Recall Landau's Theory

$$\sum_{\frac{1}{V}\sum_i \sigma_i = M} e^{-\mathcal{H}(\sigma)/kT} =: e^{-VU_{eff}(M)}$$

sum only over spins s.t. the average $= M$.

To find the partition function (or free energy), just remove the constraint on the average:

$$Z = e^{-VF} = \sum_{\text{all } \sigma} e^{-\mathcal{H}(\sigma)/kT} = \sum_M e^{-VU_{eff}(M)}$$

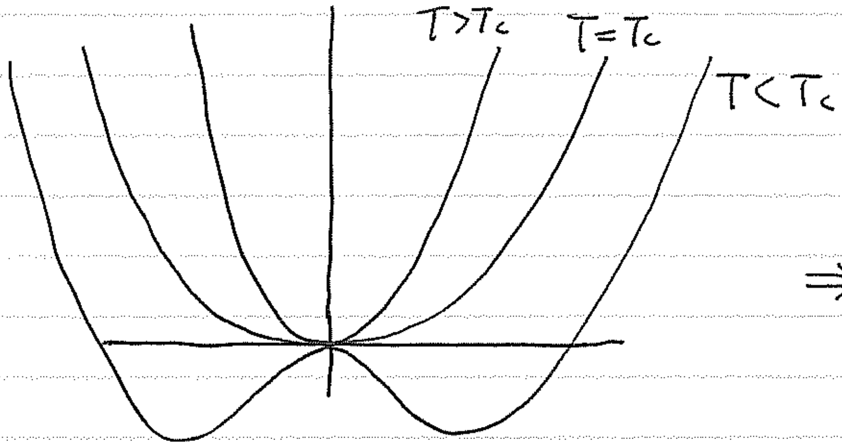
$$\underset{\text{dominated by}}{\overset{V\to\infty}{\simeq}} e^{-VU_{eff}(M_*)} \qquad \therefore\ F \simeq U_{eff}(M_*)$$

where M_* is the minimum of $U_{eff}(M)$.

$$\langle \sigma_i \rangle = \frac{\sum_{\sigma:\text{all}} \sigma_i e^{-\mathcal{H}(\sigma)/kT}}{Z} = \frac{\sum_{\sigma:\text{all}} \left(\frac{1}{V}\sum_i \sigma_i\right) e^{-\mathcal{H}(\sigma)/kT}}{Z}$$

$$= \frac{\sum_M M e^{-VU_{eff}(M)}}{\sum_M e^{-VU_{eff}(M)}} \simeq M_* .$$

e.g. $$U_{eff}(M) = U_{eff}(0) + \underset{\vee\atop 0}{a}(T-T_c)M^2 + \underset{\vee\atop 0}{b}M^4 - MH$$



$\Rightarrow$ mean field result.

e.g. $$U_{eff}(M) = U_{eff}(0) + a(T-T_c)^{7/4}M^2 + bM^{16} - MH$$

$\Rightarrow$ Correct critical exponents of 2d Ising model

$$\beta = \frac{1}{8}, \quad \gamma = \frac{7}{4}, \quad \ldots$$

Can we <u>compute</u> $U_{eff}(M)$?

This idea to separate the sum $\sum_\sigma e^{-\mathcal{H}(\sigma)/kT}$ into steps.

is the essence of $\boxed{\text{renormalization group}}$.

The above was, however, too coarse to compute some quantities

For example, $\langle \sigma_i \sigma_j \rangle$ cannot be computed this way:

Necessary information was washed away at the moment

the first step $\sum_{\frac{1}{V}\sum_i \sigma_i = M} e^{-\mathcal{H}(\sigma)/kT}$ was taken.

