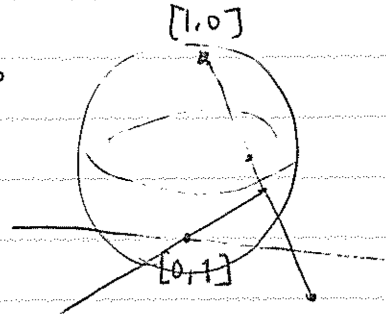
Projective Ward Identities and Solutions

$$\mathbb{CP}^1 = (\mathbb{C}^2 \setminus 0)/\mathbb{C}^\times = \{(z_1, z_2) \neq (0,0)\}\Big/_{(z_1,z_2)\sim(\lambda z_1, \lambda z_2)}^{\lambda\neq 0} \ni [z_1, z_2]$$

Relation to $\mathbb{C}_z$



$$z = z_1/z_2$$

$$\left(w = z^{-1} = z_2/z_1 \right)$$

Global conformal transformations of $S^2 = \mathbb{CP}^1$: $SL(2,\mathbb{C})$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad ad-bc=1$$

In z-coordinate : $\quad z \longmapsto \dfrac{az_1+bz_2}{cz_1+dz_2} = \dfrac{az+b}{cz+d}$.

[To be precise, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{C})$ acts trivially.
Thus the conformal group is $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm 1\}$]

3 generators $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

$$g_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \rightsquigarrow \delta z = \frac{d}{dt} g_t(z)\Big|_0 = 1 \qquad \text{translation}$$

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \rightsquigarrow \delta z = z \qquad \text{dilatation/rotation}$$

$$g_t = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \rightsquigarrow \delta z = z^2 \qquad \text{special conformal transformations.}$$

These are called the <u>projective conformal transformations.</u>

<u>Note</u> $\quad T(z) \sim \text{reg} \quad \text{as} \quad z \to 0 \,,$

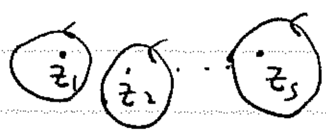
$T(z) \sim \dfrac{1}{z^4} \quad \text{as} \quad z \to \infty \,,$ Thus

$$\boxed{\oint_0 dz\, \epsilon(z)\, T(z) = \oint_\infty dz\, \epsilon(z)\, T(z) = 0}$$

for $\epsilon(z) = 1, z, z^2$ (i.e. for proj. conf.)

$$\sum_{i=1}^{s} \langle \mathcal{O}_1(z_1) \cdots \delta_\epsilon \mathcal{O}_i(z_i) \cdots \mathcal{O}_s(z_s) \rangle$$

$$= \int \frac{dz}{2\pi i} \epsilon(z) \langle T(z) \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle$$



$$= \oint_\infty \frac{dz}{2\pi i} \epsilon(z) \langle T(z) \mathcal{O}_1 \cdots \mathcal{O}_s \rangle - \oint_0 \frac{dz}{2\pi i} \epsilon(z) \langle T(z) \mathcal{O}_1 \cdots \mathcal{O}_s \rangle$$

$$= 0 - 0$$

$\therefore$

$$\boxed{\sum_{i=1}^{s} \langle \mathcal{O}_1(z_1) \cdots \delta_\epsilon \mathcal{O}_i(z_i) \cdots \mathcal{O}_s(z_s) \rangle = 0}$$

for projective conformal transformation $\epsilon(z) = 1, z, z^2$

An operator $\mathcal{O}$ is <u>quasi-primary</u> if it transforms in the same way as primaries, under projective conformal transformations $\epsilon(z) = 1, z, z^2$:

$$\delta_\epsilon \mathcal{O}(z) = \epsilon(z)\partial_z \mathcal{O}(z) + \Delta \epsilon'(z)\mathcal{O}(z)$$

$$\delta_{\bar\epsilon} \mathcal{O}(z) = \bar\epsilon(\bar z)\partial_{\bar z} \mathcal{O}(z) + \tilde\Delta \bar\epsilon'(\bar z)\mathcal{O}(z).$$

- Since $\delta_\epsilon \mathcal{O}(z) = \sum_{n\geq -1} \frac{1}{(n+1)!}\left(\frac{d^{n+1}}{dz^{n+1}}\epsilon(z)\right)(L_n\mathcal{O})(z)$

  for a general $\epsilon(z)$, $\mathcal{O}$ is quasi-primary if and only if $\boxed{L_1\mathcal{O} = \tilde L_1\mathcal{O} = 0}$

- Any descendant of a primary (i.e. any local operator) is a linear compbination of quasi-primaries and the derivatives of quasi-primaries.

