Notes on Coordinate/metric dependence of operators

To define a primary operator, we need to choose a holomorphic coordinate, say z, in a neighbourhood of the insertion point p.

Write $\mathcal{O}_{\{z\}}(p)$ if we want to emphasize the coordinate used.

Under change of coordinate, $z \mapsto w(z)$, a primary of $(\Delta, \tilde{\Delta})$ transforms as

$$\mathcal{O}_{\{z\}}(p) = \left(\frac{dw}{dz}(p)\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}(p)\right)^{\tilde{\Delta}} \mathcal{O}_{\{w\}}(p)$$

(In this sense one may say $\mathcal{O} = \mathcal{O}_{\{z\}}(dz)^{\Delta}(d\bar{z})^{\tilde{\Delta}}$ is coordinate independent.)

The descendants $\left(L_{-k_1} \cdots L_{-k_s} \tilde{L}_{-l_1} \cdots \tilde{L}_{-l_t} \mathcal{O}\right)_{\{z\}}$ transforms in a more complicated way, because the definition of L_{-k}

$$\left(L_{-k}\mathcal{O}\right)_{\{z\}}(p) = \oint_p \frac{dz}{2\pi i} \left(z - z(p)\right)^{-k+1} T_{zz}(z)\, \mathcal{O}_{\{z\}}(p)$$

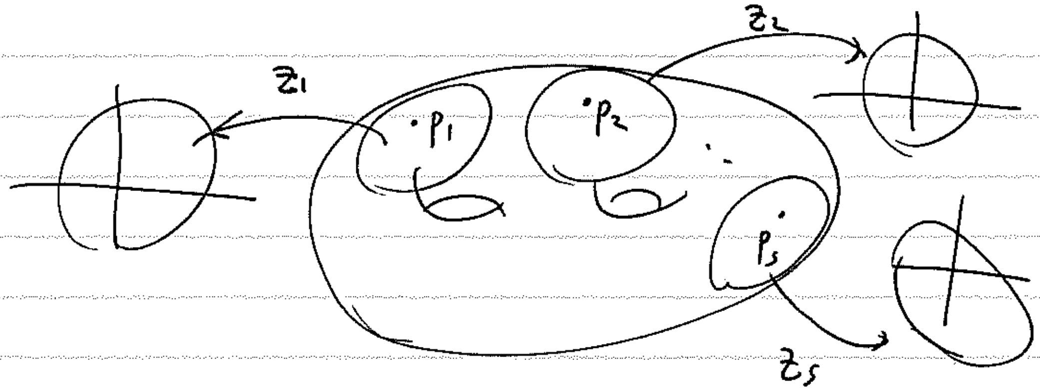
involves the choice of coordinate.

On the other hand, these operators are invariant under Weyl transformation $g \to e^{\phi} g$

$$\left\langle \mathcal{O}_{1\{z_1\}}(p_1) \cdots \mathcal{O}_{s\{z_s\}}(p_s) \right\rangle_{\Sigma, e^{\phi} g} = \left\langle \mathcal{O}_{1\{z_1\}}(p_1) \cdots \mathcal{O}_{s\{z_s\}}(p_s) \right\rangle_{\Sigma, g}.$$

We may drop from the notation.

where z_i is a holomorphic coordinate in a nhd of p_i



Relation to the previous remark that

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\Sigma, e^{\phi} g} = \prod_{i=1}^{s} e^{-\frac{d_i}{2}\phi(p_i)} \langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_s(p_s) \rangle_{\Sigma, g}$$

for $\mathcal{O}_i$ primary of dimension $\Delta_i + \tilde{\Delta}_i = d_i$.

In this remark, we have chosen a particular holo. coordinate z_i s.t. the metric $= |dz_i|^2$ at p_i. That is,

$$\mathcal{O}_i(p_i) = \mathcal{O}_{i\{z_i\}}(p_i) \quad \text{for} \quad g(p_i) = |dz_i|^2(p_i) \quad \text{on RHS}$$

and $\mathcal{O}_i(p_i) = \mathcal{O}_{i\{w_i\}}(p_i)$ for $e^{\phi(p_i)} g(p_i) = |dw_i|^2(p_i)$ on LHS.