But we may separate the sum into FINER STEPS.

e.g. Square lattice $i, j, k \cdots = (x, y) \quad x \in \mathbb{Z},\ y \in \mathbb{Z}$

Fix the spins at even sites $\sigma_{ev} = \left\{ \sigma_{(x,y)} \,\middle|\, \begin{matrix} x \in 2\mathbb{Z} \\ y \in 2\mathbb{Z} \end{matrix} \right\}$

and sum over spins at all other sites

$$\sum_{\left\{ \sigma_{(x,y)} \,\middle|\, \begin{matrix} x \text{ or } y \\ \text{is odd} \end{matrix} \right\}} e^{-\mathcal{H}(\sigma)/kT} = e^{-\mathcal{H}_{ev}(\sigma_{ev})/kT}$$

"effective Hamiltonian" for even sites spins.

$$Z = \sum_{\sigma\,\text{all}} e^{-\mathcal{H}(\sigma)/kT} = \sum_{\sigma_{ev}} \left(\sum_{\{\sigma_{(x,y)} \mid x \text{ or } y \text{ is odd}\}} e^{-\mathcal{H}(\sigma)/kT} \right)$$

$$= \sum_{\sigma_{ev}} e^{-\mathcal{H}_{ev}(\sigma_{ev})/kT}$$

If i and j are even sites (i.e. $i = (2n_x^i, 2n_y^i)$, $j = (2n_x^j, 2n_y^j)$)

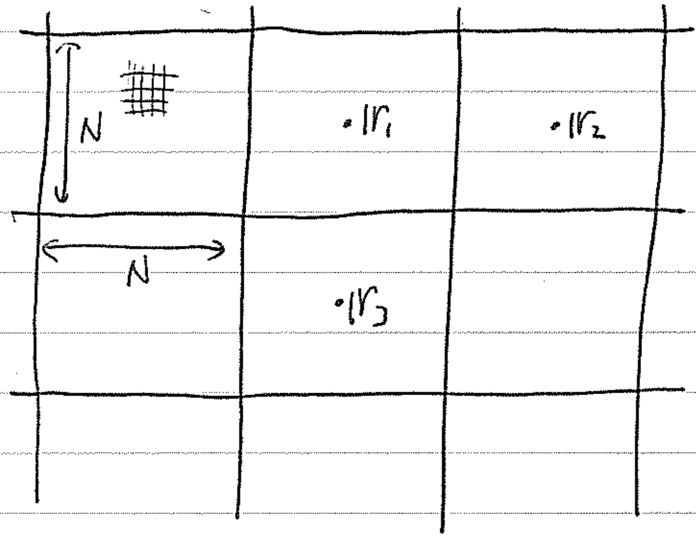
$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{\sigma:\text{all}} \sigma_i \sigma_j \, e^{-\mathcal{H}(\sigma)/kT}}{\sum_{\sigma:\text{all}} e^{-\mathcal{H}(\sigma)/kT}}$$

$$= \frac{\sum_{\sigma_{ev}} \sigma_i \sigma_j \, e^{-\mathcal{H}_{ev}(\sigma_{ev})/kT}}{\sum_{\sigma_{ev}} e^{-\mathcal{H}_{ev}(\sigma_{ev})/kT}}$$

Thus, we can use the "effective Hamiltonian" $\mathcal{H}_{ev}(\sigma_{ev})$ to compute anything about even site spins.

[But of course, if i or j ~~are~~ is not even sites, we cannot use $\mathcal{H}_{ev}(\sigma_{ev})$ to compute $\langle \sigma_i \sigma_j \rangle$.]

Alternatively :



Consider a big grid s.t. each square contains N^2 lattice points. For each square, assign a number $M(\mathbb{r})$ ($\mathbb{r}$ = center of the square).

$$e^{-\mathcal{H}_{(N)}(M(\mathbb{r}))/kT} = \sum_{\substack{\frac{1}{N^2}\sum_{i \in \text{square with center } \mathbb{r}} \sigma_i = M(\mathbb{r})}} e^{-\mathcal{H}(\sigma)/kT}$$

Then $Z = \sum_{M(\cdot)} e^{-\mathcal{H}_{(N)}(M(\cdot))/kT}$

If $|i-j| \gg N$, we may approximate $\langle \sigma_i \sigma_j \rangle$ by

$$\langle M(\mathbb{r}_i)\, M(\mathbb{r}_j) \rangle = \frac{\sum_{M(\cdot)} M(\mathbb{r}_i)\, M(\mathbb{r}_j)\, e^{-\mathcal{H}_{(N)}(M(\cdot))/kT}}{\sum_{M(\cdot)} e^{-\mathcal{H}_{(N)}(M(\cdot))/kT}}$$