[☹ induction of <u>level</u>

($L_{-n_1}\cdots L_{-n_r}\phi$ has level $n_1+\cdots+n_r$)]

If $\mathcal{O}_1, \cdots, \mathcal{O}_s$ are quasi-primary,

$G = G(z_1, \cdots, z_s) = \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle$ obeys the

projective Ward identities

$$\sum_{i=1}^{s} \frac{\partial}{\partial z_i} G = 0 \qquad (P_{-1})$$

$$\sum_{i=1}^{s} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i \right) G = 0 \qquad (P_0)$$

$$\sum_{i=1}^{s} \left(z_i^2 \frac{\partial}{\partial z_i} + 2\Delta_i z_i \right) G = 0 \qquad (P_1)$$

also

$$\sum_{i=1}^{s} \frac{\partial}{\partial \bar{z}_i} G = 0 \qquad (\tilde{P}_{-1})$$

$$\sum_{i=1}^{s} \left(\bar{z}_i \frac{\partial}{\partial \bar{z}_i} + \tilde{\Delta}_i \right) G = 0 \qquad (\tilde{P}_0)$$

$$\sum_{i=1}^{s} \left(\bar{z}_i^2 \frac{\partial}{\partial \bar{z}_i} + 2\tilde{\Delta}_i \bar{z}_i \right) G = 0 \qquad (\tilde{P}_1)$$

Simple cases

$\underline{s=1}$ $\quad G(z) = \langle \mathcal{O}(z) \rangle$

$P_{-1}, \tilde{P}_{-1}$: $\quad \frac{\partial}{\partial z} G = \frac{\partial}{\partial \bar{z}} G = 0 \quad \therefore G$ is a constant.

$P_0, \tilde{P}_0$: $\quad \Delta G = \tilde{\Delta} G = 0 \quad \therefore G \neq 0$ iff $\Delta = \tilde{\Delta} = 0$

$$\boxed{\text{In a unitary CFT, } \langle \mathcal{O} \rangle = 0 \text{ unless } \mathcal{O} = id}$$

$\underline{s=2}$ $\quad G = \langle \mathcal{O}_1(z_1)\, \mathcal{O}_2(z_2) \rangle$

$P_{-1}, \tilde{P}_{-1}$: $\quad G = f(z, \bar{z}) \Big|_{z = z_1 - z_2,\ \bar{z} = \bar{z}_1 - \bar{z}_2}$

$P_0, \tilde{P}_0$: $\quad \left(z_1 \frac{\partial}{\partial z} - z_2 \frac{\partial}{\partial z} + \Delta_1 + \Delta_2 \right) f \Big|_{\substack{z = z_1 - z_2 \\ \bar{z} = \bar{z}_1 - \bar{z}_2}} = \left(\bar{z}_1 \frac{\partial}{\partial \bar{z}} - \bar{z}_2 \frac{\partial}{\partial \bar{z}} + \tilde{\Delta}_1 + \tilde{\Delta}_2 \right) f \Big|_{\substack{z = z_1 - z_2 \\ \bar{z} = \bar{z}_1 - \bar{z}_2}} = 0$

f is homogeneous $\sim z^{-(\Delta_1 + \Delta_2)} \bar{z}^{-(\tilde{\Delta}_1 + \tilde{\Delta}_2)}$

$P_1, \tilde{P}_1$: $\quad \left(z_1^2 \frac{\partial}{\partial z} - z_2^2 \frac{\partial}{\partial z} + 2\Delta_1 z_1 + 2\Delta_2 z_2 \right) f \Big|_{\cdots} = \left(\bar{z}_1^2 \frac{\partial}{\partial \bar{z}} - \bar{z}_2^2 \frac{\partial}{\partial \bar{z}} + 2\tilde{\Delta}_1 \bar{z}_1 + 2\tilde{\Delta}_2 \bar{z}_2 \right) f \Big|_{=} = 0$

$(z_1 + z_2)(z_1 - z_2) \frac{\partial f}{\partial z} = (z_1 + z_2)(-\Delta_1 - \Delta_2) f$