The z_i and w_i may be related by $w_i = e^{\frac{\phi(p_i)}{2}} z_i$.

Then $$\mathcal{O}_{i\{z_i\}}(p_i) = \left(\frac{dw_i}{dz_i}\right)^{\Delta_i} \left(\frac{d\bar{w}_i}{d\bar{z}_i}\right)^{\tilde{\Delta}_i} \mathcal{O}_{i\{w_i\}}(p_i) = e^{\frac{\phi(p_i)}{2}(\Delta_i + \tilde{\Delta}_i)} \mathcal{O}_{i\{w_i\}}(p_i)$$

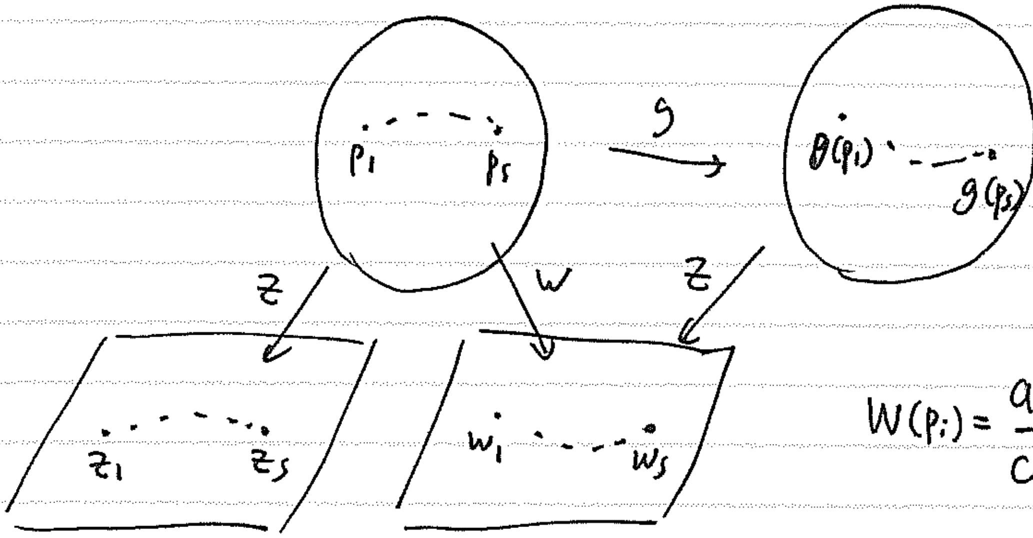
i.e. $\mathcal{O}_{i\{w_i\}}(p_i) = e^{-\frac{\phi(p_i)}{2} d_i} \mathcal{O}_{i\{z_i\}}(p_i)$. The remark follow from this.

Global form of projective Ward identity

$\mathcal{O}_1, \dots, \mathcal{O}_s$ quasi-primaries, $\quad z \mapsto w = \dfrac{az+b}{cz+d}$

$$\left\langle \mathcal{O}_{1\{z\}}(p_1) \cdots \mathcal{O}_{s\{z\}}(p_s) \right\rangle_{\mathbb{CP}^1}$$

$$= \prod_{i=1}^{s} \left(\frac{dw}{dz}(p_i)\right)^{\Delta_i} \left(\frac{d\bar{w}}{d\bar{z}}(p_i)\right)^{\tilde{\Delta}_i} \left\langle \mathcal{O}_{1\{w\}}(p_1) \cdots \mathcal{O}_{s\{w\}}(p_s) \right\rangle_{\mathbb{CP}^1}$$



$$w(p_i) = \frac{a\,z(p_i)+b}{c\,z(p_i)+d} = z(g(p_i))$$

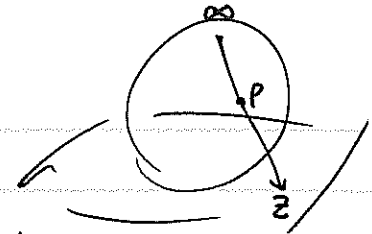
$p_1, \dots, p_s$ in coordinate w $\iff$ $g(p_1), \dots, g(p_s)$ in coordinate z

