Critical Exponents

$$m_1 \propto |t-t_c| \propto |T-T_c| \qquad t = \tanh K, \quad K = \frac{J}{kT}$$

$\therefore$ correlation length $\xi = \frac{1}{m_1} \propto |T-T_c|^{-1}$

$$\therefore \boxed{\nu = 1} \qquad \left(\xi = |T-T_c|^{-\nu}\right)$$

Other exponents :

$\underline{T \gtrsim T_c}$ $\qquad M \sim \chi_m H$ $\qquad$ (as $H \sim 0$)

spontaneous magnetization, external field, magnetic suscebtibility

$$\chi_m \sim (T-T_c)^{-\gamma}$$

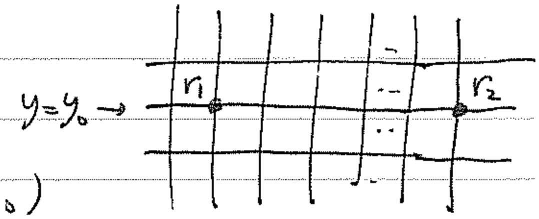
$\underline{T \lesssim T_c}$ $\qquad |M| \sim (T_c - T)^{\beta}$ $\qquad$ (at $H=0$)

$\underline{T = T_c}$ $\qquad \langle \sigma_{r_1} \sigma_{r_2} \rangle = \frac{1}{|r_1 - r_2|^{\eta}}$ $\qquad$ (at $H=0$)

$$\boxed{\gamma = \frac{7}{4}, \quad \beta = \frac{1}{8}, \quad \eta = \frac{1}{4}}$$

All can be found by studying $\langle \sigma_{r_1} \sigma_{r_2} \rangle$.

Two point function $\langle \sigma_{r_1} \sigma_{r_2} \rangle$



Consider $r_1 = (x_1, y_0)$, $r_2 = (x_2, y_0)$

$$\langle \sigma_{r_1} \sigma_{r_2} \rangle = \frac{\sum_{\sigma} \sigma_{r_1} \sigma_{r_2} e^{-\beta \mathcal{H}}}{Z}$$

$$= \frac{1}{Z} \sum_{\sigma_{\cdot,M}, \cdots, \sigma_{\cdot,1}} T_{\sigma_{\cdot 1} \sigma_{\cdot M}} T_{\sigma_{\cdot M} \sigma_{\cdot M-1}} \cdots T_{\sigma_{\cdot y_0+1} \sigma_{\cdot y_0}} \sigma_{x_1 y_0} \sigma_{x_2 y_0} T_{\sigma_{\cdot y_0} \sigma_{\cdot y_0-1}} \cdots$$
$$\cdots T_{\sigma_{\cdot 2}, \sigma_{\cdot,1}}$$

$$= \frac{1}{Z} \mathrm{Tr}_{(\mathbb{C}^2)^{\otimes N}} \left(T^{M-y_0+1} U_{x_1, x_2} T^{y_0-1} \right)$$

$$(U_{x_1, x_2})_{\sigma'_{\cdot}, \sigma_{\cdot}} = \delta_{\sigma'_{\cdot}, \sigma_{\cdot}} \sigma_{x_1} \sigma_{x_2}$$

when $T_{\sigma'_{\cdot}, \sigma_{\cdot}} = \prod_x e^{K \sigma_x \sigma'_x + K \sigma_x \sigma_{x+1}}$

When T is expressed as $(2\sinh(2K))^{\frac{N}{2}} \prod_x e^{\tilde{K} \sigma^1_x} \prod_x e^{K \sigma^3_x \sigma^3_{x+1}}$

$$U_{x_1 x_2} = \sigma^3_{x_1} \sigma^3_{x_2}$$

When T is expressed as $(2\sinh(2K))^{\frac{N}{2}} \prod_x e^{\tilde{K} \sigma^3_x} \prod_x e^{K \sigma^1_x \sigma^1_{x+1}}$

$$U_{x_1 x_2} = \sigma^1_{x_1} \sigma^1_{x_2}$$

Jordan-Wigner : $a_x = \left(\prod_{x'=1}^{x-1} \sigma^3_{x'}\right)\cdot \sigma^-_x$ $\qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^+ + \sigma^-$

