A QFT is obtained from a CFT by perturbation

$$S = S_{CFT} + \int d^dx \sum_i \underset{\text{coupling.}}{\lambda_i} \underset{\text{local operator}}{\mathcal{O}_i(x)}$$

What happens depends very much on the dimension $d_{\mathcal{O}_i}$ of the operator $\mathcal{O}_i$

$$[\Delta S]_{CFT} = 0, \quad [d^dx]_{CFT} = \text{mass}^{-d}, \quad [\mathcal{O}_i]_{CFT} = \text{mass}^{d_{\mathcal{O}_i}} \quad \therefore [\lambda_i]_{CFT} = \text{mass}^{d - d_{\mathcal{O}_i}}$$

In a n.h.d. of the CFT (i.e. of $\lambda_i = 0 \ \forall i$), the β-function behaves as

$$\beta_i(\lambda) = (d_{\mathcal{O}_i} - d)\lambda_i + O(\lambda^2)$$

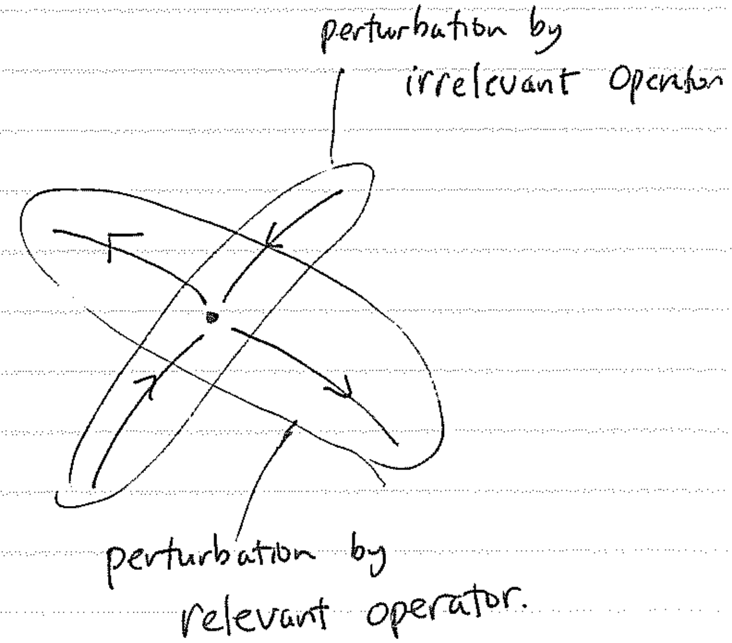
$$\therefore \quad \lambda_i' = \lambda_i\left(\frac{\Lambda'}{\Lambda}, \lambda\right) \sim \left(\frac{\Lambda'}{\Lambda}\right)^{d_{\mathcal{O}_i} - d} \lambda_i \quad \xrightarrow{\Lambda'/\Lambda \to 0} 0 \quad \text{if } d_{\mathcal{O}_i} > d$$

$$\xrightarrow{\Lambda'/\Lambda \to 0} \infty \quad \text{if } d_{\mathcal{O}_i} < d.$$

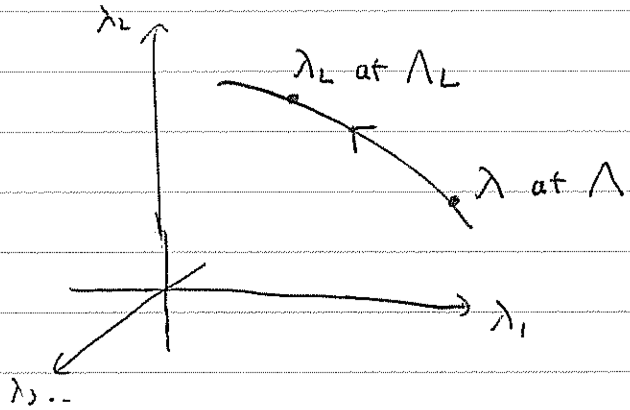
$\mathcal{O}$ is <u>irrelevant</u> if $d_{\mathcal{O}} > d$

