§ OPE of primaries

Operator product expansion

$$\mathcal{O}_I(z)\,\mathcal{O}_J(0) = \sum_K f_{IJ}^{\ K}(z)\,\mathcal{O}_K(0)$$

If $\mathcal{O}_I, \mathcal{O}_J, \mathcal{O}_K$'s have definite conformal weights, $(\Delta_I, \tilde{\Delta}_I), \cdots$
the "structure functions" should be of the form

$$f_{IJ}^{\ K}(z) \propto z^{\Delta_K-\Delta_I-\Delta_J} \cdot \bar{z}^{\tilde{\Delta}_K-\tilde{\Delta}_I-\tilde{\Delta}_J}$$

We are interested in OPE of primaries, $\phi_n(z)\phi_m(0) = \cdots$,
since others follow from operations of L_{-l}'s, $\tilde{L}_{-l}$'s.
The $\mathcal{O}_K$'s on the RHS are primaries <u>AND descendants</u>.
It should be of the form

$$\phi_n(z)\,\phi_m(0) = \sum_p \sum_{\{k\},\{\bar{k}\}} C_{nm}^{p\{k\}\{\bar{k}\}}\, z^{\Delta_p+|k|-\Delta_n-\Delta_m} \cdot \bar{z}^{\tilde{\Delta}_p+|\bar{k}|-\tilde{\Delta}_n-\tilde{\Delta}_m} \times \phi_p^{\{k\}\{\bar{k}\}}(0) \quad \text{—} (☆)$$

label of primaries

$\{k\} = \{k_1 \leq k_2 \leq \cdots \leq k_s\}$

$\{\bar{k}\} = \{\bar{k}_1 \leq \bar{k}_2 \leq \cdots \leq \bar{k}_t\}$

$|k| = \sum_i k_i$

$|\bar{k}| = \sum_j \bar{k}_j$

$\{k\},\{\bar{k}\}$ descendant of ϕ_p

$$L_{-\{k\}}\tilde{L}_{-\{\bar{k}\}}\phi_p = L_{-k_1}\cdots L_{-k_s}\tilde{L}_{-\bar{k}_1}\cdots\tilde{L}_{-\bar{k}_t}\phi_p$$

Claim $C_{nm}^{p\{k\}\{\bar{k}\}} = C_{nm}^{p}\beta_{nm}^{p\{k\}}\bar{\beta}_{nm}^{p\{\bar{k}\}}$

where $C_{nm}^{p} = C_{\bar{p}nm} = \langle p|\phi_n(1)|m\rangle$

and $\beta_{nm}^{p\{k\}}$ $(\bar{\beta}_{nm}^{p\{\bar{k}\}})$ is determined

just by $\Delta_n, \Delta_m, \Delta_p, \{k\}$ $(\tilde{\Delta}_n, \tilde{\Delta}_m, \tilde{\Delta}_p, \{\bar{k}\})$.

To see this, let us consider

$$\langle \phi_{\bar{p}}^{\{k\}\{\bar{k}\}}(\infty)\phi_n(z)\phi_m(0)\rangle = \langle p|L_{k_s}\cdots L_{k_1}\tilde{L}_{\bar{k}_t}\cdots\tilde{L}_{\bar{k}_1}\phi_n(z)|m\rangle$$

$$\overset{L_k|m\rangle = \tilde{L}_k|m\rangle = 0 \quad k>0}{=} \langle p|[L_{k_s},[\cdots[L_{k_1},[\tilde{L}_{\bar{k}_t},[\cdots,[\tilde{L}_{\bar{k}_1},\phi_n(z)]\cdots]|m\rangle$$

$$= \hat{\mathcal{L}}_{k_1}\cdots\hat{\mathcal{L}}_{k_s}\hat{\tilde{\mathcal{L}}}_{\bar{k}_1}\cdots\hat{\tilde{\mathcal{L}}}_{\bar{k}_t}\underbrace{\langle p|\phi_n(z)|m\rangle}_{= C_{\bar{p}nm}z^{\Delta_p-\Delta_n-\Delta_m}\bar{z}^{\tilde{\Delta}_p-\tilde{\Delta}_n-\tilde{\Delta}_m}}$$