$$a^+_x = \left(\prod_{x'=1}^{x-1} \sigma^3_{x'}\right)\cdot \sigma^+_x$$

(Then $\{a_x, a^+_{x'}\} = \delta_{x,x'}$, $\{a_x, a_{x'}\} = \{a^+_x, a^+_{x'}\} = 0$

and $T = (2\sinh(2K))^{\frac{N}{2}} \prod_x e^{\tilde{K}[a^+_x a_x]} \prod_x e^{K(a_x - a^+_x)\cdot(a_{x+1} + a^+_{x+1})}$)

$$\sigma^1_x = \left(\prod_{x'=1}^{x-1} \sigma^3_{x'}\right)\cdot\left(a^+_x + a_x\right) = (-1)^F_1 \cdots (-1)^F_{x-1}(a^+_x + a_x)$$

$$\therefore U_{x_1 x_2} = \sigma^1_{x_1}\sigma^1_{x_2} = (-1)^F_1 \cdots (-1)^F_{x_1-1}(a^+_{x_1} + a_{x_1})(-1)^F_1 \cdots (-1)^F_{x_2-1}(a^+_{x_2} + a_{x_2})$$

$$= (a^+_{x_1} + a_{x_1})(-1)^F_{x_1}(-1)^F_{x_1+1} \cdots (-1)^F_{x_2-1}(a^+_{x_2} + a_{x_2})$$

<u>Kernel function for $U_{x_1x_2}T$:</u>

$$(U_{x_1x_2}T)(\bar{\eta},\eta) = \left(\bar{\eta}_{x_1} + \frac{\partial}{\partial\bar{\eta}_{x_1}}\right)\left(\bar{\eta}_{x_2} + \frac{\partial}{\partial\bar{\eta}_{x_2}}\right) T(\bar{\eta}',\eta)$$

where $\bar{\eta}' = (\bar{\eta}_1, \cdots, \bar{\eta}_{x_1-1}, -\bar{\eta}_{x_1}, -\bar{\eta}_{x_1+1}, \cdots, -\bar{\eta}_{x_2-1}, \bar{\eta}_{x_2}, \cdots \bar{\eta}_N)$

(c.f. $T(\bar{\eta},\eta) = (\sinh(2K))^N \exp\left(\sum_p A_p \bar{\eta}_p \eta_p + B_p \bar{\eta}_{-p}\bar{\eta}_p + C_p \eta_{-p}\eta_p\right)$

$= (\sinh(2K))^N \exp\left(\sum_{x x'} \bar{\eta}_x A_{xx'} \eta_{x'} + \bar{\eta}_x B_{xx'} \bar{\eta}_{x'} + \eta_x C_{xx'} \eta_{x'}\right)$)

Thus

$$\langle \sigma_{x_1,y_0}\, \sigma_{x_2,y_0} \rangle = \frac{1}{Z} \int \prod_{y=1}^{M} \prod_{x=1}^{N} \left(d\bar{\eta}_{x,y}\, d\eta_{x,y+1}\, e^{-\bar{\eta}_{x,y}\eta_{x,y+1}} \right) \times$$

$$\times\, T(\bar{\eta}_{\cdot,M}, \eta_{\cdot,M})\, T(\bar{\eta}_{\cdot,M-1}, \eta_{\cdot,M-1}) \cdots \left(\bar{\eta}_{x_1,y_0} + \frac{\partial}{\partial \bar{\eta}_{x_1,y_0}} \right) \left(\bar{\eta}_{x_2,y_0} + \frac{\partial}{\partial \bar{\eta}_{x_2,y_0}} \right) T(\bar{\eta}'_{\cdot,y_0}, \eta_{\cdot,y_0})$$

$$\times\, T(\bar{\eta}_{\cdot,y_0-1}, \eta_{\cdot,y_0-1}) \cdots T(\bar{\eta}_{\cdot,1}, \bar{\eta}_{\cdot,1})$$

$$= \frac{1}{Z} \int \prod_{x,y} \left(d\bar{\eta}_{x,y}\, d\eta_{x,y+1} \right) e^{A'} \left(\bar{\eta}_{x_1,y_0} + \eta_{x_1,y_0+1} \right) \left(\bar{\eta}_{x_2,y_0} + \eta_{x_2,y_0+1} \right)$$