<u>relevant</u> if $d_{\mathcal{O}} < d$

<u>marginal</u> if $d_{\mathcal{O}} = d$



RG Equation



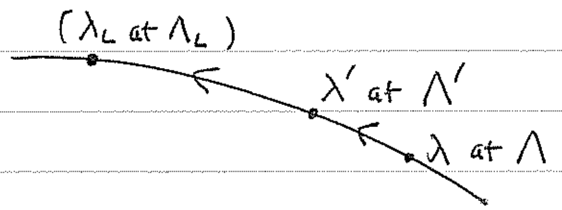
Suppose we have an RG trajectory.

The effective action at Λ_L

$S_{\Lambda_L}(\varphi)$ can be regarded as a function of

ϕ, λ, Λ and Λ_L. write $S_{\Lambda_L}(\phi) = S_{\Lambda_L}(\varphi, \lambda, \Lambda)$.

It does not change if (Λ, λ) is replaced by (Λ', λ') on the same trajectory. (Of course!)



Suppose $\bar{\lambda}(t)$ is a solution to $\dfrac{d\bar{\lambda}_i}{dt} = \beta_i(\bar{\lambda})$ with initial condition $\bar{\lambda}(0) = \lambda$. Then

$$\boxed{S_{\Lambda_L}(\varphi, \bar{\lambda}(t), e^t\Lambda) = S_{\Lambda_L}(\phi, \lambda, \Lambda) \quad \forall t}$$

This is the RG equation (in the integral form).

[Its differential form is obtained by taking t-derivative:

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\sum_i\beta_i(\lambda)\frac{\partial}{\partial\lambda_i}\right]S_{eff}=0.$$

But we will use the integral form, because its more convenient]

In particular, we are interested in the effective potential at $\Lambda_L=0$; $U_{eff}(\phi,\lambda,\Lambda)$. The RGE is

$$\boxed{U_{eff}(\phi,\bar\lambda(t),e^{+t}\Lambda)\quad\text{is}\quad t\text{-independent}}$$

In a n.h.d. of a CFT

$$\frac{d\bar\lambda_i}{dt}=(d_{O_i}-d)\bar\lambda_i\quad\Rightarrow\quad\bar\lambda_i(t)=e^{(d_{O_i}-d)t}\lambda_i$$

this includes $\displaystyle\frac{d\sqrt{K}}{dt}=-\gamma_\phi\sqrt{K}\quad\Rightarrow\quad\sqrt{K}(t)=e^{-\gamma_\phi t}\sqrt{K(0)}$

anomalous dimension of ϕ (a number)

Include $\sqrt{K}$ into ϕ; $\bar\phi_c(t)=\sqrt{K(t)}\phi=e^{-\gamma_\phi t}\bar\phi_c(0)$

Consider a particular perturbation where only one $\lambda_i \mathcal{O}_i$ is turned on. (call it $\lambda \mathcal{O}$).

also denot $\phi_c = \sqrt{K}\phi$ by M $(\gamma_\phi = \gamma_M)$

Then the RG Eqn is

$$\boxed{U_{eff}\left(e^{-\gamma_M t} M,\ e^{-(d-d_0)t}\lambda,\ e^{t}\Lambda\right) \text{ is } t\text{-independent}}$$

Dimensional analysis (canonical dim):

$U_{eff}(M)$ enters S_{eff} by $\int d^dx\, U_{eff}$. Thus $[U_{eff}] = \text{mass}^{-d}$

$[M] = \text{mass}^{d_M^c}$. Thus $M\Lambda^{-d_M^c}$ is dimensionless.