$\mathbb{r}_i$ ($\mathbb{r}_j$) : Center of the square i (j) belongs to.

$$\mathcal{H}(\sigma) \to \mathcal{H}_{ev}(\sigma_{ev})$$

$$\mathcal{H}(\sigma) \to \mathcal{H}_{(N)}(M(\cdot))$$

are called <u>renormalization group transformations</u>

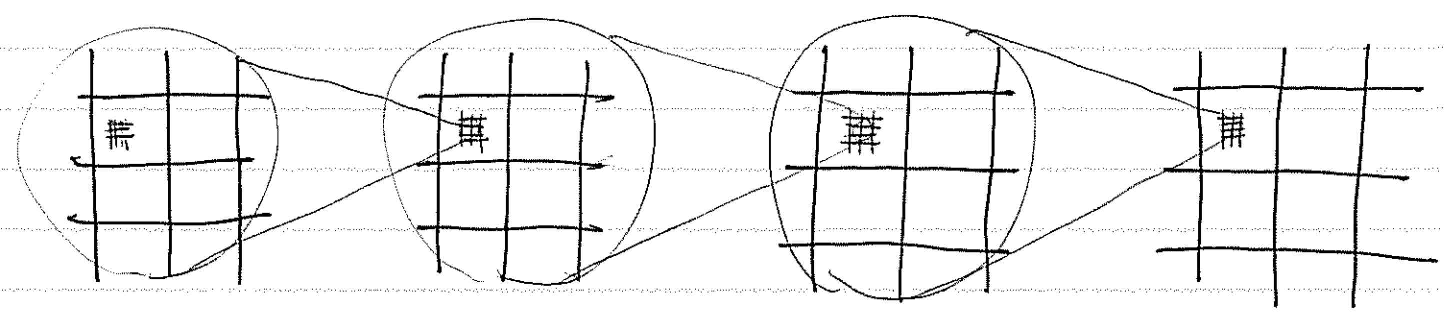
One may iterate these:

$$\mathcal{H}(\sigma) \to \mathcal{H}_{ev} \to (\mathcal{H}_{ev})_{ev} \to ((\mathcal{H}_{ev})_{ev})_{ev} \to \cdots$$

or

$$\mathcal{H} \to \mathcal{H}_{(N)} \to (\mathcal{H}_{(N)})_{(N)} \to ((\mathcal{H}_{(N)})_{(N)})_{(N)} \to \cdots$$

<u>renormalization group flow</u> (of effective Hamiltonians)



Sequence of coarse grainings.

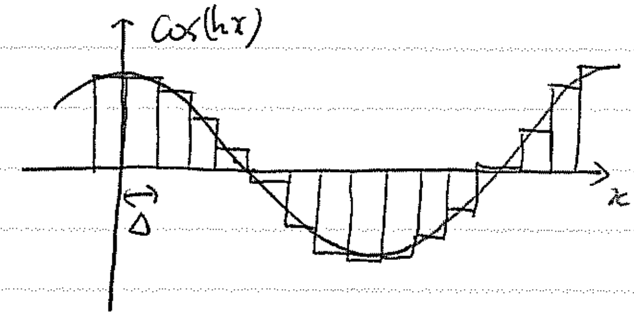
One can do it also in <u>momentum space</u>

Before doing it, Notice some elementary fact:

$e^{ikx} = \cos(kx) + i\sin(kx)$

is nearly constant over distances

$|\Delta x| \ll \frac{1}{k}$



[piecewise constant function is a good approximation if $\Delta \ll \frac{1}{k}$.]

$$\phi(x) = \int_{|k| \leq \Lambda} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \hat{\phi}(k)$$

... Well approximated by a piecewise constant function

with $\Delta \ll \frac{1}{\Lambda}$

Making Δ larger and larger

$\Longleftrightarrow$ Λ smaller and smaller.