$\therefore (\Delta_1 - \Delta_2)(z_1 - z_2) f(z_1 - z_2, \bar{z}_1 - \bar{z}_2) = (\tilde{\Delta}_1 - \tilde{\Delta}_2)(\bar{z}_1 - \bar{z}_2) f(z_1 - z_2, \bar{z}_1 - \bar{z}_2)$

$= 0$

$f = 0$ unless $\quad \underset{(= \Delta)}{\Delta_1 = \Delta_2}, \quad \underset{(= \tilde{\Delta})}{\tilde{\Delta}_1 = \tilde{\Delta}_2}$

$$\therefore \boxed{\langle \mathcal{O}_1(z_1)\, \mathcal{O}_2(z_2) \rangle = \frac{c\, \delta_{\Delta_1, \Delta_2}\, \delta_{\tilde{\Delta}_1, \tilde{\Delta}_2}}{(z_1 - z_2)^{2\Delta} (\bar{z}_1 - \bar{z}_2)^{2\tilde{\Delta}}}}$$

General case ($s \geq 3$)

$$G = \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \rangle$$

If we omit the terms $\sum_i \Delta_i G$, $\sum_i \tilde{\Delta}_i G$ in $P_0, \tilde{P}_0$ and $\sum_i 2\Delta_i z_i G$, $\sum_i 2\tilde{\Delta}_i \bar{z}_i G$ in $P_1, \tilde{P}_1$, the equations are

$$\left.\begin{aligned} &\sum_{i=1}^{s} z_i^n \frac{\partial}{\partial z_i} G = 0 \qquad (P_n^{\text{omit}}) \\ &\sum_{i=1}^{s} \bar{z}_i^n \frac{\partial}{\partial \bar{z}_i} G = 0 \qquad (\tilde{P}_n^{\text{omit}}) \end{aligned}\right\} n = 0, 1, \overset{-1}{\cancel{2}}.$$

That would mean $G(z_1, \cdots, z_s)$ is invariant under the uniform $SL(2,\mathbb{C})$ action

$$z_i \longmapsto g(z_i) = \frac{a z_i + b}{c z_i + d} \qquad i = 1, 2, \cdots, s.$$

What are $SL(2,\mathbb{C})$ invariant functions?

$$\underbrace{\frac{(z_i - z_j)(z_k - z_\ell)}{(z_i - z_\ell)(z_k - z_j)}}_{\text{For distinct labels } i,\, j,\, k,\, \ell}$$ are invariant. (Called cross-ratios)

Check $\longrightarrow$

$$\frac{\left(\frac{a z_i + b}{c z_i + d} - \frac{a z_j + b}{c z_j + d}\right)\left(\frac{a z_k + b}{c z_k + d} - \frac{a z_\ell + b}{c z_\ell + d}\right)}{\left(\quad i \qquad \ell \quad\right)\left(\quad k \qquad j \quad\right)} = \frac{(z_i - z_j)(z_k - z_\ell)}{(z_i - z_\ell)(z_k - z_j)}$$

OK

$$\frac{(c z_j + d)(a z_i + b) - (c z_i + d)(a z_j + b)}{(c z_i + d)(c z_j + d)} = \frac{z_i - z_j}{(c z_i + d)(c z_j + d)}$$

Questions : Are they all ? Which of them are independent ?

Answer : Yes. Some $(S-3)$ of them.

- Any 3 distinct points in $\mathbb{CP}^1$ can be sent to $0, 1, \infty$ by an $SL(2,\mathbb{C})$ transformation.

proof Suppose z_1, z_2, z_3 are distinct.

Consider $z \mapsto g(z) = \dfrac{(z_1 - z)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z)}$

This does $g(z_1) = 0$, $g(z_2) = 1$, $g(z_3) = \infty$.

Is this an $SL(2,\mathbb{C})$ transformation ?

$$g(z) = \frac{(z_2 - z_3)z + z_1(z_3 - z_2)}{(z_2 - z_1)z + z_3(z_1 - z_2)} = \frac{az + b}{cz + d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} z_2 - z_3 & z_1(z_3 - z_2) \\ z_2 - z_1 & z_3(z_1 - z_2) \end{pmatrix}$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha^2 \{ (z_2 - z_3) z_3 (z_1 - z_2) - z_1 (z_3 - z_2)(z_2 - z_1) \}$$

$$= \alpha^2 \underbrace{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}_{\neq 0}$$

since z_1, z_2, z_3 are distinct.