$$\therefore \left\langle \mathcal{O}_{1\{w\}}(p_1) \cdots \mathcal{O}_{s\{w\}}(p_s) \right\rangle_{\mathbb{CP}^1} = \left\langle \mathcal{O}_{1\{z\}}(g(p_1)) \cdots \mathcal{O}_{s\{z\}}(g(p_s)) \right\rangle_{\mathbb{CP}^1}$$

$$\boxed{\left\langle \mathcal{O}_{1\{z\}}(p_1) \cdots \mathcal{O}_{s\{z\}}(p_s) \right\rangle_{\mathbb{CP}^1} = \prod_{i=1}^{s} \left(\frac{dw}{dz}(p_i)\right)^{\Delta_i} \left(\frac{d\bar{w}}{d\bar{z}}(p_i)\right)^{\tilde{\Delta}_i} \left\langle \mathcal{O}_{1\{z\}}(g(p_1)) \cdots \mathcal{O}_{s\{z\}}(g(p_s)) \right\rangle_{\mathbb{CP}^1}}$$

$\xrightarrow{\text{infinitesimal form}}$ differential eqns $\quad P_0, P_{\pm 1}, \tilde{P}_0, \tilde{P}_{\pm 1}$.

Operator at $\infty \in \mathbb{CP}^1$



We will consider correlation functions on $\mathbb{CP}^1$ s.t. an operator is inserted at $\underset{z}{\underset{\shortparallel}{\infty}} \in \mathbb{CP}^1$. Of course, we should use a regular coordinate there, e.g. $w = z^{-1}$.

Thus when we write

$$\left\langle \mathcal{O}_1(\infty)\, \mathcal{O}_2(z_2) \cdots \mathcal{O}_s(z_s) \right\rangle_{\mathbb{CP}^1}$$

we mean $\left\langle \mathcal{O}_{1\{z^{-1}\}}(\infty)\, \mathcal{O}_{2\{z\}}(z_2) \cdots \mathcal{O}_{s\{z\}}(z_s) \right\rangle_{\mathbb{CP}^1}$

Note that if $\mathcal{O}_i$ has $(\Delta, \tilde{\Delta})$, $\mathcal{O}_{\{z\}} = \left(\frac{dz^{-1}}{dz}\right)^{\Delta} \left(\frac{d\bar{z}^{-1}}{d\bar{z}}\right)^{\tilde{\Delta}} \mathcal{O}_{\{z^{-1}\}}$

$$= (-z^{-2})^{\Delta} (-\bar{z}^{-2})^{\tilde{\Delta}}\, \mathcal{O}_{\{z^{-1}\}}$$

i.e. $\mathcal{O}_{\{z^{-1}\}} = z^{2\Delta} \bar{z}^{2\tilde{\Delta}}\, \mathcal{O}_{\{z\}}$ (up to an overall phase).

Thus
$$\boxed{\begin{aligned} &\left\langle \mathcal{O}_1(\infty)\, \mathcal{O}_2(z_2) \cdots \mathcal{O}_s(z_s) \right\rangle_{\mathbb{CP}^1} \\ &= \lim_{z_1 \to \infty} z_1^{2\Delta_1} \bar{z}_1^{2\tilde{\Delta}_1} \left\langle \mathcal{O}_1(z_1)\, \mathcal{O}_2(z_2) \cdots \mathcal{O}_s(z_s) \right\rangle_{\mathbb{CP}^1} \end{aligned}}$$

where $\left\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_s(z_s) \right\rangle_{\mathbb{CP}^1} := \left\langle \mathcal{O}_{1\{z\}}(z_1) \cdots \mathcal{O}_{s\{z\}}(z_s) \right\rangle_{\mathbb{CP}^1}$ on RHS.