RG in momentum space

$$S(\phi, m, g_4, g_6, \cdots) = \int d^dx \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g_4\phi^4 + \frac{1}{6!}g_6\phi^6 + \cdots \right\}$$

Some action for a field ϕ
(or a set of fields)

Assume $\displaystyle \phi(x) = \int_{|k| \leq \Lambda_0} \frac{d^dk}{(2\pi)^d} e^{ikx} \hat{\phi}(k)$ [Λ_0 : UV cut-off]

Decompose it as $\displaystyle = \underbrace{\int_{|k| \leq \Lambda_1} \frac{d^dk}{(2\pi)^d} e^{ikx} \hat{\phi}(k)}_{\phi_L} + \underbrace{\int_{\Lambda_1 \leq |k| \leq \Lambda_0} \frac{d^dk}{(2\pi)^d} e^{ikx} \hat{\phi}(k)}_{\phi_H}$

Low frequency part — High frequency part.

$$\int \mathcal{D}\phi_H \, e^{-S(\phi_L + \phi_H)} = e^{-S_L(\phi_L)}$$

Path-integral only over High frequency modes

"effective action" at Λ_1

$$S(\phi) \longrightarrow S_L(\phi_L)$$

is the renormalization group transformation.

How does $S_L(\phi_L)$ look like? [c.f QFT II]

In general

$$S_L(\phi_L) = \int d^d x \left\{ \frac{1}{2} K_L (\partial_\mu \phi_L)^2 + \frac{1}{2} m_L^2 \phi_L^2 + \frac{1}{4!} g_{4,L} \phi_L^4 + \frac{1}{6!} g_{6,L} \phi_L^6 + \cdots \right.$$

$$\underbrace{\cdots\cdots + \tilde{K}_L \left((\partial_\mu \phi_L)^2 \right)^2 + \tilde{\tilde{K}}_L \phi_L^2 (\partial_\mu \phi_L)^2 + \cdots}_{\text{terms that were not in original } S(\phi) \text{ may be generated.}}$$

$K_L,\ m_L^2,\ g_{4,L},\ \cdots\cdots,\ \tilde{K}_L,\ \tilde{\tilde{K}}_L,\ \cdots\cdots$

are some (computable) functions of $m, g_4, g_6, \cdots, \Lambda_1, \Lambda_0$

We may interate:

$$\Lambda_0 \longrightarrow \Lambda_1 \longrightarrow \Lambda_2 \longrightarrow \Lambda_3 \longrightarrow \cdots$$

$$\begin{pmatrix} m \\ g \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} K_1 \\ m_1 \\ g_1 \\ \vdots \\ \tilde{K}_1 \\ \tilde{\tilde{K}}_1 \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} K_2 \\ m_2 \\ g_2 \\ \vdots \\ \tilde{K}_2 \\ \tilde{\tilde{K}}_2 \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} K_3 \\ m_3 \\ g_3 \\ \vdots \\ \tilde{K}_3 \\ \tilde{\tilde{K}}_3 \\ \vdots \end{pmatrix} \longrightarrow \cdots$$

RG flow of Coupling constants

Dimensional analysis (e.g. scalar fields)

$$S(\phi) = \int d^dx \left\{ \frac{1}{2}\underbrace{(\partial_\mu \phi)^2}_{} + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g_4 \phi^4 + \cdots \right.$$

$S(\phi)$: dimensionless

d^dx: $(\text{mass})^{-d}$

this must have dimension $(\text{mass})^d$

∂_μ has dim = mass^1 $\quad \therefore 2(\text{dim of } \phi) = (d-2)$

$\therefore$ (canonical) dimension of ϕ is $d_\phi^c = \frac{d-2}{2}$.