$$\hat{\mathcal{L}}_k = z^{k+1}\frac{\partial}{\partial z} + \Delta_n(k+1)z^k$$

$$\hat{\tilde{\mathcal{L}}}_{\bar{k}} = \bar{z}^{\bar{k}+1}\frac{\partial}{\partial \bar{z}} + \tilde{\Delta}_n(\bar{k}+1)\bar{z}^{\bar{k}}$$

homogeneous $\sim \begin{matrix} z^k \\ \bar{z}^{\bar{k}} \end{matrix}$

$$= \overset{C_{nm}^{p}}{\overset{\|}{C_{\bar{p}nm}}}\alpha_{nm}^{p\{k\}}\bar{\alpha}_{nm}^{p\{\bar{k}\}}z^{\Delta_p+|k|-\Delta_n-\Delta_m}\cdot\bar{z}^{\tilde{\Delta}_p+|\bar{k}|-\tilde{\Delta}_n-\tilde{\Delta}_m}$$

where $\alpha_{nm}^{p\{k\}}$ $(\bar{\alpha}_{nm}^{p\{\bar{k}\}})$ is a # that depends only on

$\Delta_n, \Delta_p-\Delta_n-\Delta_m, \{k\}$ $(\tilde{\Delta}_n, \tilde{\Delta}_p-\tilde{\Delta}_n-\tilde{\Delta}_m, \{\bar{k}\})$.

On the other hand, using the OPE expression (☆), it is also

$$= \sum_{\{k'\},\{\bar{k}'\}} C_{nm}^{p\{k'\}\{\bar{k}'\}} z^{\Delta_p+|k'|-\Delta_n-\Delta_m} \bar{z}^{\tilde{\Delta}_p+|\bar{k}'|-\tilde{\Delta}_n-\tilde{\Delta}_m} \langle \phi_{\bar{p}}^{\{k\}\{\bar{k}\}}(\infty)\, \phi_p^{\{k'\}\{\bar{k}'\}}(0) \rangle$$

Here we note

$$\langle \phi_{\bar{p}}^{\{k\}\{\bar{k}\}}(\infty)\, \phi_p^{\{k'\}\{\bar{k}'\}}(0) \rangle$$

$$= \langle p | L_{k_s} \cdots L_{k_1} \tilde{L}_{\bar{k}_t} \cdots \tilde{L}_{\bar{k}_1} L_{-k'_1} \cdots L_{-k'_{s'}} \tilde{L}_{-\bar{k}'_1} \cdots \tilde{L}_{-\bar{k}'_{t'}} | p \rangle$$

$$= M_p^{\{k\}\{k'\}} \cdot \bar{M}_p^{\{\bar{k}\}\{\bar{k}'\}}$$

$$(\nwarrow)$$

some matrix that depends only on $\Delta_p, \{k\}, \{k'\}$

$(\tilde{\Delta}_p, \{\bar{k}\}, \{\bar{k}'\})$

$\neq 0$ only if $|k| = |k'|$ $\left(|\bar{k}| = |\bar{k}'| \right)$

By comparison we have an equation

$$\boxed{\sum_{\{k'\}\{\bar{k}'\}} M_p^{\{k\}\{k'\}} \bar{M}_p^{\{\bar{k}\}\{\bar{k}'\}} C_{nm}^{p\{k'\}\{\bar{k}'\}} = C_{nm}^{p} \alpha_{nm}^{p\{k\}} \bar{\alpha}_{nm}^{p\{\bar{k}\}}}$$

Claim $\Leftrightarrow$ It has a solution of the form $C_{nm}^{p\{k\}\{\bar{k}\}} = C_{nm}^{p} \beta_{nm}^{p\{k\}} \bar{\beta}_{nm}^{p\{\bar{k}\}}$

where $\sum_{\{k'\}} M_p^{\{k\}\{k'\}} \beta_{nm}^{p\{k'\}} = \alpha_{nm}^{p\{k\}}$, $\sum_{\{\bar{k}'\}} \bar{M}_p^{\{\bar{k}\}\{\bar{k}'\}} \bar{\beta}_{nm}^{p\{\bar{k}'\}} = \bar{\alpha}_{nm}^{p\{\bar{k}\}}$.

This would be obvious if $M_p^{\{k\}\{k'\}}$ and $\bar{M}_p^{\{\bar{k}\}\{\bar{k}'\}}$ were invertible matrices (just put $\beta = M^{-1}\alpha$, $\bar{\beta} = \bar{M}^{-1}\bar{\alpha}$).

