A quick review (of conformal invariance, energy momentum tensor, Ward identity, $L_n, \tilde{L}_n$, conformal weights (dimension, spin), primary and descendant operators (states) ...)

E-M tensor

In a general QFT, energy-momentum tensor is defined by

$$\delta\left[\mathcal{D}_g X\, e^{-S_E(g,X)}\right] = \mathcal{D}_g X\, e^{-S_E(g,X)} \frac{1}{4\pi}\int_\Sigma \sqrt{g}\, d^2\sigma\, \delta g^{\mu\nu} T_{\mu\nu}$$

Diffeo-invariance $\Rightarrow$ $\nabla^\mu T_{\mu\nu} = 0$ (AFOO)

CFT

a CFT of central charge c is a QFT s.t.

$$T^\mu_\mu = -\frac{c}{12} R$$

OPE of T_{zz}

If we write $T_{zz}^{hol} := T_{zz} + \frac{c}{24}\left(2\partial_z^2 \log g_{z\bar{z}} - (\partial_z \log g_{z\bar{z}})^2\right)$

and rewrite $T_{zz}^{hol} = T_{zz}$, then

- $\partial_{\bar{z}} T_{zz} = 0$ (AFOO)
- $$T_{zz}(z)\, T_{ww}(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T_{ww}(w) + \frac{1}{z-w}\partial_w T_{ww}(w) + \text{reg}$$
- $T_{zz} = (\zeta'(z))^2 T_{\zeta\zeta} + \frac{c}{12}\{\zeta, z\}$ for a finite conf. transf $z \mapsto \zeta(z)$

$$\{\zeta, z\} = \frac{\zeta'''(z)}{\zeta'(z)} - \frac{3}{2}\left(\frac{\zeta''(z)}{\zeta'(z)}\right)^2 \quad \text{Schwarzian derivative.}$$

Conformal transf. of local operators

$$\epsilon = \epsilon^z(z)\frac{\partial}{\partial z} \quad :$$

$$\delta_\epsilon \mathcal{O}(p) = \frac{1}{2\pi i}\oint_p dz\, \epsilon^z(z)\, T_{zz}(z)\, \mathcal{O}(p)$$

For a choice of coordinate system $\{z\}$

$$\boxed{(L_n\mathcal{O})(p) := \frac{1}{2\pi i}\oint_p dz\,(z - z(p))^{n+1}\, T_{zz}(z)\,\mathcal{O}(p).}$$

- They obey $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$.

- $$T_{zz}(z)\,\mathcal{O}(w) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-2}(L_n\mathcal{O})(w)$$

- $$\delta_\epsilon \mathcal{O}(w) = \sum_{n\geq -1}\frac{1}{(n+1)!}\left(\left(\frac{d}{dw}\right)^{n+1}\epsilon(w)\right)(L_n\mathcal{O})(w)$$

$\mathcal{O}(p)$: local operator $\Leftrightarrow$ $\delta_\epsilon\mathcal{O}$ is a <u>finite sum</u> of derivatives of $\epsilon(w)$.

$\Leftrightarrow$ $L_n\mathcal{O} = 0$ at sufficiently large n

i.e. $\exists N_\mathcal{O}\in\mathbb{Z}$ s.t. $L_n\mathcal{O}=0$ $\forall n \geq N_\mathcal{O}$.

$>$

Same for $T_{\bar z\bar z}$, $\rightsquigarrow$ $\overset{\text{anti-hol}}{T_{\bar z\bar z}} \rightleftarrows T_{\bar z\bar z}$ $\rightsquigarrow$ $\tilde{L}_n$.

