0-d QFT

$S(x_1, \cdots, x_n)$ a function of $\mathbb{R}^n$ ($\nearrow \infty$ as $|x| \mapsto \infty$ "fast enough")

Then we have a QFT

partition function $Z = \int_{\mathbb{R}^n} dx_1 \cdots dx_n \, e^{-S(x_1, \cdots, x_n)}$

Correlation function for $f(x_1, \cdots, x_n)$

$$\langle f \rangle = \int_{\mathbb{R}^n} d^n x \, e^{-S(x)} f(x) \Big/ Z$$

Free field theory : $S(x)$ is a positive definite quadratic function of variables X.

e.g. 1-variable

$$S(x) = \frac{1}{2} A X^2 \qquad A > 0$$

$$Z(A) = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} A X^2} = \sqrt{\frac{2\pi}{A}}$$

Gaussian integral

real
n-variables

$$S(x_1,\cdots,x_n)=\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j \qquad \text{assume } A_{ij}=A_{ji}$$

$A>0$ i.e. all eigenvalues >0

$$Z(A)=\int_{\mathbb{R}^n} dx_1\cdots dx_n\, e^{-S(x_1,\cdots,x_n)} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

What is $\langle f\rangle$? for some function $f(x_1,\cdots,x_n)$.

Introduce

$$Z(A,J)=\int_{\mathbb{R}^n} dx_1\cdots dx_n\, e^{-\frac{1}{2}x^T\cdot Ax+J^T\cdot x}$$

$$\frac{\partial}{\partial J_i}Z(A,J)=\int_{\mathbb{R}^n} dx_1\cdots dx_n\, e^{-\frac{1}{2}x^TAx+J^Tx}\, x_i$$

$$\frac{\partial}{\partial J_{i_1}}\cdots\frac{\partial}{\partial J_{i_s}}Z(A,J)=\int \cdots\; x_{i_1}\cdots x_{i_s}$$

$$\therefore\ \langle x_{i_1}\cdots x_{i_s}\rangle=\frac{1}{Z(A)}\frac{\partial}{\partial J_{i_1}}\cdots\frac{\partial}{\partial J_{i_s}}Z(A,J)\Big|_{J=0}$$

So we want to know $Z(A,J)$.

$$-\frac{1}{2}x^T\cdot Ax+J^Tx=-\frac{1}{2}(x-A^{-1}J)^T A\,(x-A^{-1}J)+\frac{1}{2}J^TA^{-1}J$$

$$Z(A,J)=\int d^nx\, e^{-\frac{1}{2}(x-\cdots)^TA\,(x-\cdots)+\frac{1}{2}J^TA^{-1}J}$$

$$=Z(A)\cdot e^{\frac{1}{2}J^TA^{-1}J}$$

$$\frac{1}{Z(A)}\frac{\partial}{\partial J_i} Z(A,J) = \frac{\partial}{\partial J_i} e^{\frac{1}{2} J^T A^{-1} J} = \left\{ \frac{1}{2}\left(A^{-1}J\right)_i + \frac{1}{2}\left(J^T A^{-1}\right)_i \right\} e^{\frac{1}{2} J^T A^{-1} J}$$

$$= \sum_k A^{-1}_{ik} J_k \, e^{\frac{1}{2} J^T A^{-1} J} \xrightarrow{J=0} 0 \qquad \therefore \langle X_i \rangle = 0$$

$$\frac{1}{Z(A)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} Z(A,J) = A^{-1}_{ij}\, e^{\frac{1}{2} J^T A^{-1} J} + \sum_k A^{-1}_{ik} J_k \sum_l A^{-1}_{il} J_l \, e^{\frac{1}{2} J^T A^{-1} J}$$

$$\xrightarrow{J=0} A^{-1}_{ij}$$

$$\therefore \langle X_i X_j \rangle = A^{-1}_{ij}$$

each $\frac{\partial}{\partial J_i}$ $\begin{cases} \cdot \text{ brings down } (A^{-1}J)_i \\ \cdot \text{ or absorb } J_i \text{ from } (A^{-1}J)_\cdot \text{, yielding } A^{-1}_{\cdot i} \end{cases}$

When we set $J_\cdot = 0$, what remains are A^{-1}_{ij}

from each pair of $\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}$

pairing or contraction

$$\therefore \langle X_i X_j X_k \rangle = 0$$

$$\langle X_i X_j X_k X_l \rangle = X_i X_j X_k X_l + X_i X_j X_k X_l + X_i X_j X_k X_l$$

$$\overset{\text{i.e.}}{=} A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk}$$