where $A' = -\sum_{x,y} \bar{\eta}_{x,y}\eta_{x,y+1} + \sum_{x,x',y} \left(\bar{\eta}'_{x,y} A_{x,x'} \eta_{x',y} + \bar{\eta}'_{x,y} B_{x,x'} \bar{\eta}_{x',y} + \eta_{x,y} C_{x,x'} \eta_{x',y} \right)$

where
$$\bar{\eta}'_{x,y} = \begin{pmatrix} \bar{\eta}_{1M} & \cdots & \bar{\eta}_{NM} \\ & & \\ & -\bar{\eta}_{x_1,y_0} \; -\bar{\eta}_{x_1+1,y_0} \cdots -\bar{\eta}_{x_2-1,y_0} & \\ & & \\ \bar{\eta}_{11} & \cdots & \bar{\eta}_{N1} \end{pmatrix}$$

Locally, $A' = A|_{\eta \to \eta',\, \bar{\eta} \to \bar{\eta}'}$ if we define

for $x_1 < x < x_2$

$$\eta'_{x,y} = \begin{cases} -\eta_{x,y} & y \geq y_0 + 1 \\ \eta_{x,y} & y \leq y_0 \end{cases}$$

$$\bar{\eta}'_{x,y} = \begin{cases} -\bar{\eta}_{x,y} & y \geq y_0 \\ \bar{\eta}_{x,y} & y \leq y_0 - 1 . \end{cases}$$

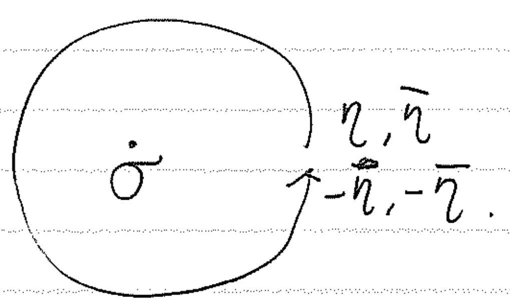
We find that $\sigma_{r_1} \sigma_{r_2}$ introduces a cut

in the fermion system!

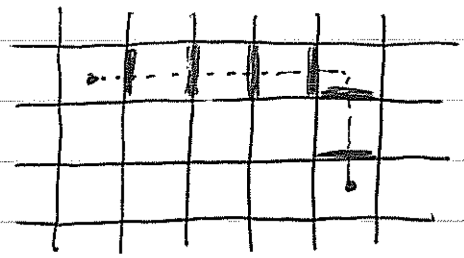
[Originated from Jordan-Wigner

$$U_{x_1 x_2} = (a^\dagger_{x_1} + a_{x_1})(-1)^F_{x_1}(-1)^F_{x_1+1} \cdots (-1)^F_{x_2-1}(a^\dagger_{x_2} + a_{x_2})$$
]

i.e.



Two point function $\langle \mu_{r_1} \mu_{r_2} \rangle$



$K \to -K$ at the thick edges.

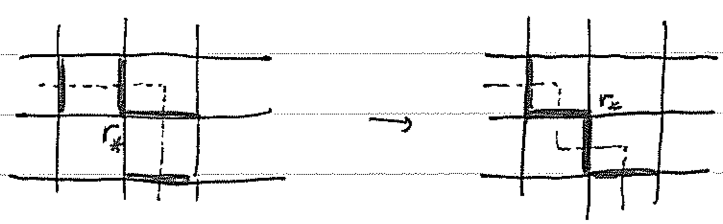
$t = \tanh(K) \to -\tanh(K)$

In $Z = 2^V (\cosh(K))^E \int \prod_r dU_r dD_r dR_r dL_r \, e^{\hat{A}}$

$$\hat{A} = \sum_r (t L_{r+\hat{x}} R_r + t D_{r+\hat{y}} U_r) + $$
$$+ \sum_r (RU + RD + UL + LD + LR + DU)_r ,$$

do $t \to -t$ at the corresponding edge.

