What kind of operators the GSO projected Majorana fermion has?

— Let's compute the partition function to find the spectrum of states!

[$\Longrightarrow$ spectrum of operators by State/Operator Correspondence.]

$$Z = \frac{1}{2} Z_{AP,AP} + \frac{1}{2} Z_{AP,P} + \frac{1}{2} Z_{P,AP} \mp \frac{1}{2} Z_{P,P}$$

$$= \frac{1}{2} \mathrm{Tr}_{\mathcal{H}_{NS\text{-}NS}} \left(q^{H_R} \bar{q}^{H_L} \right) + \frac{1}{2} \mathrm{Tr}_{\mathcal{H}_{NS\text{-}NS}} \left((-1)^F q^{H_R} \bar{q}^{H_L} \right)$$

$$+ \frac{1}{2} \mathrm{Tr}_{\mathcal{H}_{R\text{-}R}} \left(q^{H_R} \bar{q}^{H_L} \right) + \frac{1}{2} \mathrm{Tr}_{\mathcal{H}_{R\text{-}R}} \left((-1)^F q^{H_R} \bar{q}^{H_L} \right)$$

<u>NS-NS</u>

$$\psi_-(t,\sigma) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r(t) e^{ir\sigma}, \quad \psi_+(t,\sigma) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \tilde{\psi}_r(t) e^{-ir\sigma}$$

reality $\psi_-^* = \psi_- \Rightarrow \psi_r^\dagger = \psi_{-r}$

$\psi_+^* = \psi_+ \Rightarrow \tilde{\psi}_r^\dagger = \tilde{\psi}_{-r}$.

$$L = \frac{1}{4\pi}\int_0^{2\pi} d\sigma \left(i\bar{\psi}_-(\partial_t+\partial_\sigma)\psi_- + i\bar{\psi}_+(\partial_t-\partial_\sigma)\psi_+ \right)$$

$$= \frac{1}{4\pi}\sum_{r,r'}\int_0^{2\pi} d\sigma \{ i\psi_{r'} e^{ir'\sigma}(\dot{\psi}_r + ir\psi_r)e^{ir\sigma} + i\tilde{\psi}_{r'}e^{-ir'\sigma}(\dot{\tilde{\psi}}_r + ir\tilde{\psi}_r)e^{-ir\sigma} \}$$

$$= \frac{1}{2}\sum_r \{ i\psi_{-r}\dot{\psi}_r - r\psi_{-r}\psi_r + i\tilde{\psi}_{-r}\dot{\tilde{\psi}}_r - r\tilde{\psi}_{-r}\tilde{\psi}_r \}$$

$$= \sum_{r>0} \{ i\psi_{-r}\dot{\psi}_r - r\psi_{-r}\psi_r + i\tilde{\psi}_{-r}\dot{\tilde{\psi}}_r - r\tilde{\psi}_{-r}\tilde{\psi}_r \}$$

$$\Rightarrow \quad \{\psi_r, \psi_s\} = \delta_{r+s,0}, \quad \{\tilde{\psi}_r, \tilde{\psi}_s\} = \delta_{r+s,0}$$

$$\{\psi_r, \tilde{\psi}_s\} = 0.$$

Hamiltonian for the r-th sector :

$$H_r = r\,\frac{\psi_{-r}\psi_r - \psi_r\psi_{-r}}{2} + r\,\frac{\tilde{\psi}_{-r}\tilde{\psi}_r - \tilde{\psi}_r\tilde{\psi}_{-r}}{2}$$

$$= r\left(\psi_{-r}\psi_r - \frac{1}{2}\right) + r\left(\tilde{\psi}_{-r}\tilde{\psi}_r - \frac{1}{2}\right)$$

$$H = \sum_{r>0} H_r .$$

$$[H, \psi_r] = -r\psi_r, \quad [H, \tilde{\psi}_r] = -r\tilde{\psi}_r$$

$\psi_{r>0}$, $\tilde{\psi}_{r>0}$ decreases energy by r. (annihilation op.)

$\psi_{-r<0}$, $\tilde{\psi}_{-r<0}$ increases 〃 (creation op.)