<u>Wick contraction rule</u>

<u>def</u>: normal ordered product.

$$\mathcal{O}(x_1,\cdots,x_n) \rightsquigarrow \, :\mathcal{O}(x_1,\cdots,x_n):$$

For polynomial $\mathcal{O}(x_1,\cdots,x_n)$

$$\mathcal{O}(x_1,\cdots,x_n) = \; : \text{all possible contractions of } X_i\text{'s in } \mathcal{O}:$$

Motivation: $\langle X_i X_j \rangle = A^{-1}_{ij}$ is singular as $i \to j$ in high dimensions

e.g. $1 = \; :1:$

$$X_i = \; :X_i:$$

$$X_i X_j = \; :X_i X_j: + \overbrace{X_i X_j}$$

i.e. $:X_i X_j: \stackrel{\text{def}}{=} X_i X_j - \overbrace{X_i X_j}$

$$= X_i X_j - A^{-1}_{ij}$$

$$= X_i X_j - \langle X_i X_j \rangle .$$

$$X_i X_j X_k = \; :X_i X_j X_k: + :\overbrace{X_i X_j} X_k: + :\overbrace{X_i X_j X_k}: + :X_i \overbrace{X_j X_k}:$$

i.e. $:X_i X_j X_k: = X_i X_j X_k - \overbrace{X_i X_j} X_k - \overbrace{X_i X_k} X_j - \overbrace{X_j X_k} X_i$

$$X_i X_j X_k X_l = \; :X_i X_j X_k X_l: + \left(:\overbrace{X_i X_j} X_k X_l: + 5 \text{ other terms} \right)$$

$$+ \overbrace{X_i X_j}\,\overbrace{X_k X_l} + X_i X_j X_k X_l + X_i X_j X_k X_l .$$

i.e. $:X_i X_j X_k X_l: = X_i X_j X_k X_l - \left(\overbrace{X_i X_j} \left(X_k X_l - \overbrace{X_k X_l} \right) + 5 \text{ other terms} \right)$

$$- \overbrace{X_i X_j}\,\overbrace{X_k X_l} - X_i X_j X_k X_l - X_i X_j X_k X_l$$

Claim $\langle :X_{i_1} \cdots X_{i_s}: \rangle = 0 \qquad s \geq 1.$

e.g. $\langle :X_i: \rangle = \langle X_i \rangle = 0$

$\langle :X_i X_j: \rangle = \langle (X_i X_j - \langle X_i X_j \rangle) \rangle = \langle X_i X_j \rangle - \langle X_i X_j \rangle = 0$

$\langle :X_i X_j X_k X_l: \rangle = ?$

look at $X_i X_j X_k X_l = \cdots \text{RHS} -$

We know $\langle X_i X_j X_k X_l \rangle = \overbrace{X_i X_j}\,\overbrace{X_k X_l} + X_i X_j X_k X_l + X_i X_j X_k X_l$

$\qquad //$

& $\langle \text{RHS} \rangle = \langle :X_i X_j X_k X_l: \rangle$

we know $\langle :X_i X_j: \rangle = 0$

$+ \cancel{\langle :\overbrace{X_i X_j} X_k X_l: + 5 \text{ terms}} \rangle$

$+ \langle \overbrace{X_i X_j}\,\overbrace{X_k X_l} + X_i X_j X_k X_l + X_i X_j X_k X_l \rangle$

same

$\therefore \quad \langle :X_i X_j X_k X_l: \rangle = 0$

Exercise $:X_i X_j::X_k X_l: = X_i X_j X_k X_l + X_i X_j X_k X_l$

$+ :X_i X_j X_k X_l: + :X_i X_j X_k X_l:$

$+ :X_i X_j X_k X_l: + :X_i X_j X_k X_l:$

avoids self contractions

$\overbrace{X_i X_j}, \overbrace{X_k X_l}$

$+ :X_i X_j X_k X_l:$

Fermions — Anticommuting variables

We want to consider $\int d\psi_1 \cdots d\psi_n \, e^{-S(\psi_1, \cdots, \psi_n)} f(\psi_1, \cdots, \psi_n)$

for <u>anticommuting variables</u> $\psi_1, \cdots, \psi_n$

$$\psi_i \psi_j = -\psi_j \psi_i \quad \forall i, j$$

<u>Note</u> $\psi_i \psi_j = -\psi_j \psi_i \Rightarrow (\psi_i)^2 = 0$

In particular, a function of $\psi_1, \cdots, \psi_n$ is a polynomial & has at most 2^n terms:

$$f(\psi_1, \cdots, \psi_n) = \underset{1}{f_0} + \underset{n}{\sum_i \psi_i f_i} + \underset{\binom{n}{2}}{\sum_{i<j} \psi_i \psi_j f_{ij}} + \cdots + \underset{1}{\psi_1 \cdots \psi_n f_{1 \cdots n}}$$

$$1 + n + \binom{n}{2} + \binom{n}{3} + \cdots + 1 = (1+1)^n = 2^n.$$

f is called $\begin{cases} \text{even (bosonic)} \\ \text{odd (fermionic)} \end{cases}$ if it consists of $\begin{cases} \text{even} \\ \text{odd} \end{cases}$ powers of anticommuting variables.

1	even
ψ	odd
$1+\psi$	neither

$$(-1)^{|f|} = \begin{cases} 1 & f \text{ even} \\ -1 & f \text{ odd} \end{cases}$$

Differentiation

$$\frac{\partial}{\partial\psi_i}\psi_j = \delta_{ij}$$

$$\frac{\partial}{\partial\psi_1}(\psi_1\psi_2) = \psi_2 \quad , \quad \frac{\partial}{\partial\psi_1}(\psi_2\psi_1) = -\frac{\partial}{\partial\psi_1}(\psi_1\psi_2) = -\psi_2$$

$$= -\psi_2\frac{\partial}{\partial\psi_1}\psi_1$$

$$\rightsquigarrow \quad \frac{\partial}{\partial\psi_i}(\psi_j f) = -\psi_j\frac{\partial}{\partial\psi_i}f \quad \text{if } i\neq j.$$

in general $\quad \frac{\partial}{\partial\psi_i}(AB) = \frac{\partial A}{\partial\psi_i}B + (-1)^{|A|}A\frac{\partial B}{\partial\psi_i}$

Integration

Single Variable $\quad \int d\psi\, f(\psi)$ Want ... ① linear in f

② $\int d\psi\, f(\psi+\eta) = \int d\psi\, f(\psi)$

$f(\psi) = a + \psi b \qquad$ ① $\Rightarrow \int d\psi\, f(\psi) = \alpha a + \beta b$

② : $\quad f(\psi+\eta) = a + (\psi+\eta)b = (a+\eta b) + \psi b$

$\therefore \quad \alpha(a+\eta b) + \cancel{\beta b} = \cancel{\alpha a} + \cancel{\beta b} \quad \Rightarrow \underline{\alpha = 0\ !}$

Thus, $\quad \int d\psi\,(a+\psi b) := b \qquad \begin{pmatrix}\alpha=0\\ \beta=1\end{pmatrix}$

i.e. $\quad \int d\psi\cdot 1 = 0 \quad , \quad \int d\psi\cdot\psi = 1.$

2 variables $\quad \int d\psi_1 d\psi_2\,(f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_1\psi_2 f_{12})$

$$= \int d\psi_1 d\psi_2\,(-\psi_2\psi_1 f_{12}) = -f_{12}.$$

n - variables $\quad f = f_0 + \sum_i \psi_i f_i + \cdots + \psi_1 \cdots \psi_n f_{1\cdots n}$

$$\int d\psi_1 \cdots d\psi_n\, f(\psi_1 \cdots \psi_n) = \int d\psi_1 \cdots d\psi_n\, \psi_1 \cdots \psi_n f_{1\cdots n}$$

$$= (-1)^{\frac{n(n-1)}{2}} f_{1\cdots n}.$$ Take out the top component with sgn.

Change of variables

$$\psi' = \alpha\psi \qquad \int d\psi' \psi' = 1 = \int d\psi\, \psi$$

$$\int d\psi' \psi' = \int d\psi' \alpha\psi$$

$\boxed{\therefore\ d\psi' = \frac{1}{\alpha} d\psi}$

i.e. $\boxed{d(\alpha\psi) = \frac{1}{\alpha} d\psi}$

$$\psi_i' = \sum_{j=1}^{n} \alpha_{ij} \psi_j$$

e.g. $n=2$ $\quad \int d\psi_1' d\psi_2' \underbrace{\psi_2' \psi_1'}_{=} = 1 = \int d\psi_1 d\psi_2\, \psi_2 \psi_1$

$$(\alpha_{21}\psi_1 + \alpha_{22}\psi_2)(\alpha_{11}\psi_1 + \alpha_{12}\psi_2) = \alpha_{21}\alpha_{12}\psi_1\psi_2 + \alpha_{22}\alpha_{11}\psi_2\psi_1$$

$$= (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\psi_2\psi_1 = \det\alpha \cdot \psi_2\psi_1$$

$$\therefore\ d\psi_1' d\psi_2' = \frac{1}{\det\alpha} d\psi_1 d\psi_2$$

$$\boxed{d\psi_1' \cdots d\psi_n' = \frac{1}{\det(\alpha)} d\psi_1 \cdots d\psi_n}$$