$$\therefore\ U_{eff}(M,\lambda,\Lambda) = \Lambda^d\, u(M\Lambda^{-d_M^c}, \lambda)$$

$$\overset{\text{RGE}}{=} e^{dt}\Lambda^d u(\underbrace{e^{-\gamma_M t - d_M^c t}}_{= e^{-d_M t}} M\Lambda^{-d_M^c},\ e^{-(d-d_0)t}\lambda)$$

$$d_M = \underset{\text{canonical}}{d_M^c} + \underset{\text{anomalous}}{\gamma_M}$$

This means $u(x,y) = x^{\frac{d}{d_M}} f\left(x^{-\frac{d-d_0}{d_M}} y\right)$

or

$$= y^{\frac{d}{d-d_0}}\, g\left(x\, y^{-\frac{d_M}{d-d_0}}\right)$$

$$U_{\text{eff}} = \Lambda^{d} u(M\Lambda^{-d_M^c}, \lambda) \overset{\substack{\text{forget } \Lambda \ (\overset{\text{set}}{=} 1) \\ \downarrow}}{=} u(M, \lambda)$$

$$= M^{\frac{d}{d_M}} f\left(\lambda M^{-\frac{d-d_O}{d_M}}\right)$$

$$\overset{\text{or}}{=} \lambda^{\frac{d}{d-d_O}} g\left(M\lambda^{-\frac{d_M}{d-d_O}}\right)$$

Critical exponents

$$\lambda \sim (T - T_c)$$

[e.g. 2d Ising $\quad \mathcal{O} = \psi_+\psi_- \quad (d_{\mathcal{O}} = 1)$

$$\Delta S = \int d^2x \, (T - T_c)\, \psi_+\psi_-$$]

Fix λ and minimize $U_{\text{eff}}(M, \lambda)$:

Use the expression $U_{\text{eff}} = \lambda^{\frac{d}{d-d_O}} g\left(M\lambda^{-\frac{d_M}{d-d_O}}\right)$

$$M_* \lambda^{-\frac{d_M}{d-d_O}} = x_* . \quad (x_* = \text{minimum of } g(x).).$$

$$\Rightarrow \quad M_* = x_* \lambda^{\frac{d_M}{d-d_O}} \sim (T - T_c)^{\beta}$$

$$\boxed{\beta = \frac{d_M}{d - d_O}}$$

Turn on the external field H:

$$\tilde{U}_{eff}(M,H) = U_{eff}(M) - MH$$

$$0 = \frac{\partial U_{eff}}{\partial M}(M_*) - H = \underbrace{\lambda^{\frac{d}{d-d_0}} \cdot \lambda^{-\frac{d_M}{d-d_0}}}_{\lambda^{\frac{d-d_M}{d-d_0}}} g'(M\lambda^{-\frac{d_M}{d-d_0}}) - H$$

$$\therefore \quad M_* = \lambda^{\frac{d_M}{d-d_0}} m(\lambda^{-\frac{d-d_M}{d-d_0}} H) \qquad \left(\begin{array}{l} m(y) = \text{inverse of } g'(x) \\ g'(m(y)) = y \end{array} \right)$$

$T > T_c$ (paramagnetic): we expect $M_* \sim \chi_m H$ as $H \sim 0$

i.e. $m(x) \sim x$ as $x \sim 0$

$$\chi_m = \left.\frac{\partial M_*}{\partial H}\right|_{H=0} = \lambda^{-\frac{d-2d_M}{d-d_0}} m'(0) \sim (T-T_c)^{-\gamma}$$

$$\therefore \quad \boxed{\gamma = \frac{d-2d_M}{d-d_0}}$$

$T = T_c$ (critical pt): $\lambda = 0$

$$U_{eff}|_{\lambda=0} = M^{\frac{d}{d_M}} f(\lambda M^{-\frac{d-d_0}{d_M}})\Big|_{\lambda=0} = M^{\frac{d}{d_M}} f(0)$$

$$0 = \frac{\partial U_{eff}|_{\lambda=0}}{\partial M} - H \quad \Rightarrow \quad M_*^{\frac{d-d_M}{d_M}} \propto H$$

$$M_* \propto H^{\frac{d_M}{d-d_M}} \sim H^{1/\delta} \qquad \boxed{\delta = \frac{d-d_M}{d_M}}$$