Change in the path



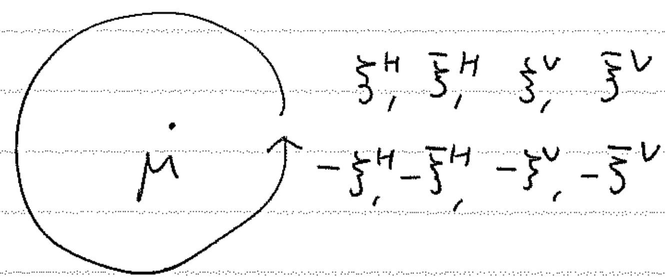
The change can be undone by

$$U_r, D_r, R_r, L_r \to \begin{cases} -U_{r_*}, -D_{r_*}, -R_{r_*}, -L_{r_*} & \text{at } r = r_* \\ U_r, D_r, R_r, L_r & \text{others.} \end{cases}$$

But the sign flip occurs if there were an (odd) insertion of U_{r*}, D_{r*}, R_{r*}, L_{r*}.

This also shows that the path

introduces a cut in the fermion system



If we compute $\langle \sigma_{r_1} \sigma_{r_2} \rangle$, using the expression obtained, we find the exponents $\gamma = \frac{7}{4}$, $\beta = \frac{1}{8}$, $\eta = \frac{1}{4}$.

e.g. $$M \left(:= \sqrt{\lim_{r \to \infty} \langle \sigma_0 \sigma_r \rangle} \right) = \left[1 - \sinh(2K)^{-4} \right]^{\frac{1}{8}} \qquad T < T_c$$

$$\langle \sigma_{r_1} \sigma_{r_2} \rangle \sim \frac{1}{|r_1 - r_2|^{\frac{1}{4}}} \qquad \text{at } T = T_c$$

But here we do not try to follow these computations.

Instead, we study the critical theory.

This turns out to be enough to find these exponents.

Massless Majorana Fermion

[recall : Critical 2d Ising = Massless Majorana fermion with a GSO projection]

$$S = \frac{1}{4\pi}\int dt\, d\sigma \left(i\psi_-(\partial_t+\partial_\sigma)\psi_- + i\psi_+(\partial_t-\partial_\sigma)\psi_+ \right)$$

This is obtained from Dirac fermion

$$S_{\text{Dirac}} = \frac{1}{2\pi}\int dt\, d\sigma \left(i\overline{\psi_-}(\partial_t+\partial_\sigma)\psi_- + i\overline{\psi_+}(\partial_t-\partial_\sigma)\psi_+ \right),$$

By setting $\psi_\pm = \overline{\psi_\pm}$ and <u>dividing by 2</u>. (*)

In other words, writing $\psi_\pm = (\psi_\pm^1 + i\psi_\pm^2)/\sqrt{2}$

$\overline{\psi_\pm} = (\psi_\pm^1 + i\psi_\pm^2)/\sqrt{2}$

We obtain <u>two copies of Majorana fermions</u> decoupled from each other.

Dirac = Majorana $\oplus$ Majorana.

↑

CFT of $c=1$

$\Rightarrow$ Majorana fermion ~~is~~ must be a CFT of $c=\frac{1}{2}$

*) "dividing by 2".

$$S_E = \sum_{i,j} \overline{\psi}_i A_{ij} \psi_j \Rightarrow \langle \psi_i \overline{\psi}_j \rangle = A^{-1}_{ij}$$

$$S_E = \underbrace{\frac{1}{2}}_{\uparrow} \sum_{ij} \psi_i B_{ij} \psi_j \Rightarrow \langle \psi_i \psi_j \rangle = B^{-1}_{ij}$$

Thus, if $\langle \psi_-(z) \overline{\psi}_-(w) \rangle \sim \frac{1}{z-w}$ for $S = \frac{1}{2\pi} \int i \overline{\psi}_- (\partial_t + \partial_\sigma) \psi_- + \cdots$

then $\langle \psi_-(z) \psi_-(w) \rangle \sim \frac{1}{z-w}$ for $S = \underbrace{\frac{1}{4\pi}} \int i \psi_- (\partial_t + \partial_\sigma) \psi_- + \cdots$

Let's confirm $c = \frac{1}{2}$

$$T_{zz} = -\frac{1}{2} \psi_- \partial_z \psi_- \quad , \quad T_{\bar{z}\bar{z}} = -\frac{1}{2} \psi_+ \partial_{\bar{z}} \psi_+$$

For Dirac: $T_{zz} = -\frac{1}{2} \overline{\psi}_- \partial_z \psi_- + \frac{1}{2} \partial_z \overline{\psi}_- \psi_-$ for $S = \frac{1}{2\pi} \int \cdots$