O.K. //

and uniquely

Suppose g_1 & g_2 do the same $\begin{pmatrix} z_1 \to 0 \\ z_2 \to 1 \\ z_3 \to \infty \end{pmatrix}$

Then $g = g_1 g_2^{-1}$ $\begin{pmatrix} 0 \to 0 \\ 1 \to 1 \\ \infty \to \infty \end{pmatrix}$

$$g(z) = \frac{az+b}{cz+d} \qquad \left.\begin{array}{l} 0 \to 0 : \quad b = 0 \\ 1 \to 1 : \quad a+b = c+d \\ \infty \to \infty : \quad c = 0 \end{array}\right\} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

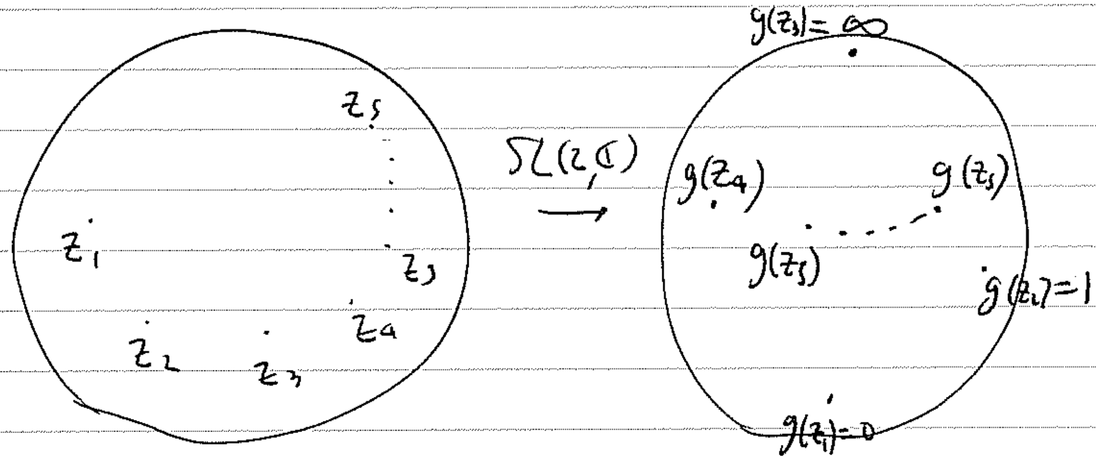
$$\therefore g_1 = \pm g_2$$

//.

The space $\{(z_1, \cdots, z_s) \mid \text{distinct}\} / SL(2,\mathbb{C}) = \mathcal{M}_{0,s}$ ← genus = 0

is parametrized by the position of $z_4, z_5, \cdots z_s$

<u>after</u> z_1, z_2, z_3 is sent to $0, 1, \infty$ by $SL(2,\mathbb{C})$.



i.e. by $\quad x_i = g(z_i) = \dfrac{(z_1 - z_i)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_i)} \qquad i = 4, 5, \cdots, s.$

$\therefore \ \mathcal{M}_{0,s} = \left\{ (x_4, \cdots, x_s) \,\middle|\, \begin{matrix}\text{distinct} \\ \neq 0, 1, \infty\end{matrix} \right\} \qquad \left(\dim_{\mathbb{C}} \mathcal{M}_{0,s} = s - 3 \right)$

[This leads to <u>the Answer</u>. Cross-ratios are <u>all</u>.

Independent ones are, say, $\dfrac{(z_1 - z_i)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_i)} = x_i \qquad i = 4, 5, \cdots, s.$]

$\therefore$ General soln of $P_n^{\text{omit}}, \tilde{P}_n^{\text{omit}}$ $\underset{n = 0, 1, 2}{}$ is $\ G = f(x_4, x_5, \cdots, x_s)$.

A general solution of $P_n, \tilde{P}_n$ $n=0,1,-1$

G a solution $\Rightarrow$ $G \cdot \underbrace{f(x_4, \dots x_s)}_{\text{any function of the cross-ratios.}}$ is also a solution

$P_{-1}, \tilde{P}_{-1}$: G is a function of $z_i - z_j$, $\bar{z}_i - \bar{z}_j$.