The ground state is the state $|0\rangle$ annihilated by

$\psi_r, \tilde{\psi}_r \quad \forall r>0.$ $\qquad \psi_r|0\rangle = \tilde{\psi}_r|0\rangle = 0.$

The ground state energy : $\displaystyle\sum_{r>0}\left\{r\left(-\tfrac{1}{2}\right)+r\left(-\tfrac{1}{2}\right)\right\}$

$$= -\sum_{r>0} r = -\left(\tfrac{1}{2}+\tfrac{3}{2}+\tfrac{5}{2}+\cdots\right)$$

$$= -\zeta(-1,\tfrac{1}{2}) = -\frac{1}{24} \qquad \left[\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}\right]$$

$$\therefore \; H_R = \sum_{r>0} r\left(\psi_{-r}\psi_r - \tfrac{1}{2}\right) = \sum_{r>0} r\psi_{-r}\psi_r - \frac{1}{48}$$

$$H_L = \sum_{r>0} r\left(\tilde{\psi}_{-r}\tilde{\psi}_r - \tfrac{1}{2}\right) = \sum_{r>0} r\tilde{\psi}_{-r}\tilde{\psi}_r - \frac{1}{48}$$

$\left(\rightsquigarrow \quad c = \tfrac{1}{2} \text{ also}\right)$

$$Z_{AP,AP} = \mathrm{Tr}_{\mathcal{H}_{NS\text{-}NS}}\left(q^{H_R}\bar{q}^{H_L}\right), \quad Z_{AP,P} = \mathrm{Tr}_{\mathcal{H}_{NS\text{-}NS}}\left((-1)^F q^{H_R}\bar{q}^{H_L}\right)$$

define $(-1)^F = 1$ on the ground state $|0\rangle$.

$\overset{\shortparallel}{(-1)^{F_R}(-1)^{F_L}}$

<u>The Natural choice</u>

$$\mathrm{Tr}_{\mathcal{H}_{NS\text{-}NS}}(\pm 1)^F q^{H_R}\bar{q}^{H_L} = \underbrace{\mathrm{Tr}_{\mathcal{H}_{NS}}(\pm 1)^{F_R} q^{H_R}}_{\overset{\shortparallel}{q^{-\frac{1}{48}}\prod_{r>0}(1\pm q^r)}} \cdot \underbrace{\mathrm{Tr}_{\mathcal{H}_{NS}}(\pm 1)^{F_L}\bar{q}^{H_L}}_{\overset{\shortparallel}{\bar{q}^{-\frac{1}{48}}\prod_{r>0}(1\pm \bar{q}^r)}}$$

R-R $\quad \psi_-(t,\sigma) = \sum_{n\in\mathbb{Z}} \psi_n(t)\, e^{in\sigma}, \quad \psi_+(t,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{\psi}_n(t)\, e^{-in\sigma}$

reality $\quad \psi_n^* = \psi_{-n}, \quad \tilde{\psi}_n^* = \tilde{\psi}_{-n}$

in particular $\quad \psi_0^* = \psi_0, \quad \tilde{\psi}_0^* = \tilde{\psi}_0.$

$$L = \frac{i}{2}\psi_0\dot{\psi}_0 + \frac{i}{2}\tilde{\psi}_0\dot{\tilde{\psi}}_0 + \sum_{n>0}\left\{ i\psi_{-n}\dot{\psi}_n - n\psi_{-n}\psi_n + i\tilde{\psi}_{-n}\dot{\tilde{\psi}}_n - n\tilde{\psi}_{-n}\tilde{\psi}_n \right.$$

$$\longrightarrow \quad \{\psi_n, \psi_m\} = \delta_{n+m,0}, \quad \{\tilde{\psi}_n, \tilde{\psi}_m\} = \delta_{n+m,0}$$

$$\{\psi_n, \tilde{\psi}_m\} = 0.$$

$\psi_{n>0}, \tilde{\psi}_{n>0}$ decreases energy.

$\psi_{-n<0}, \tilde{\psi}_{-n<0}$ increases energy.

$\psi_0, \tilde{\psi}_0$ do not change energy!

What is the ground state?