However there is a possibility that $M_p^{\{k\},\{k'\}}$, $\bar{M}_p^{\{\bar{k}\}\{\bar{k}'\}}$ are not invertible (important!).

This is so even in unitary theory we are considering.

Some of the vectors $L_{-\{k\}}\bar{L}_{-\{\bar{k}\}}|p\rangle$ may be vanishing.
Then $\overset{(-)}{M}{}_p^{\{\overset{(-)}{k}\}\{\overset{(-)}{k}\}} = 0$ for that $\{k\}$ or $\{\bar{k}\}$.
See later.

Even in such a case, since

$$C_{nm}^{p}\, \alpha_{nm}^{p\{k\}}\, \bar{\alpha}_{nm}^{p\{\bar{k}\}} = \langle p| L_{\{k\}} \bar{L}_{\{\bar{k}\}}\, \phi_n(1)|m\rangle$$

$$= \langle p| L_{\{k\}} \bar{L}_{\{\bar{k}\}} |\text{some state}\rangle,$$

$\alpha_{nm}^{p\{k\}}$ is in the image of $M_p^{\{k\}\{k'\}}$

and $\bar{\alpha}_{nm}^{p\{\bar{k}\}}$ 〃 $\bar{M}_p^{\{\bar{k}\}\{\bar{k}'\}}$.

Thus there must be some $\beta_{nm}^{p\{k'\}}$, $\bar{\beta}_{nm}^{p\{\bar{k}'\}}$ s.t.

$$\alpha_{nm}^{p\{k\}} = \sum_{\{k'\}} M_p^{\{k\}\{k'\}} \beta_{nm}^{p\{k'\}}$$

$$\bar{\alpha}_{nm}^{p\{\bar{k}\}} = \sum_{\{\bar{k}'\}} M_p^{\{\bar{k}\}\{\bar{k}'\}} \bar{\beta}_{nm}^{p\{\bar{k}'\}}.$$

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Note again $M_\Delta^{\{h\}\{h'\}} = \left(L_{-\{h\}}|\Delta\rangle, L_{-\{h'\}}|\Delta\rangle \right) = 0$

unless $|h| = |h'|$.

i.e. $M_\Delta^{\{h\}\{h'\}}$ decomposes into blocks by the <u>level</u>.

<u>Level 0</u> $\quad M_\Delta^{\{0\},\{0\}} = \langle\Delta|\Delta\rangle = 1$

<u>level 1</u> $\quad M_\Delta^{1,1} = \langle\Delta|L_1 L_{-1}|\Delta\rangle = \langle\Delta|2L_0|\Delta\rangle = 2\Delta$

<u>level 2</u> $\quad M_\Delta^{2,2} = \langle\Delta|L_2 L_{-2}|\Delta\rangle = \langle\Delta|\left(4L_0 + \frac{c}{12}(8-2)\right)|\Delta\rangle = 4\Delta + \frac{c}{2}$

$$M_\Delta^{2,\{1,1\}} = M_\Delta^{\{1,1\},2} = \langle\Delta|L_1 L_1 L_{-2}|\Delta\rangle = \langle\Delta|L_1(3L_{-1})|\Delta\rangle = 3\cdot 2\Delta$$

$$M_\Delta^{\{1,1\},\{1,1\}} = \langle\Delta|L_1 L_1 L_{-1} L_{-1}|\Delta\rangle = \langle\Delta|L_1(2L_0 + L_{-1}L_1)L_{-1}|\Delta\rangle$$

$$= 2(\Delta+1)\langle\Delta|L_1 L_{-1}|\Delta\rangle + \langle\Delta|L_1 L_{-1} 2\underbrace{L_0}_{=\Delta}|\Delta\rangle$$

$$= 2(2\Delta+1)\cdot 2\Delta$$

$$\therefore M_{\Delta\,\text{level }2} = \begin{pmatrix} 4\Delta + \frac{c}{2} & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix}$$