Term $g_n \phi^n$ it must have dim $(\text{mass})^d$

$(\phi)^n$ has dim $(\text{mass})^{n d_\phi^c}$

$\therefore g_n$ must have $(\text{mass})^{d - n d_\phi^c}$

(e.g. $m^2 = g_2$ has dim $(\text{mass})^{d-2d_\phi^c} = \text{mass}^{d-(d-2)} = \text{mass}^2$)

Let us neutralize the dimension of coupling by writing $g_n = \Lambda^{d-nd_\phi^c} \underbrace{\lambda_n}_{\text{dimensionless.}}$ in the effective action at Λ

Similarly for term like $\tilde{K} \phi^2 (\partial_\mu \phi)^2$, ----

$$\tilde{K} = \Lambda^{d-2-4d_\phi^c} \lambda_{\tilde{K}}$$

e.g. effective action at Λ in $d=4$ scalar theory looks like:

$$S_\Lambda(\phi) = \int d^4x \left\{ \frac{1}{2} K (\partial_\mu \phi)^2 + \frac{1}{2}\lambda_2 \Lambda^2 \phi^2 + \frac{1}{4!}\lambda_4 \phi^4 + \frac{1}{6!}\frac{\lambda_6}{\Lambda^2}\phi^6 + \cdots \right.$$

$$\left. + \frac{\lambda_{\tilde{K}}}{\Lambda^4}\left((\partial_\mu\phi)^2\right)^2 + \frac{\lambda_{\tilde{\tilde{K}}}}{\Lambda^2}\phi^2(\partial_\mu\phi)^2 + \cdots \right\}$$

Let $\lambda = (\lambda_i)$ be the tuple of all dimensionless couplings (including K in kinetic term).

Then, RG flow takes the form

$$(\Lambda, \lambda) \longrightarrow (\Lambda', \lambda') \qquad \Lambda' < \Lambda$$

$$\lambda'_i = \lambda_i\left(\frac{\Lambda'}{\Lambda}, \lambda\right)$$

Infinitesimal RG flow:

$$\Lambda' \frac{d}{d\Lambda'} \lambda_i\left(\frac{\Lambda'}{\Lambda}, \lambda\right)\Bigg|_{\Lambda'=\Lambda} = \text{dimensionless} = \beta_i(\lambda)$$

function of $\lambda = (\lambda_i)$ only.

— <u>The β-functions</u>

For K (the coefficient of kinetic term), there is a special notation.

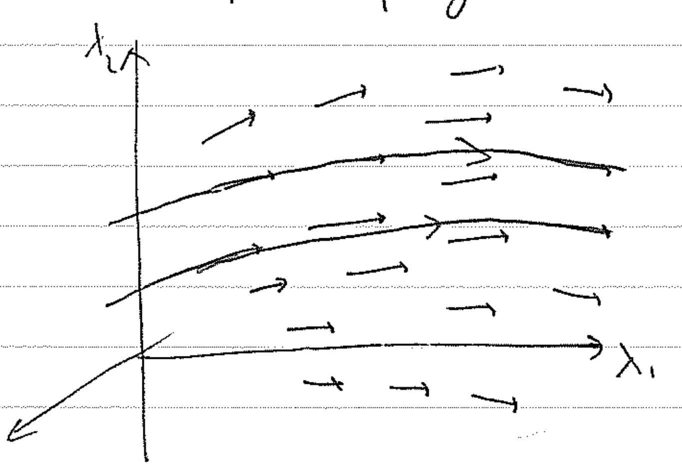
$$\Lambda' \frac{d}{d\Lambda'} K\left(\frac{\Lambda'}{\Lambda}, \lambda\right)\Big|_{\Lambda'=\Lambda} = -2K \cdot \underbrace{\gamma(\lambda)}_{\text{anomalous dimension}}$$

$$\left(\Leftrightarrow \quad \Lambda' \frac{d}{d\Lambda'} \log K^{\frac{1}{2}} \Big|_{\Lambda'=\Lambda} = -\gamma(\lambda) \right)$$

Note: If we define $\phi_c = K^{\frac{1}{2}} \phi$, then ϕ_c has the standard kinetic term

$$S_\Lambda = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi_c)^2 + \cdots \right\}$$

β-functions define a vector field on the space of coupling constants.