Conformal weights, dimension, spin

$\mathcal{O}$ has conformal weight $(\Delta, \tilde{\Delta})$ if $L_0 \mathcal{O} = \Delta \mathcal{O}$, $\tilde{L}_0 \mathcal{O} = \tilde{\Delta} \mathcal{O}$,

The dimension of $\mathcal{O}$ is $d_{\mathcal{O}} = \Delta + \tilde{\Delta}$

The spin of $\mathcal{O}$ is $s_{\mathcal{O}} = \Delta - \tilde{\Delta}$

Primary operators

$\mathcal{O}$ is a primary operator if

$$\underline{L_n \mathcal{O} = \tilde{L}_n \mathcal{O} = 0 \quad \forall n \geq 1}$$

(· An Operator of the form $L_{-n_1} \cdots L_{-n_k} \tilde{L}_{-m_1} \cdots \tilde{L}_{-m_\ell} \mathcal{O}$ is a descendant of $\mathcal{O}$.)

· A primary operator of conformal weight $(\Delta, \tilde{\Delta})$ has OPE with T:

$$T_{zz}(z)\, \mathcal{O}(w) = \frac{\Delta}{(z-w)^2} \mathcal{O}(w) + \frac{1}{z-w} \partial_w \mathcal{O}(w) + \text{reg.}$$

(Similar for $T_{\bar{z}\bar{z}}(\bar{z}) \mathcal{O}(w)$)

· For a finite conformal transf $z \mapsto \zeta(z)$

$$\mathcal{O}(z) \to (\zeta'(z))^{\Delta} (\bar{\zeta}'(\bar{z}))^{\tilde{\Delta}} \mathcal{O}(\zeta(z))$$

§ CFTs on a 2-sphere S^2

A CFT (or a more general QFT) can be formulated on a general Riemannian manifold (Σ, g).

[Sometimes, additional structures, such as orientation, spin structure, etc need to be specified as well.]

A CFT depends essentially only on the conformal structure:

- If c = central charge,

$$Z(\Sigma, e^{\phi} g) = e^{c S_L(g,\phi)} Z(\Sigma, g)$$

where $S_L(g,\phi) = \frac{1}{48\pi} \int_{\Sigma} \sqrt{g}\, d^2\sigma \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi + R(g)\phi \right)$

"Liouville action"

- If $\mathcal{O}_1, \cdots, \mathcal{O}_s$ are primaries of dimension $d_{\mathcal{O}_1}, \cdots, d_{\mathcal{O}_s}$,

$$\left\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \right\rangle_{\Sigma, e^{\phi} g} = \prod_{i=1}^{s} e^{-\frac{d_{\mathcal{O}_i}}{2} \phi(p_i)} \left\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \right\rangle_{\Sigma, g}.$$

a $\underline{\text{Conformal class of 2d Riemannian mfd}}$ $=$ $\underline{\text{a Riemann surface}}$

$\overset{\text{if oriented}}{\Longrightarrow}$ a 2d Complex manifold.

There are ∞-ly many Riemann surfaces.

e.g. $\mathbb{R}^2 \cong \mathbb{C}$ (just one)

$$S^2 \cong \mathbb{CP}^1 = \{(z_1, z_2) \neq 0\} \Big/ \underset{(z_1,z_2)\sim(\lambda z_1, \lambda z_2)}{\lambda \neq 0} = (\mathbb{C}^2 \setminus 0) \big/ \mathbb{C}^\times$$

(just one)

$$T^2_\tau \cong \mathbb{C} / \mathbb{Z} \oplus \tau\mathbb{Z} \quad (\infty\text{-ly many})$$

parametrized by τ.

$$T^2_\tau \cong T^2_{\tau'} \quad \text{iff} \quad \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \quad \tau' = \frac{a\tau + b}{c\tau + d}.$$

⋮

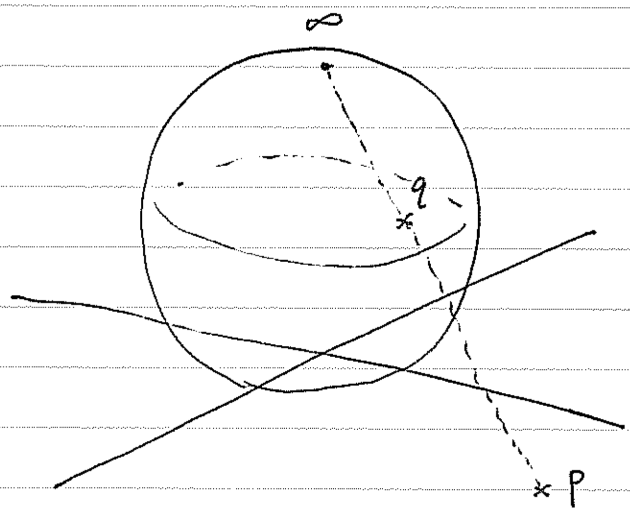
We shall study $\underline{\text{CFTs on } S^2 \cong \mathbb{CP}^1}$ in detail.