Note also $\alpha_{nm}^{p\{k\}} = \hat{\mathcal{L}}_{k_1} \cdots \hat{\mathcal{L}}_{k_s} z^{\Delta_p - \Delta_n - \Delta_m} \Big|_{z=1}$

level 0 $\quad \alpha_{nm}^{p\{\cdot\}} = z^{\Delta_p - \Delta_n - \Delta_m} \Big|_{z=1} = 1$

$$\therefore \beta_{nm}^{p\{\cdot\}} = M_p^{\{\cdot\},\{\cdot\}\,-1} \alpha_{nm}^{p\{\cdot\}} = 1$$

level 1 $\quad \alpha_{nm}^{p\{1\}} = \hat{\mathcal{L}}_1 z^{\Delta_p - \Delta_n - \Delta_m} \Big|_{z=1}$

$$= \left(z^2 \frac{\partial}{\partial z} + \Delta_n \cdot 2 \cdot z \right) z^{\Delta_p - \Delta_n - \Delta_m} \Big|_{z=1}$$

$$= \Delta_p - \Delta_n - \Delta_m + 2\Delta_n = \Delta_p + \Delta_n - \Delta_m$$

$$(M_p^{-1})^{\{1\}\{1\}} = (M_p^{\{1\}\{1\}})^{-1} = \frac{1}{2\Delta_p}$$

$$\therefore \boxed{\beta_{nm}^{p\{1\}} = \frac{\Delta_p + \Delta_n - \Delta_m}{2\Delta_p}}$$

right part

The case $M_p \big|_{level=1}$ is <u>not</u> invertible $\Leftrightarrow \Delta_p = 0 \Leftrightarrow \phi_p = id$

but then $C_{\bar{p}nm} \neq 0$ only if $\Delta_n = \Delta_m \quad \therefore \alpha_{nm}^{p\{1\}} = 0$

<u>So it is ok.</u>

(We don't have to consider $\beta_{nm}^{p\{1\}}$).

level 2

$$\alpha^{p\{2\}}_{nm} = \hat{\mathcal{L}}_2 \, z^{\Delta_p-\Delta_n-\Delta_m} \Big|_{z=1}$$

$$= \left(z^3 \frac{\partial}{\partial z} + \Delta_n 3 z^2 \right) z^{\Delta_p-\Delta_n-\Delta_n} = \Delta_p + 2\Delta_n - \Delta_m$$

$$\alpha^{p\{1,1\}}_{nm} = \hat{\mathcal{L}}_1 \hat{\mathcal{L}}_1 \, z^{\Delta_p-\Delta_n-\Delta_m} \Big|_{z=1}$$

$$= \left(z^2 \frac{\partial}{\partial z} + 2\Delta_n z \right)\left(z^2 \frac{\partial}{\partial z} + 2\Delta_n z \right) z^{\Delta_p-\Delta_n-\Delta_m} \Big|_{z=1}$$

$$= \left(z^2 \frac{\partial}{\partial z} + 2\Delta_n z \right) z^{\Delta_p-\Delta_n-\Delta_n+1} \Big|_{z=1} \times (\Delta_p+\Delta_n-\Delta_m)$$

$$= (\Delta_p+\Delta_n-\Delta_m+1)(\Delta_p+\Delta_n-\Delta_m)$$

$$\begin{pmatrix} \beta^{p\{2\}}_{nm} \\ \beta^{p\{1,1\}}_{nm} \end{pmatrix} = \begin{pmatrix} 4\Delta_p + \frac{c}{2} & 6\Delta_p \\ 6\Delta_p & 4\Delta_p(2\Delta_p+1) \end{pmatrix}^{-1} \begin{pmatrix} \Delta_p + 2\Delta_n - \Delta_m \\ (\Delta_p+\Delta_n-\Delta_m+1)(\Delta_p+\Delta_n-\Delta_m) \end{pmatrix}$$

if $M_p \big|_{\text{level } 2}$ is invertible

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4 point functions

$$G^{\ell k}_{nm}(x,\bar{x}) = \langle \phi_k(\infty)\, \phi_\ell(1)\, \phi_n(x)\, \phi_m(0) \rangle$$

$$\overset{\substack{OPE\\ \downarrow}}{=} \sum_{p,\{k\},\{\bar{k}\}} C^p_{nm}\, \beta^{p\{k\}}_{nm}\, \bar{\beta}^{p\{\bar{k}\}}_{nm}\, x^{\Delta_p+|k|-\Delta_n-\Delta_m} \cdot \bar{x}^{\tilde{\Delta}_p+|\bar{k}|-\tilde{\Delta}_n-\tilde{\Delta}_m}$$

$$\cdot \underbrace{\langle \phi_k(\infty)\, \phi_\ell(1)\, \phi_p^{\{k\}\{\bar{k}\}}(0) \rangle}_{C_{p\ell k}\, \alpha^{\bar{p}\{k\}}_{\ell k}\, \bar{\alpha}^{\bar{p}\{\bar{k}\}}_{\ell k}}$$