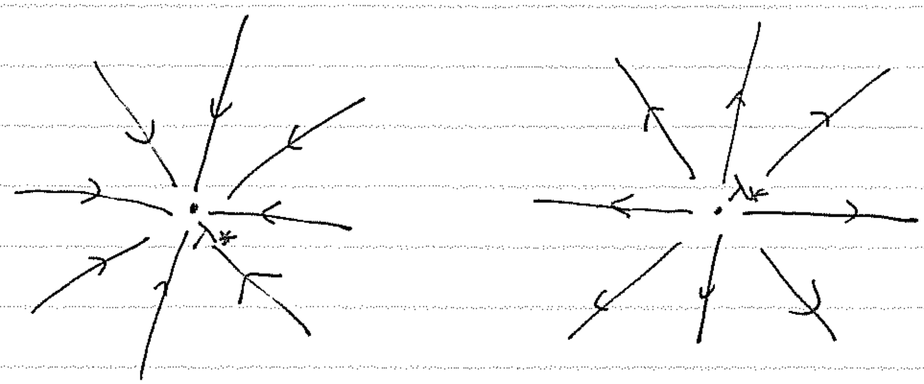


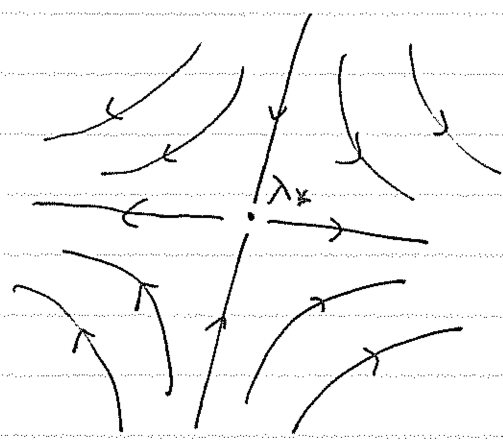
RG flows are ~~integral~~ phase curves of this vector field.

There may a zero of β-functions.

$$\beta_i(\lambda_*) = 0 \quad \forall i.$$

Such a point corresponds to a fixed point of RG flow (i.e. fixed point of RG transformation).

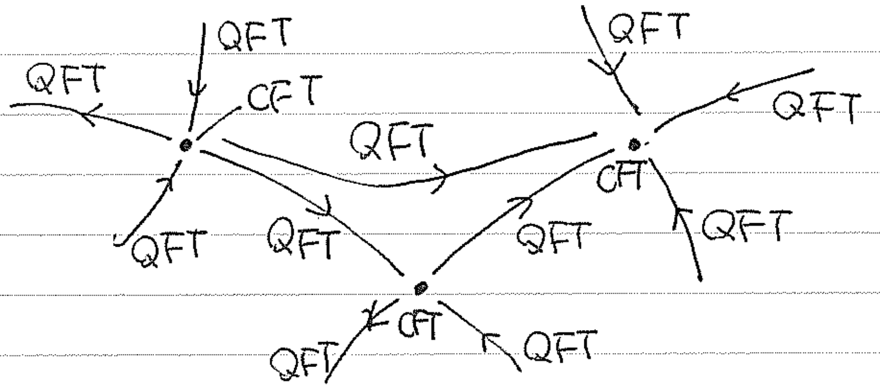




The theory at such a fixed point is a <u>scale invariant</u> theory.

In all known examples, it is also <u>conformally invariant</u>. i.e. it is a <u>CFT</u>.

A QFT is an RG trajectory



The trajectory always has some end point (IR fixed point).

either
- empty theory (massive theory)
- non-trivial CFT

It may also have some starting point (UV fixed point).

Usually, we consider a free theory perturbed by some interaction terms. e.g. $S = \underbrace{\int \frac{1}{2}(\partial_\mu \phi)^2}_{S_{free}} + \underbrace{\int \phi^4}_{\Delta S \text{ (perturbation)}}$

- In many cases, it comes back to the free theory
- In many other cases, it flows to a massive theory (i.e. no IR degrees of freedom. e.g. 4d YM, 2d $O(N)$ sigma model).
- But in some cases, it flows to a non-trivial IR fixed point.

e.g. 4d SQCD $SU(N_c)$, $N_c \leq N_f < 3N_c$ 2d NLSM $M_d \subset \mathbb{CP}^{N-1}$ $3 \leq d < N$