$\therefore$ For Majorana with $S = \underbrace{\frac{1}{4\pi}} \int \cdots$, we have

$$T_{zz} = \underbrace{\frac{1}{2}} \left(-\frac{1}{2} \psi_- \partial_z \psi_- + \frac{1}{2} \partial_z \psi_- \psi_- \right) = -\frac{1}{2} \psi_- \partial_z \psi_-$$

Similar for $T_{\bar{z}\bar{z}}$

Then $\langle T_{zz}(z)\, T_{ww}(w) \rangle = \frac{1}{4} \langle :\psi_-(z)\partial_z \psi_-(z): \; :\psi_-(w)\partial_w \psi_-(w): \rangle$

$$= \frac{1}{4}\psi_-(z)\partial_z\psi_-(z)\,\psi_-(w)\,\partial_w\psi_-(w) + \frac{1}{4}\psi_-(z)\,\partial_z\psi_-(z)\,\psi_-(w)\,\partial_w\psi_-(w)$$

$$= -\frac{1}{4}\cdot\frac{1}{z-w}\cdot\frac{2}{(z-w)^3} + \frac{1}{4}\,\frac{1}{(z-w)^2}\cdot\frac{-1}{(z-w)^2}$$

$$= \frac{1}{4}\,\frac{1}{(z-w)^4} = \frac{c/2}{(z-w)^4} \quad \text{for} \quad c = \frac{1}{2}$$

Similarly $\langle T_{\bar z\bar z}(\bar z)\, T_{\bar w\bar w}(w) \rangle = \frac{1}{4}\,\frac{1}{(\bar z-\bar w)^4}$ //.

One can also confirm the full OPE :

$$\begin{cases} T_{zz}(z)\,T_{ww}(w) = \dfrac{1/4}{(z-w)^4} + \dfrac{2}{(z-w)^2}T_{ww}(w) + \dfrac{1}{z-w}\partial_w T_{ww}(w) + \text{reg} \\[2ex] T_{\bar z\bar z}(z)\,T_{\bar w\bar w}(w) = \dfrac{1/4}{(\bar z-\bar w)^4} + \dfrac{2}{(\bar z-\bar w)^2}T_{\bar w\bar w}(w) + \dfrac{1}{\bar z-\bar w}\partial_{\bar w} T_{\bar w\bar w}(w) + \text{reg}. \end{cases}$$

$$T_{zz}(z)\,\psi_-(w) = -\frac{1}{2}\psi_-(z)\partial_z\psi_-(z)\psi_-(w) - \frac{1}{2}\psi_-(z)\partial_z\psi_-(z)\psi_-(w) - \frac{1}{2}:\psi_-(z)\partial_z\psi_-(z)\psi_-(w):$$

$$= \frac{1}{2}\frac{1}{(z-w)^2}\psi_-(z) + \frac{1}{2}\frac{1}{z-w}\partial_z\psi_-(z) - \frac{1}{2}:\psi_-(z)\partial_z\psi_-(z)\psi_-(w):$$

$$= \frac{1}{2}\frac{1}{(z-w)^2}\psi_-(w) + \frac{1}{z-w}\partial_w\psi_-(w) + \frac{3}{4}\partial_w^2\psi_-(w) + O(z-w)$$

$\therefore$ ψ_- is a primary operator of $\Delta = \frac{1}{2}$

also, ψ_+ is a primary operator of $\tilde{\Delta} = \frac{1}{2}$

The energy operator $\varepsilon = \sigma_i\sigma_j$ ($\langle i,j\rangle$: nearest neighbor)

is coupled to K as $-\beta\mathcal{H} = \sum_{\langle i,j\rangle} K \underbrace{\sigma_i\sigma_j}_{=\varepsilon}$.

On the other hand, K enters into the near critical action as

$$S_{2d\ \text{Ising}} = \frac{1}{4\pi}\int\left\{ i\psi_-(\partial_t+\partial_\sigma)\psi_- + i\psi_+(\partial_t-\partial_\sigma)\psi_+ + \underline{A(K-K_c)\psi_+\psi_-} \right\}$$

Thus

$$\boxed{\begin{array}{l} \varepsilon \sim \psi_+\psi_- \\ \text{It is a primary operator of } (\Delta,\tilde{\Delta}) = \left(\frac{1}{2},\frac{1}{2}\right) \end{array}}$$

What about the spin operator ?