… Two motivations.

① relation to CFT on $\mathbb{R}^2 \cong \mathbb{C}$

② string theory (the leading term in string perturbation theory).

relation to CFT on $\mathbb{R}^2$



$$\mathbb{R}^2 \cong S^2 \setminus \{\infty\}$$

by stereographic projection.

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\mathbb{R}^2,\ \text{flat metric}} = \langle \mathcal{O}_1(q_1) \cdots \mathcal{O}_s(q_s) \rangle_{S^2,\ g_{S^2}}$$

for <u>any</u> set of operators $\mathcal{O}_1, \dots, \mathcal{O}_s$.

g_{S^2} : a metric on S^2 which is mapped to the flat metric on $\mathbb{R}^2$ by the stereographic projection, in a domain containing $q_1, \dots, q_s$.

Thus to study correlation functions on $\mathbb{R}^2 \cong \mathbb{C}$, we can study correlation functions on $S^2 \cong \mathbb{CP}^1$.

(advantage : Compactness of $S^2 \cong \mathbb{CP}^1$.)

In what follows $\langle \cdots \rangle$ means $\langle \cdots \rangle_{S^2, g_{S^2}}$ unless otherwise stated.

Also, $T(z) := T_{zz}(z)$, $\overline{T}(\bar{z}) := T_{\bar{z}\bar{z}}(z)$, etc.

Asymptotic behaviour of $T(z)$ as $z \to \infty$

$$\boxed{\begin{array}{l} T(z) \sim \dfrac{1}{z^4} \quad \text{as } |z| \to \infty \\[2ex] \text{i.e. } \langle T(z) \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle \sim \dfrac{\text{const}}{z^4} \quad \text{as } |z| \to \infty. \end{array}}$$

<u>proof</u> One can show that $T(z)$ does not depend on the choice of metric [see the handout].

In particular

$$\langle T(z) \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle = \langle T(z) \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle_{S^2, \tilde{g}_{S^2}}$$

where $\tilde{g}_{S^2}$ is flat in a domain including $z_1, \cdots, z_s$ only.

i.e. It does not need to be flat near z.

Now use $T_{zz} = (w'(z))^2 T_{ww} + \frac{c}{12}\{w, z\}$ for $w(z) = z^{-1}$.

This finds $T(z) = T_{zz}(z) = z^{-4} T_{ww}(w)$.

Note that $\langle T_{ww}(w) \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle_{S^2, \tilde{g}_{S^2}}$ is regular as $w \to 0$

as long as the metric $\tilde{g}_{S^2}$ is non-singular at ∞ $(w=0)$.

Then, $\langle T(z) \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle = \langle T(z) \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{S^2, \tilde{g}_{S^2}}$

$$= z^{-4} \langle T_{ww}(w) \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_{S^2, \tilde{g}_{S^2}} \sim c\, z^{-4}$$

as $|z| \to \infty$ //

Ward identities on S^2 (i.e. on $\mathbb{R}^2$)

Recall $T(z)\mathcal{O}(w) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-2} (L_n\mathcal{O})(w)$ as $z\to w$

$\uparrow$ This sum is bounded above for a local operator $\mathcal{O}$.

i.e. $\exists N_{\mathcal{O}}$ s.t. $L_n\mathcal{O}=0 \quad \forall n > N_{\mathcal{O}}$.