C.f. Commuting variables

$$x_i' = \sum_j \alpha_{ij} x_j$$

$$\Rightarrow\ dx_1' \cdots dx_n' = \det(\alpha)\, dx_1 \cdots dx_n$$

$S(\psi_1, \dots, \psi_n)$ action (some function of $\psi_1, \dots, \psi_n$)

$$Z = \int d\psi_1 \cdots d\psi_n \, e^{-S(\psi_1, \dots, \psi_n)}$$

$$\langle f \rangle = \int d\psi_1 \cdots d\psi_n \, e^{-S(\psi_1, \dots, \psi_n)} f(\psi_1, \dots, \psi_n) \Big/ Z$$

<u>Example</u> <u>Free field theory for a pair of variables $\bar{\psi}, \psi$</u>

$$S = \bar{\psi} a \psi \qquad 0 \neq a \in \mathbb{C}$$

$$Z = \int d\bar{\psi} d\psi \, e^{-\bar{\psi} a \psi} = \int d\bar{\psi} d\psi \Big(1 - \bar{\psi} a \psi \underbrace{+ \tfrac{1}{2}(-\bar{\psi} a \psi)^2 + \cdots}_{0}\Big)$$

$$= a$$

$$\langle \psi \bar{\psi} \rangle = \int \frac{d\bar{\psi} d\psi \, e^{-\bar{\psi} a \psi} \psi \bar{\psi}}{Z} = \frac{1}{a}$$

$$\langle \psi \rangle = \langle \bar{\psi} \rangle = 0$$

<u>Free field theory of n pairs</u> $\bar{\psi}_1, \psi_1, \bar{\psi}_2, \psi_2, \dots, \bar{\psi}_n, \psi_n$

$$S = \sum_{i,j} \bar{\psi}_i A_{ij} \psi_j$$

$$Z = \int d\bar{\psi}_1 d\psi_1 \cdots d\bar{\psi}_n d\psi_n \, e^{-\sum_{i,j} \bar{\psi}_i A_{ij} \psi_j} = ?$$

Change variables $\psi_i' = \sum_j A_{ij}\psi_j$

$$\Rightarrow \quad S = \sum_i \bar{\psi}_i \psi_i' \quad , \quad d\psi_1' \cdots d\psi_n' = \frac{1}{\det(A)} d\psi_1 \cdots d\psi_n$$

$$Z = \int (-1)^{\frac{n(n-1)}{2}} d\bar{\psi}_1 \cdots d\bar{\psi}_n \underbrace{d\psi_1 \cdots d\psi_n}_{= \det(A)\, d\psi_1' \cdots d\psi_n'} e^{-\sum_i \bar{\psi}_i \psi_i'}$$

$$= \det A \int d\bar{\psi}_1 d\psi_1' d\bar{\psi}_2 d\psi_2' \cdots d\bar{\psi}_n d\psi_n' \prod_i e^{-\bar{\psi}_i \psi_i'}$$

$$= \det A \int \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i' e^{-\bar{\psi}_i \psi_i'} = \det A .$$

$$\langle \psi_i \psi_j \rangle , \langle \bar{\psi}_i \bar{\psi}_j \rangle , \langle \psi_i \bar{\psi}_j \rangle = ?$$

$$Z(A, \eta, \bar{\eta}) = \int \underbrace{d\bar{\psi}_1 \cdots d\psi_n}_{d^{2n}\psi} e^{-\sum_{i,j} \bar{\psi}_i A_{ij} \psi_j + \sum_i (\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i)}$$

$$\frac{\partial}{\partial \bar{\eta}_i} Z(A, \eta, \bar{\eta}) = \int d^{2n}\psi \, e^{-\bar{\psi}^T A \psi + \bar{\eta}^T \psi + \bar{\psi}^T \eta} \psi_i$$

$$\frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \bar{\eta}_i} Z(A, \eta, \bar{\eta}) = \int d^{2n}\psi \, e^{-\bar{\psi}^T A \psi + \bar{\eta}^T \psi + \bar{\psi}^T \eta} \underbrace{(-\bar{\psi}_j)\psi_i}_{= \psi_i \bar{\psi}_j}$$

⋮