quasi-primaries

<u>Note</u>

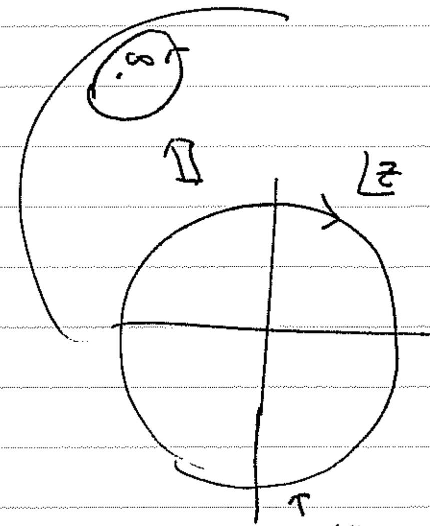
$$\langle \mathcal{O}_i(z_1)\, \mathcal{O}_j(z_2) \rangle = \frac{C_{ij}}{(z_1 - z_2)^{2\Delta_i} (\bar{z}_1 - \bar{z}_2)^{2\tilde{\Delta}_i}} \qquad \neq 0 \text{ only if } \begin{array}{l} \Delta_i = \Delta_j \\ \tilde{\Delta}_i = \tilde{\Delta}_j \end{array}$$

$$\boxed{\therefore \langle \mathcal{O}_i(\infty)\, \mathcal{O}_j(0) \rangle = \lim_{z_1 \to \infty} z_1^{2\Delta_i} \bar{z}_1^{2\tilde{\Delta}_i} \frac{C_{ij}}{(z_1 - 0)^{2\Delta_i} (\bar{z}_1 - 0)^{2\tilde{\Delta}_i}} = C_{ij}}$$

$$\langle \mathcal{O}_i(z_1)\, \mathcal{O}_j(z_2)\, \mathcal{O}_k(z_3) \rangle = \frac{C_{ijk}}{\begin{array}{c}(z_1 - z_2)^{\Delta_i + \Delta_j - \Delta_k} (z_2 - z_3)^{\Delta_j + \Delta_k - \Delta_i} (z_1 - z_3)^{\Delta_k + \Delta_i - \Delta_j} \\ \times (\bar{z}_1 - \bar{z}_2)^{\tilde{\Delta}_i + \tilde{\Delta}_j - \tilde{\Delta}_k} (\bar{z}_2 - \bar{z}_3)^{\tilde{\Delta}_j + \tilde{\Delta}_k - \Delta_i} (\bar{z}_1 - \bar{z}_3)^{\tilde{\Delta}_k + \tilde{\Delta}_i - \tilde{\Delta}_j}\end{array}}$$

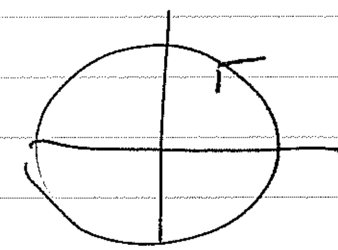
$$\boxed{\begin{aligned} \langle \mathcal{O}_i(\infty)\, \mathcal{O}_j(z)\, \mathcal{O}_k(0) \rangle &= \lim_{z_1 \to \infty} z_1^{2\Delta_i} \bar{z}_1^{2\tilde{\Delta}_i} \langle \mathcal{O}_i(z_1)\, \mathcal{O}_j(z)\, \mathcal{O}_k(0) \rangle \\ &= C_{ijk}\, z^{\Delta_i - \Delta_j - \Delta_k}\, \bar{z}^{\tilde{\Delta}_i - \tilde{\Delta}_j - \tilde{\Delta}_k} \end{aligned}}$$

Note that



$$\left(L_{-n}\mathcal{O}\right)(\infty) := \left(L_{-n}\mathcal{O}\right)_{\{w\}}(\infty)$$