The Ward identity on S^2;

$$\boxed{\begin{aligned}&\langle T(z)\mathcal{O}_1(z_1)\cdots\mathcal{O}_s(z_s)\rangle \\ &= \sum_{i=1}^{s}\sum_{n\geq -1}^{N_{\mathcal{O}_i}} (z-z_i)^{-n-2}\langle \mathcal{O}_1(z_1)\cdots(L_n\mathcal{O}_i)(z_i)\cdots\mathcal{O}_s(z_s)\rangle\end{aligned}}$$

proof The RHS has the correct pole structure as $z\to z_i \quad \forall i$.

Thus the difference $f_{z_1\cdots z_s}(z) = (\mathrm{LHS})-(\mathrm{RHS})$

has no pole ① as a function of z.

Also
$$\left.\begin{aligned}(\mathrm{LHS}) &\sim \frac{1}{z^4} \text{ as } |z|\to\infty \\ (\mathrm{RHS}) &\sim \frac{1}{z} \text{ (or smaller) as } |z|\to\infty\end{aligned}\right\}\ \underbrace{f_{z_1\cdots z_s}(z)\sim\frac{1}{z} \text{ (or smaller) as } |z|\to\infty}_{②}$$

① & ② means $f_{z_1\cdots z_s}(z)\equiv 0$. //

※ Let $f(z)$ be a holomorphic function of $z \in \mathbb{C}$

i.e. no pole, s.t. $f(z) \to$ a regular function of $w = z^{-1}$.

This means that $f(z)$ extends to a holomorphic function of $\mathbb{CP}^1$.

But, on a compact complex manifold, there is no holomorphic function other than constant functions.

$\therefore$ $f(z)$ must be a constant.

If, in addition, we know $f(z) \overset{z \to \infty}{\sim} \frac{1}{z}$ or smaller, $f(z)$ must be zero everywhere, $f(z) \equiv 0$.

On a compact space, a continuous function must have a minimum and a maximum, but a non-constant holomophic function can have neither.

If ϕ_i is a primary operator with Conformal Weight $(\Delta_i, \tilde{\Delta}_i)$

$$T(z)\,\phi_i(w) = \sum_{n=-\infty}^{0} (z-w)^{-n-2} (L_n \phi_i)(w) \quad \text{as} \quad z \to w$$

with $L_{-1}\phi_i = \partial_z \phi_i$, $L_0 \phi_i = \Delta_i \phi_i$.

Thus

$$\boxed{\begin{aligned}&\langle T(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle \\ &= \sum_{i=1}^{s} \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_1(z_1) \cdots \phi_s(z_s) \rangle\end{aligned}}$$

Also, again for primaries $\phi_1(z_1), \cdots, \phi_s(z_s)$,

$$\boxed{\begin{aligned}&\langle T(z)\, T(z_1') \cdots T(z_t')\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle \\ &= \sum_{i=1}^{s} \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \langle T(z_1') \cdots T(z_t')\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle \\ &+ \sum_{j=1}^{t} \left(\frac{2}{(z-z_j')^2} + \frac{1}{z-z_j'} \frac{\partial}{\partial z_j'} \right) \langle T(z_1') \cdots T(z_t')\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle \\ &+ \sum_{j=1}^{t} \frac{c/2}{(z-z_j')^4} \langle T(z_1') \cdots \hat{j} \cdots T(z_t')\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle\end{aligned}}$$

$T(z_j')$ omit.

For primaries $\phi_1, \cdots, \phi_s$

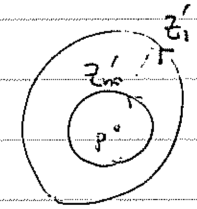
$$\therefore \langle T(z_1') \cdots T(z_r') \phi_1(z_1) \cdots \phi_s(z_s) \rangle$$

Can be expressed in terms of $\langle \phi_1(z_1) \cdots \phi_s(z_s) \rangle$.