$P_0, \tilde{P}_0$: G is homogeneous of degree $-\sum_i \Delta_i$, $-\sum_i \tilde{\Delta}_i$:

$$\boxed{G = \prod_{i<j} (z_i - z_j)^{\gamma_{ij}} (\bar{z}_i - \bar{z}_j)^{\tilde{\gamma}_{ij}} f(x_4, \dots, x_s)}$$

$P_0, \tilde{P}_0$: $\sum_{i<j} \gamma_{ij} = -\sum_i \Delta_i$, $\sum_{i<j} \tilde{\gamma}_{ij} = -\sum_i \tilde{\Delta}_i$

$P_1, \tilde{P}_1$:

$$\boxed{\begin{aligned} &\forall i \quad \sum_{j<i} \gamma_{ji} + \sum_{i<k} \gamma_{ik} = -2\Delta_i \\ &\forall i \quad \sum_{j<i} \tilde{\gamma}_{ji} + \sum_{i<k} \tilde{\gamma}_{ik} = -2\tilde{\Delta}_i \end{aligned}}$$

This is the general solution.

$\underline{S=2}$ $\left.\begin{aligned} \gamma_{12} + 2\Delta_1 = 0 \\ \gamma_{12} + 2\Delta_2 = 0 \end{aligned}\right\} \Rightarrow$ $\Delta_1 = \Delta_2 (= \Delta)$ required and $\gamma_{12} = -2\Delta$

similarly for $\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\gamma}_{12}$.

$$\therefore \langle O_1(z_1) O_2(z_2) \rangle \sim \frac{1}{(z_1 - z_2)^{2\Delta} (\bar{z}_1 - \bar{z}_2)^{2\tilde{\Delta}}}$$ reproduced.

$\underline{S=3}$

$$\gamma_{12}+\gamma_{13} = -2\Delta_1 \qquad\qquad \gamma_{12} = -\Delta_1-\Delta_2+\Delta_3$$
$$\gamma_{12}+\gamma_{23} = -2\Delta_2 \quad \Rightarrow \quad \gamma_{23} = \Delta_1-\Delta_2-\Delta_3$$
$$\gamma_{13}+\gamma_{23} = -2\Delta_3 \qquad\qquad \gamma_{13} = -\Delta_1+\Delta_2-\Delta_3$$

$$\therefore \quad \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\rangle$$

$$= \frac{\text{const}}{(z_1-z_2)^{\Delta_1+\Delta_2-\Delta_3}(z_2-z_3)^{\Delta_2+\Delta_3-\Delta_1}(z_3-z_1)^{\Delta_3+\Delta_1-\Delta_2}(\bar z_1-\bar z_2)^{\tilde\Delta_1+\tilde\Delta_2-\tilde\Delta_3}(\bar z_2-\bar z_3)^{\tilde\Delta_2+\tilde\Delta_3-\tilde\Delta_1} \times (\bar z_3-\bar z_1)^{\tilde\Delta_3+\tilde\Delta_1-\Delta_2}}$$

$\underline{S=4}$

$$\left.\begin{aligned}
\gamma_{12}+\gamma_{13}+\gamma_{14} &= -2\Delta_1\\
\gamma_{12}+\gamma_{23}+\gamma_{24} &= -2\Delta_2\\
\gamma_{13}+\gamma_{23}+\gamma_{34} &= -2\Delta_3\\
\gamma_{14}+\gamma_{24}+\gamma_{34} &= -2\Delta_4
\end{aligned}\right\}$$

4 equations for 6 variables

$\downarrow$

2 kernels

$\uparrow$

satisfy $P_n^{\text{omit}}, \tilde P_n^{\text{omit}}$ $n=0,\pm1$.

$\therefore$ functions of cross-ratios.

One can take <u>any</u> solution.

e.g. If $\Delta_1=\Delta_2=\Delta^{(1)}$, $\Delta_3=\Delta_4=\Delta^{(2)}$

One solution is $\gamma_{12}=\Delta^{(1)}\times(-2)$. $\gamma_{34}=-2\Delta^{(2)}$, others $=0$

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle = \frac{1}{(z_1-z_2)^{2\Delta^{(1)}}(z_3-z_4)^{2\Delta^{(2)}}}f(x_4)$$

(if we ignore $\bar z_i$ dependence)