- It must be annihilated by $\psi_n, \tilde{\psi}_n \quad \forall n>0.$
- $\psi_0, \tilde{\psi}_0$ acts on it and produce another ground state.

$$\{\psi_0, \psi_0\} = \{\tilde{\psi}_0, \tilde{\psi}_0\} = 1, \quad \{\psi_0, \tilde{\psi}_0\} = 0$$

2d. real Clifford algebra $\longleftrightarrow$ 1 $d_{\mathbb{C}}$ Clifford algebra.

$\rightarrow$ Two fold degeneracy

Write $\quad \eta = \frac{1}{\sqrt{2}}(\psi_0 + i\tilde{\psi}_0)$, $\quad \bar{\eta} = \frac{1}{\sqrt{2}}(\psi_0 - i\tilde{\psi}_0)$

$$\{\eta, \bar{\eta}\} = 1, \quad \eta^2 = \bar{\eta}^2 = 0$$

$|0\rangle$ ann. by η, $\quad \bar{\eta}|0\rangle$ ann by $\bar{\eta}$

$$|0\rangle \underset{\eta}{\overset{\bar{\eta}}{\rightleftarrows}} \bar{\eta}|0\rangle$$

$$\psi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\psi}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

$(-1)^F = ?$ $\quad$ there is no natural choice

Pick one $\quad$ say $+1$ on $|0\rangle$, $\quad -1$ on $\bar{\eta}|0\rangle$.

$$H = \sum_{n>0}\left(n\,\frac{\psi_{-n}\psi_n - \psi_n\psi_{-n}}{2} + n\,\frac{\tilde{\psi}_{-n}\tilde{\psi}_n - \tilde{\psi}_n\tilde{\psi}_{-n}}{2} \right)$$

$$= \sum_{n>0} n\left(\psi_{-n}\psi_n - \frac{1}{2}\right) + n\left(\tilde{\psi}_{-n}\tilde{\psi}_n - \frac{1}{2}\right)$$

The ground state energy $= -\sum_{n>0} n = -\left(-\frac{1}{12}\right) = \frac{1}{12}$

$$H_R = \sum_{n>0} n(\psi_{-n}\psi_n) + \frac{1}{24}$$

$$H_L = \sum_{n>0} n\,\tilde{\psi}_{-n}\tilde{\psi}_n + \frac{1}{24}$$

$$Z_{P,AP} = \mathrm{Tr}_{\mathcal{H}_{RR}} q^{H_R} \bar{q}^{H_L} = (1+1) \left| q^{\frac{1}{24}} \prod_{n>0} (1+q^n) \right|^2$$

2-fold degeneracy

$$Z_{P,P} = \mathrm{Tr}_{\mathcal{H}_{RR}} (-1)^F q^{H_R} \bar{q}^{H_L} = (1-1) \left| q^{\frac{1}{24}} \prod_{n>0} (1-q^n) \right|^2 = 0$$

$$Z = \frac{1}{2} Z_{AP,AP} + \frac{1}{2} Z_{AP,P} + \frac{1}{2} Z_{P,AP} \mp \frac{1}{2} Z_{P,P}$$

$$= \frac{1}{2} \left| q^{-\frac{1}{48}} \prod_{r>0} (1+q^r) \right|^2 + \frac{1}{2} \left| q^{-\frac{1}{48}} \prod_{r>0} (1-q^r) \right|^2$$

$$+ \frac{1}{2} \cdot 2 \left| q^{\frac{1}{24}} \prod_{n>0} (1+q^n) \right|^2 \mp \frac{1}{2} \cdot 0$$

$$= \left| \frac{1}{2} q^{-\frac{1}{48}} \prod_{r>0} (1+q^r) + \frac{1}{2} q^{-\frac{1}{48}} \prod_{r>0} (1-q^r) \right|^2 \qquad =: |A|^2$$

$$+ \left| \frac{1}{2} q^{-\frac{1}{48}} \prod_{r>0} (1+q^r) - \frac{1}{2} q^{-\frac{1}{24}} \prod_{r>0} (1-q^r) \right|^2 \qquad =: |B|^2$$

$$+ \left| q^{\frac{1}{24}} \prod_{n>0} (1+q^n) \right|^2 \qquad =: |C|^2$$

$$A = q^{-\frac{1}{48}} \left\{ \frac{1}{2} \prod_{r>0} (1+q^r) + \frac{1}{2} \prod_{r>0} (1-q^r) \right\}$$