Therefore, correlation functions of descendant operators

$$\phi^{\{k\},\{\tilde{k}\}}(p) = \left(L_{-k_1} \cdots L_{-k_m} \tilde{L}_{-\tilde{k}_1} \cdots \tilde{L}_{-\tilde{k}_n} \phi \right)(p)$$

$$= \oint_p \prod_{i=1}^{m} \frac{dz_i'}{2\pi i} (z_i' - z(p))^{-k_i+1} \prod_{j=1}^{n} \frac{d\bar{z}_j''}{-2\pi i} (\bar{z}_j'' - \bar{z}(p))^{-\tilde{k}_j+1}$$



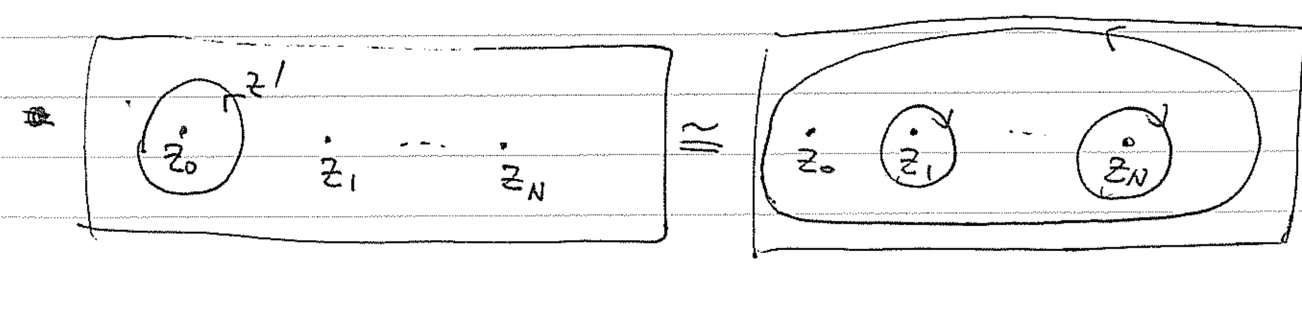
$$T(z_1') \cdots T(z_m') \bar{T}(\bar{z}_1'') \cdots \bar{T}(\bar{z}_n'') \phi(p)$$

can be expressed in terms of correlation functions of primaries.

<u>For example</u>

$$\langle \phi_0^{(-n)}(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle = \langle (L_{-n}\phi)(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

$$= \oint_z \frac{dz'}{2\pi i} (z' - z_0)^{-n+1} \underbrace{\langle T(z') \phi_0(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle}_{\sum_{i=0}^{N} \left(\frac{\Delta_i}{(z'-z_i)^2} + \frac{1}{z'-z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_0(z_0) \cdots \phi_N(z_N) \rangle}$$



$$= -\sum_{i=1}^{N} \oint_{z_i} \frac{dz'}{2\pi i} (z'-z_0)^{-n+1} \left(\frac{\Delta_i}{(z'-z_i)^2} + \frac{1}{z'-z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_0(z_0) \cdots \phi_N(z_N) \rangle$$

$\left(\begin{array}{l}\text{No contribution from large contour} \\ \text{since } \quad n \geqslant 1 \quad \text{and} \quad T(z') \sim \frac{1}{z'^4} \text{ as } z' \to \infty\end{array}\right)$

$$= \underbrace{\sum_{i=1}^{N} \left(\frac{(n-1)\Delta_i}{(z_i - z_0)^n} - \frac{1}{(z_i - z_0)^{n-1}} \frac{\partial}{\partial z_i} \right)}_{=: \hat{\mathcal{L}}_{-n}(z_0; z_1 \cdots z_N) \overset{\text{or simply}}{=\!=} \hat{\mathcal{L}}_{-n}} \langle \phi_0(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

$$\langle (L_{-n_1} \cdots L_{-n_r} \phi)(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

$$= \hat{\mathcal{L}}_{-n_1} \langle (L_{-n_2} \cdots L_{-n_r} \phi)(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

$$= \hat{\mathcal{L}}_{-n_1} \hat{\mathcal{L}}_{-n_2} \langle (L_{-n_3} \cdots L_{-n_r} \phi)(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

$$\vdots$$

$$= \hat{\mathcal{L}}_{-n_1} \hat{\mathcal{L}}_{-n_2} \cdots \hat{\mathcal{L}}_{-n_r} \langle \phi(z_0) \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$