$$= \sum_{p\,\{k\}\{\bar{k}\}} C^p_{nm} C_{p\ell k}\, \beta^{p\{k\}}_{nm}\, \alpha^{\bar{p}\{k\}}_{\ell k}\, x^{\Delta_p+|k|-\Delta_n-\Delta_m}\, \bar{\beta}^{p\{\bar{k}\}}_{nm}\, \bar{\alpha}^{\bar{p}\{\bar{k}\}}_{\ell k}\, \bar{x}^{\tilde{\Delta}_p+|\bar{k}|-\tilde{\Delta}_n-\tilde{\Delta}_m}$$

$$= \sum_p C^p_{nm} C_{p\ell k}\, \mathcal{F}^{\ell k}_{nm}(p|x)\, \bar{\mathcal{F}}^{\ell k}_{nm}(p|\bar{x})$$

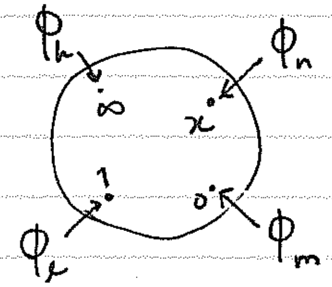
where $\mathcal{F}^{\ell k}_{nm}(p|x) = x^{\Delta_p-\Delta_n-\Delta_m} \sum_{\{k\}} \beta^{p\{k\}}_{nm}\, \alpha^{\bar{p}\{k\}}_{\ell k}\, x^{|k|}$

$$= x^{\Delta_p-\Delta_n-\Delta_m} \sum_{\{k\}\{k'\}} \beta^{p\{k\}}_{nm}\, M_{\bar{p}}^{\{k\}\{k'\}}\, \beta^{\bar{p}\{k'\}}_{\ell k}\, x^{|k|}$$

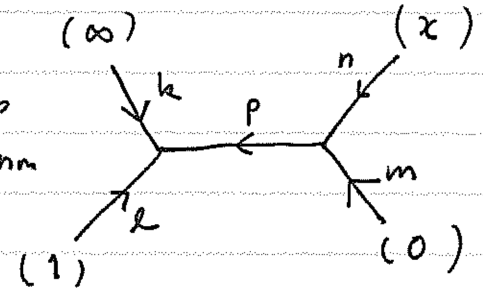
similarly $\bar{\mathcal{F}}^{\ell k}_{nm}(p|\bar{x}) = \bar{x}^{\tilde{\Delta}_p-\tilde{\Delta}_n-\tilde{\Delta}_m} \sum_{\{\bar{k}\}\{\bar{k}'\}} \bar{\beta}^{p\{\bar{k}\}}_{nm}\, \bar{M}_{\bar{p}}^{\{\bar{k}\}\{\bar{k}'\}}\, \bar{\beta}^{\bar{p}\{\bar{k}'\}}_{\ell k}\, \bar{x}^{|\bar{k}|}$

These $\mathcal{F}^{\ell k}_{nn}(p|x)$ are called the <u>Conformal blocks</u>

A picture:

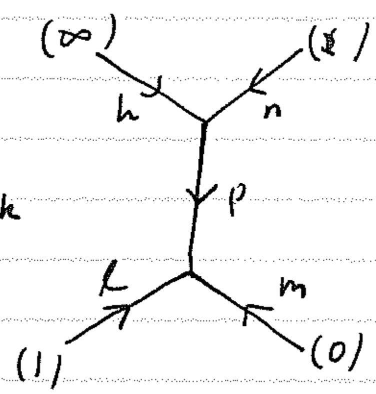


$$= \sum_p C_{plk} C^p_{nm}$$

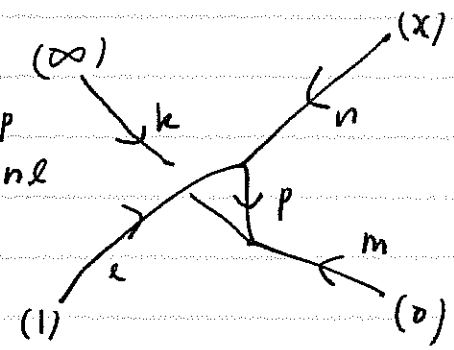


We may also consider OPE in different combinations

$$= \sum_p C_{plm} C^p_{nk}$$



$$= \sum_p C_{pmk} C^p_{nl}$$



Note: $\Leftrightarrow$ <u>Associativity</u> of Operator product algebra

$$\underbrace{\langle \phi_k(\infty)\left(\phi_l(1)\left(\phi_n(x)\,\phi_m(0)\right)\right)\rangle}_{\text{1st}} = \underbrace{\langle \phi_k(\infty)\left(\left(\phi_l(1)\,\phi_n(x)\right)\phi_m(0)\right)\rangle}_{\text{last}}$$