Specific heat

$$\left.\begin{array}{l} T>T_c \\ H=0 \end{array}\right\} \Rightarrow M=0 \qquad U_{eff}\Big|_{M=0} = \lambda^{\frac{d}{d-d_0}} g(0)$$

$$C \sim \frac{\partial^2 U_{eff}}{\partial \lambda^2} \sim \lambda^{\frac{d}{d-d_0}-2} \sim (T-T_c)^{-\alpha}$$

$$\boxed{\therefore\ \alpha = 2 - \frac{d}{d-d_0}}$$

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2 point function $\langle M(x)\, M(y) \rangle$

$$S_{eff} = \int d^dp \left\{ \Gamma^{(2)}(p)\, \phi(-p)\, \phi(p) + \cdots \right\}$$

RGEqn: $\displaystyle\int d^dp\, \Gamma^{(2)}(p, \bar{\lambda}(t), e^t\Lambda)\, \bar{K}(t)\, \phi(-p)\, \phi(p)$

$$= \int d^dp\, \Gamma^{(2)}(p, \lambda, \Lambda)\, \phi(-p)\, \phi(p)$$

i.e. $\boxed{\Gamma^{(2)}(p, \bar{\lambda}(t), e^t\Lambda)\, \bar{K}(t) \quad \text{is } t\text{-independent.}}$

dimensional analysis (canonical)

$$\underset{\text{mass}^{d_M^c}}{\underset{\uparrow}{\phi(x)}} = \int \underset{\text{mass}^{d}}{\underset{\uparrow}{d^dp}}\; e^{ixp}\; \underset{\therefore\ \text{mass}^{d_M^c - d}}{\underset{\uparrow}{\phi(p)}}$$

$$\int \underbrace{d^d p}_{\text{mass}^d} \; \underbrace{\Gamma^{(2)}(p)}_{?} \; \underbrace{\phi(-p)\,\phi(p)}_{\text{mass}^{2(d_M^c - d)}} \quad \text{is dimensionless}$$

$$\therefore \left[\Gamma^{(2)}(p)\right]_c = \text{mass}^{d-2d_M^c}$$

$$\therefore \Gamma^{(2)}(p, \lambda, \Lambda) = \Lambda^{d-2d_M^c}\, \tilde{\Gamma}^{(2)}\left(\frac{p}{\Lambda}, \lambda\right)$$

RGE: $\bar{K}(t)\, e^{(d-2d_M^c)t}\, \tilde{\Gamma}^{(2)}\left(e^{-t}\frac{p}{\Lambda}, \bar{\lambda}(t)\right)$ is t-indep.

Use $\bar{K}(t) = e^{-2\gamma_M t}$, $\bar{\lambda}(t) = e^{-(d-d_G)t}\lambda$

$$e^{(d-2d_M)t}\, \tilde{\Gamma}^{(2)}\left(e^{-t}\frac{p}{\Lambda}, e^{-(d-d_G)t}\lambda\right) \quad \text{is } t\text{-indep}$$

$$\therefore \tilde{\Gamma}^{(2)}\left(\frac{p}{\Lambda}, \lambda\right) = \left(\frac{p}{\Lambda}\right)^{d-2d_M} h\left(\lambda\left(\frac{p}{\Lambda}\right)^{-(d-d_G)}\right)$$

$$\therefore \Gamma^{(2)}(p, \lambda, \Lambda) = \Lambda^{2\gamma_M} \cdot p^{d-2d_M}\, h\left(\lambda\left(\frac{p}{\Lambda}\right)^{-(d-d_G)}\right)$$

$$\langle M(x)\, M(0)\rangle = \int \frac{d^d p}{(2\pi)^d}\, e^{ipx}\, \frac{1}{\Gamma^{(2)}(p, \lambda, \Lambda)}$$

$$= \Lambda^{-2\gamma_M} \int \frac{d^d p}{(2\pi)^d p^d}\, p^{2d_M}\, \frac{e^{ipx}}{h\left(\lambda\left(\frac{p}{\Lambda}\right)^{-(d-d_G)}\right)}$$