$$= q^{-\frac{1}{48}} \left\{ 1 + \sum_{0<r_1<r_2} q^{r_1+r_2} + \sum_{0<r_1<r_2<r_3<r_4} q^{r_1+r_2+r_3+r_4} + \cdots \right\}$$

$$B = q^{-\frac{1}{48}} \left\{ \frac{1}{2} \prod_{r>0} (1+q^r) - \frac{1}{2} \prod_{r>0} (1-q^r) \right\}$$

$$= q^{-\frac{1}{48}} \left\{ \sum_{r>0} q^r + \sum_{0<r_1<r_2<r_3} q^{r_1+r_2+r_3} + \cdots \right\}$$

$$= q^{-\frac{1}{48}+\frac{1}{2}} \left\{ \sum_{r>0} q^{r-\frac{1}{2}} + \sum_{0<r_1<r_2<r_3} q^{r_1+r_2+r_3-\frac{1}{2}} + \cdots \right\}$$

$$C = q^{\frac{1}{24}} \prod_{n>0} (1+q^n) \qquad \qquad \frac{1}{24} = -\frac{1}{48} + \frac{1}{16}$$

$$= q^{-\frac{1}{48}+\frac{1}{16}} \left(1 + \sum_{n>0} q^n + \sum_{0<n_1<n_2} q^{n_1+n_2} + \sum_{0<n_1<n_2<n_3} q^{n_1+n_2+n_3} + \cdots \right\}$$

A, B, C takes the form $q^{\Delta - \frac{c}{24}} \left(1 + \sum_{n=1}^{\infty} N_n q^n \right)$

with $\Delta = 0, \frac{1}{2}, \frac{1}{16}$

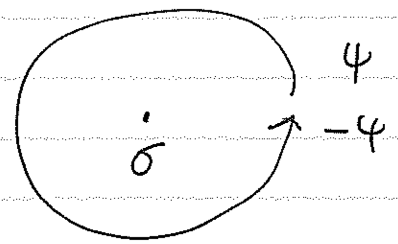
primary descendants.

$\Delta = \tilde{\Delta} = 0 \iff \mathbb{1}$

$\Delta = \tilde{\Delta} = \frac{1}{2} \longleftrightarrow \varepsilon$ as we have seen

$\Delta = \tilde{\Delta} = \frac{1}{16} \longleftrightarrow \sigma$?

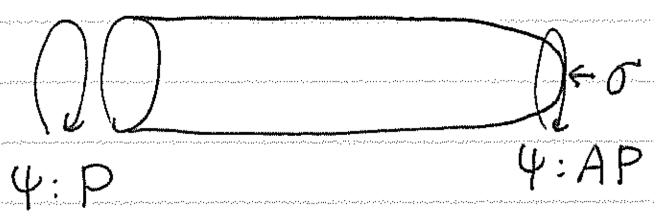
In fact, we know that



for $\psi = \psi_+$ or ψ_-.

i.e. $\psi_\pm$ is antiperiodic when it circles around σ.

Via the state/operator correspondence



$\psi_\pm$ is periodic on the state corresponding to σ.

And $\Delta = \tilde{\Delta} = \frac{1}{16}$ and descendants come from

the R-R sector ($\psi_\pm$ periodic sector)

This means that σ indeed correspond to

$\Delta = \tilde{\Delta} = \frac{1}{16}$ primary (or its descendant)

If it is indeed the primary

$$\langle \sigma(x)\,\sigma(y) \rangle \sim \frac{1}{(x-y)^{2\Delta}(\bar{x}-\bar{y})^{2\tilde{\Delta}}}$$

$$= \frac{1}{|x-y|^{\frac{1}{4}}}$$

$$\boxed{\therefore\ \eta = \frac{1}{4}}$$

Other exponents $\left(\text{e.g. } \gamma = \frac{7}{4},\ \beta = \frac{1}{8},\ \cdots \right)$

$\chi = (T-T_c)^{-\gamma},\ |M| \sim (T_c - T)^{\beta}, \cdots$

can be obtained by Renormalization Group

(or Scaling) argument, which we discuss next.