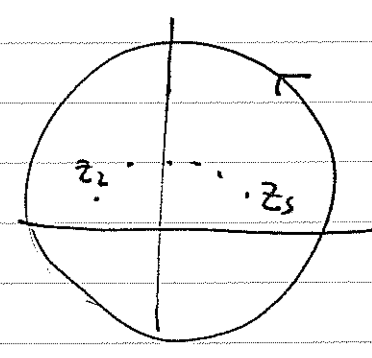
$$= \oint_{\infty} \underbrace{\frac{dw}{2\pi i} w^{-n+1} T_{ww}(w)}_{\parallel \\ -\frac{dz\, z^{-2}}{2\pi i} z^{n-1} z^{4} T_{zz}(z)} \mathcal{O}_{\{w\}}(\infty)$$



$$\overset{*}{=} -\oint_{\text{large}|z|} \left(-\frac{dz}{2\pi i}\right) z^{n+1} T_{zz}(z)\, \mathcal{O}_{\{w\}}(\infty)$$

$\leftarrow$ positive orientation

$$= \mathcal{O}_{\{w\}} L_n^{\text{large}|z|} \qquad \left(L_n^{\text{large}|z|} := \oint_{\text{large}|z|} \frac{dz}{2\pi i} z^{n+1} T_{zz}(z) \right)$$



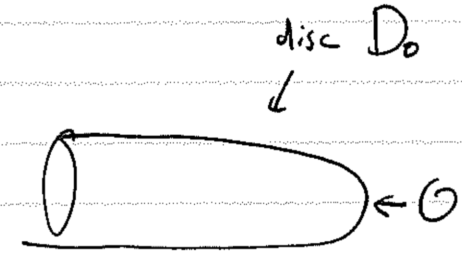
$$\therefore \left\langle \left(L_{-n}\mathcal{O}_1\right)(\infty)\, \mathcal{O}_2(z_2) \cdots \mathcal{O}_s(z_s) \right\rangle_{\mathbb{CP}^1}$$

$$= \left\langle \mathcal{O}_1(\infty)\, L_n^{\text{large}} \left(\mathcal{O}_2(z_2) \cdots \mathcal{O}_s(z_s) \right) \right\rangle_{\mathbb{CP}^1}$$

$$= \sum_{i=2}^{s} \left(z_i^{n+1} \frac{\partial}{\partial z_i} + \Delta_i (n+1) z_i^{n} \right) \left\langle \mathcal{O}_1(\infty)\, \mathcal{O}_2(z_2) \cdots \mathcal{O}_s(z_s) \right\rangle_{\mathbb{CP}^1}.$$

in particular $\left\langle \left(L_{-n}\mathcal{O}_1\right)(\infty)\, \mathcal{O}_2(0) \right\rangle_{\mathbb{CP}^1} = \left\langle \mathcal{O}_1(\infty) \left(L_n \mathcal{O}_2\right)(0) \right\rangle_{\mathbb{CP}^1}$

State/Operator Correspondence

recall $\mathcal{O} \longleftrightarrow |\mathcal{O}\rangle =$ 

i.e. $$\langle x|\mathcal{O}\rangle = \int_{X|_{\partial D_0}=x} \mathcal{D}X\, e^{-S_{D_0}(X)}\, \mathcal{O}(X)(0)$$

Inner product of two such states $|\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle$

$$\langle \mathcal{O}_1|\mathcal{O}_2\rangle = \int \mathcal{D}x\, \langle \mathcal{O}_1|x\rangle\langle x|\mathcal{O}_2\rangle$$

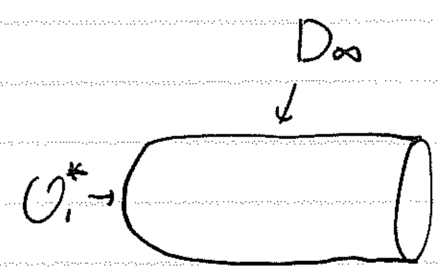
Configuration on ∂D_0

$$= \int \mathcal{D}x \left(\int_{X_1|_{\partial D_0}=x} \mathcal{D}X_1\, e^{-S_{D_0}(X_1)}\, \mathcal{O}_1(X)(0)\right)^* \int_{X_2|_{\partial D_0}=x} \mathcal{D}X_2\, e^{-S_{D_0}(X_2)}\, \mathcal{O}_2(X)(0)$$