$p = \frac{p'}{x}$

$$= \Lambda^{-2\gamma_M} \int \frac{d^d p'}{(2\pi)^d p'^d} \left(\frac{p'}{x}\right)^{2d_M} \frac{e^{ip'}}{h\left(\lambda\left(\frac{p'}{\Lambda x}\right)^{-(d-d_G)}\right)}$$

$$\therefore \langle M(x) M(0) \rangle \sim \frac{1}{|x|^{2d_M}} \frac{\tilde{h}(\lambda (x\Lambda)^{d-d_O})}{\Lambda^{2\gamma_M}}$$

<u>$T = T_c$ (On critical)</u> : $\lambda = 0$ $\quad \langle M(x) M(0) \rangle \sim \frac{1}{|x|^{2d_M}}$

as expected.

<u>$T > T_c$</u> : We expect a finite correlation length $\xi < \infty$

$$\sim e^{-|x|/\xi}$$

This behaviour is possible only if $\tilde{h}(x) \sim e^{-x^{\frac{1}{d-d_O}}}$

$$\Rightarrow \langle M(x) M(0) \rangle \sim \frac{1}{|x|^{2d_M}} e^{-|x\Lambda| \cdot \lambda^{\frac{1}{d-d_O}}}$$

$$\xi = \frac{1}{\Lambda \cdot \lambda^{\frac{1}{d-d_O}}} \sim \lambda^{-\frac{1}{d-d_O}} \sim (T - T_c)^{-\nu}$$

$$\therefore \boxed{\nu = \frac{1}{d-d_O}}$$

<u>Summary</u> :

$$\boxed{\beta = \frac{d_M}{d-d_O}, \quad \gamma = \frac{d-2d_M}{d-d_O}, \quad \delta = \frac{d-d_M}{d_M}, \quad \alpha = 2 - \frac{d}{d-d_O}, \quad \nu = \frac{1}{d-d_O}}$$

all expressed by CFT data d, d_O, d_M.

2d Ising Model

$$d = 2, \quad d_\sigma = \frac{1}{2} + \frac{1}{2} = 1, \quad d_M = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$$\boxed{\begin{array}{l} \beta = \frac{\frac{1}{8}}{2-1} = \frac{1}{8}, \quad \gamma = \frac{2-\frac{1}{4}}{2-1} = \frac{7}{4}, \quad \delta = \frac{2-\frac{1}{8}}{\frac{1}{8}} = 15 \\ \alpha = 2 - \frac{2}{2-1} = 0, \quad \nu = \frac{1}{2-1} = 1 \end{array}}$$

$$U_{eff}(M, \lambda) = M^{16} f(\lambda M^{-8}) = \lambda^2 g(M \lambda^{-\frac{1}{8}})$$

- finite & nonzero at $\lambda = 0$: need constant term in $f(x)$.
- paramagnetic $M \sim \chi_m H$ at $\lambda \neq 0$: need M^2 term in U_{eff}

  i.e. x^2 term in $g(x)$.

$$\Rightarrow \quad f(x) = 1 + x^{\frac{7}{4}}$$

$$\Downarrow \qquad \Uparrow$$

$$g(x) = x^{16} + x^2$$

Then $U_{eff}(M, \lambda) = \lambda^{\frac{7}{4}} \cdot M^2 + M^{16}$

[Our first guess!]