + 1) \rightarrow \infty.$

Thus, there is no non-trivial effect from integrating out $\psi_{\pm 3}$.

We are left with $\left(\text{writing } \sqrt{\frac{t}{2}}\,\psi_{\pm 1} = i\sqrt{\frac{4}{\pi}}\,\psi_{\mp} \right)$

$$A = -\frac{1}{\pi}\int dx\,dy \left\{ \frac{1}{2}\psi_-(\partial_x + i\partial_y)\psi_- + \frac{1}{2}\psi_+(\partial_x - i\partial_y)\psi_+ + i m_1 \psi_- \psi_+ \right\}$$

— <u>Majorana fermion</u> with mass m_1!

$\uparrow$

Dirac fermion with "reality" constraint

$$\bar{\psi}_{\mp} = \mp i\,\psi_{\mp}$$