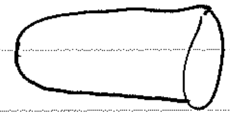
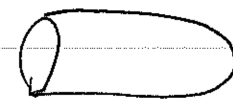
$$= \int \mathcal{D}x \int_{X_1|_{\partial D_0}=x} (\mathcal{D}X_1)^* \left(e^{-S_{D_0}(X_1)}\right)^* \mathcal{O}_1(X)(0)^* \int_{X_2|_{\partial D}=x} \mathcal{D}X_2\, e^{-S_{D_0}(X_2)}\, \mathcal{O}_2(X)(0)$$

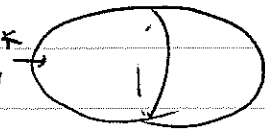
$\Big\|$ def $\mathcal{O}_1^*$ by

$$\int_{X_1^*|_{\partial D_\infty}=x} \mathcal{D}X_1^*\, e^{-S_{D_\infty}(X_1^*)}\, \mathcal{O}_1^*(X_1^*)(\infty)$$



Then, we have

$\langle O_1 | O_2 \rangle = \int \mathcal{D}x \, \Big\langle O_1^* \rightarrow$  $\Big| x \Big\rangle \Big\langle x \Big|$  $\leftarrow O_2 \Big\rangle$

$= \Big\langle O_1^* \rightarrow$  $\leftarrow O_2 \Big\rangle = \langle O_1^*(\infty)\, O_2(0) \rangle_{\mathbb{CP}^1}$

Alternatively, by $O_1^{**} = O_1$

$$\boxed{\langle O_1(\infty)\, O_2(0) \rangle_{\mathbb{CP}^1} = \langle O_1^* | O_2 \rangle}$$

$$\langle (L_{-n} O_1)(\infty)\, O_2(0) \rangle_{\mathbb{CP}^1} = \langle O_1(\infty)\, L_n^{\text{large}}\, O_2(0) \rangle_{\mathbb{CP}^1}$$

$$= \langle O_1(\infty)\, (L_n O_2)(0) \rangle_{\mathbb{CP}^1}$$

Simply corresponds to $\langle O_1^* | L_{-n}^\dagger | O_2 \rangle = \langle O_1^* | L_n | O_2 \rangle$

i.e. $L_{-n}^\dagger = L_n$

similarly $\tilde{L}_{-n}^\dagger = \tilde{L}_n$

The Hermiticity!

$$\begin{bmatrix} \text{e.g.} \quad (e^{ikX})^* = e^{-ikX} & \text{in } \sigma\text{-model on circle.} \\ \psi_\pm^* = \bar{\psi}_\pm & \text{in Dirac fermion} \end{bmatrix}$$

Notation

For a primary ϕ_n we denote $|n\rangle = |\phi_n\rangle$

(We choose orthonormal basis $\langle m|n\rangle = \delta_{m,n}$.)

Also, we denote $\phi_n^* = \phi_{\bar{n}}$.

Then $\langle \phi_n(\infty)\, \phi_m(0) \rangle = \langle \phi_n^* | \phi_m \rangle = \langle \phi_{\bar{n}} | \phi_m \rangle$

$$= \langle \bar{n} | m \rangle = \delta_{m,\bar{n}}$$

In particular

$$\langle \phi_n(z_1)\, \phi_m(z_2) \rangle \overset{\text{recall}}{=} \frac{\langle \phi_n(\infty)\, \phi_m(0) \rangle}{(z_1 - z_2)^{2\Delta_n} (\bar{z}_1 - \bar{z}_2)^{2\tilde{\Delta}_n}}$$

$$= \frac{\delta_{\bar{n},m}}{(z_1 - z_2)^{2\Delta_n} (\bar{z}_1 - \bar{z}_2)^{2\tilde{\Delta}_n}}$$