In formula: Use the $SL(2,\mathbb{C})$ Ward identity

$$G^{\ell k}_{nm}(x,\bar{x}) = \left\langle \phi_{k\{z^{-1}\}}(p_3)\, \phi_{\ell\{z\}}(p_2)\, \phi_{n\{z\}}(p_4)\, \phi_{m\{z\}}(p_1) \right\rangle_{\mathbb{CP}^1}$$

with $z(p_1)=0,\ z(p_2)=1,\ z(p_3)=\infty,\ z(p_4)=x$

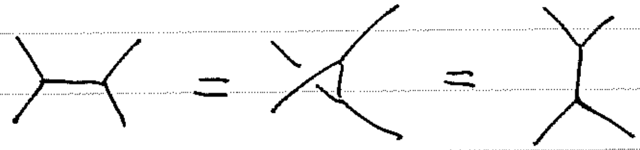
Other coordinate systems: $\zeta = 1-z$ or $w = z^{-1}$

$$G^{\ell k}_{nm}(x,\bar{x}) = \prod_{i=1}^{4}(-1)^{\Delta_i+\tilde{\Delta}_i} \left\langle \phi_{k\{\zeta^{-1}\}}(p_3)\, \phi_{\ell\{\zeta\}}(p_2)\, \phi_{n\{\zeta\}}(p_4)\, \phi_{m\{\zeta\}}(p_1) \right\rangle_{\mathbb{CP}^1}$$

$$= \prod_{i=1}^{4}(-1)^{\Delta_i+\tilde{\Delta}_i} \cdot G^{mk}_{n\ell}(1-x, 1-\bar{x})$$

$$\overset{\text{or}}{=} \prod_{i=2,4}\left(-z^{-2}(p_i)\right)^{\Delta_i}\left(-\bar{z}^{-2}(p_i)\right)^{\tilde{\Delta}_i} \left\langle \phi_{k\{w\}}(p_3)\, \phi_{\ell\{w\}}(p_2)\, \phi_{n\{w\}}(p_4)\, \phi_{m\{w^{-1}\}}(p_1) \right\rangle_{\mathbb{CP}^1}$$

$$= \prod_{i=2,4}(-1)^{\Delta_i+\tilde{\Delta}_i} \cdot x^{-2\Delta_n}\bar{x}^{-2\tilde{\Delta}_n}\, G^{\ell m}_{nk}\left(\frac{1}{x}, \frac{1}{\bar{x}}\right).$$

Thus  leads to equations

$$\sum_p C^p_{nm} C_{plk} \mathcal{F}^{lk}_{nm}(p|x) \bar{\mathcal{F}}^{lk}_{nm}(p|x)$$

$$= \prod_{i=1}^{4} (-1)^{\Delta_i+\tilde{\Delta}_i} \cdot \sum_p C^p_{nl} C_{pmk} \mathcal{F}^{mk}_{nl}(p|1-x) \bar{\mathcal{F}}^{mk}_{nl}(p|1-\bar{x})$$

$$= \prod_{i=2,4} (-1)^{\Delta_i+\tilde{\Delta}_i} \cdot x^{-2\Delta_n} \bar{x}^{-2\tilde{\Delta}_n} \sum_p C^p_{nk} C_{plm} \mathcal{F}^{lm}_{nk}\left(p\middle|\frac{1}{x}\right) \bar{\mathcal{F}}^{lm}_{nk}\left(p\middle|\frac{1}{\bar{x}}\right)$$

These are ∞ set of equations on Δ_n's, $\tilde{\Delta}_n$'s, C_{nml}'s.

Very strong constraints provided the conformal blocks $\mathcal{F}^{kl}_{nm}(p|x)$ are known.

<u>Conformal BootStrap Program</u>:

Determine possible values of Δ_n's, $\tilde{\Delta}_n$'s, C_{nml}'s using these constraints.