Review: Representation Theory of $SU(2)$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad i,j,k=1,2,3 \qquad \text{Angular Momentum}$$

$$J_i^\dagger = J_i \quad \text{(Hermitian)}.$$

We want to classify its <u>unitary representations</u>, V.

define $\quad J_\pm := J_1 \pm i J_2 \quad , \quad J_\pm^\dagger = J_\mp$

$$\begin{cases} [J_3, J_\pm] = \pm J_\pm & \cdots\ J_\pm \left\{\begin{smallmatrix}\text{raises}\\ \text{lowers}\end{smallmatrix}\right\} J_3\text{-values by } 1. \\ [J_+, J_-] = 2J_3 \end{cases}$$

Suppose $\exists\, |j\rangle \in V$ s.t. $\quad J_+|j\rangle = 0$

$$J_3|j\rangle = j|j\rangle \quad \text{for some } \underline{j \in \mathbb{R}}.$$

We have a sequence of vectors

$$|j\rangle,\ J_-|j\rangle,\ J_-^2|j\rangle,\ J_-^3|j\rangle,\ \cdots\ J_-^n|j\rangle,\ \cdots$$

$$J_3 = j\ ,\quad j-1,\quad j-2,\quad j-3,\quad \cdots\quad j-n,\quad \cdots$$

What is the norms of these vectors?

Assume $\langle j|j\rangle = 1$.

$$\| J_-^n |j\rangle \|^2 = \langle j| J_+^n J_-^n |j\rangle = ?$$

$$\| J_-|j\rangle\|^2 = \langle j|J_+J_-|j\rangle = \langle j|\big(\underbrace{[J_+,J_-]}_{2J_3\to 2j} + J_-\cancel{J_+}\big)|j\rangle \overset{\langle j|j\rangle}{=} 2j$$

$$\|J_-^2|j\rangle\|^2 = \langle j|J_+^2J_-^2|j\rangle = \langle j|J_+\big([J_+,J_-]+J_-J_+\big)J_-|j\rangle$$

$$= \langle j|J_+\big(\underbrace{[J_+,J_-]}_{2J_3\to 2(j-1)}J_- + J_-\underbrace{[J_+,J_-]}_{2J_3\to 2j} + J_-^2\cancel{J_+}\big)|j\rangle$$

$$= 2(2j-1)\langle j|J_+J_-|j\rangle = 2(2j-1)\cdot 2j$$

$$J_+J_-^n|j\rangle = \big(\underbrace{[J_+,J_-]}_{2J_3\to 2(j-(n-1))}J_-^{n-1} + J_-\underbrace{[J_+,J_-]}_{2J_3\to 2(j-(n-2))}J_-^{n-2} + \cdots + J_-^{n-1}\underbrace{[J_+,J_-]}_{2j} + J_-^n\cancel{J_+}\big)|j\rangle$$

$$= 2\left(nj - \frac{n(n-1)}{2}\right)J_-^{n-1}|j\rangle = 2n\left(j-\frac{n-1}{2}\right)J_-^{n-1}|j\rangle$$

$$\therefore \|J_-^n|j\rangle\|^2 = 2n\left(j-\frac{n-1}{2}\right)\|J_-^{n-1}|j\rangle\|^2$$

$$= 2^2n(n-1)\left(j-\frac{n-1}{2}\right)\left(j-\frac{n-2}{2}\right)\|J_-^{n-2}|j\rangle\|^2 = \cdots$$

$$= 2^n n!\left(j-\frac{n-1}{2}\right)\left(j-\frac{n-2}{2}\right)\cdots\left(j-\frac{1}{2}\right)j\,.$$

For some n it is negative unless $j\in\{0,\frac{1}{2},1,\frac{3}{2},2,\cdots\}$

We conclude that j must be a non-negative integer/half-integer.

Then, $\|J_-^n|j\rangle\|^2$ is positive for $n=0,1,2,\cdots,2j$

zero for $n=2j+1,2j+2,\cdots$.

positive definiteness of inner product $\Rightarrow$

$J_-^n|j\rangle$ must be zero for $n \geq 2j+1$.

We have a spin j representation (denoted by V_j)

Spanned by $\{ |j\rangle, J_-|j\rangle, \cdots, J_-^{2j}|j\rangle \}$. $(\dim V_j = 2j+1)$

Claim : Any unitary representation of $SU(2)$ is a direct sum of spin j representations $j \in \{0, \frac{1}{2}, 1, \cdots\}$

$$V \cong \bigoplus_{j \in \{0, \frac{1}{2}, 1, \cdots\}} V_j^{\oplus m_j}$$

proof : $\mathbb{J}^2 = J_1^2 + J_2^2 + J_3^2 = J_3^2 + \frac{1}{2}(J_- J_+ + J_+ J_-)$ commutes with J_3

diagonalize $\mathbb{J}^2$, J_3 simultaneously.

$$\mathbb{J}^2 - J_3^2 = \frac{1}{2}(J_+^\dagger J_+ + J_-^\dagger J_-) \geq 0$$

Thus on a subspace of a constant $\mathbb{J}^2$, J_3^2 must be bounded. ~~(above) and~~ J_3 must be bounded above (and below).

$\therefore \exists |j\rangle$ s.t. $J_+|j\rangle = 0$, $J_3|j\rangle = j|j\rangle$ in such a subspace //.

V_j is an example of highest weight representation.

$|j\rangle$ highest weight vector (j = highest weight)

i.e. annihilated by J_+

Another highest weight representation : <u>Verma module</u>

$$M_j = \mathrm{Span}\{ \underset{j}{v_j},\ \underset{j-1}{J_- v_j},\ \underset{j-2}{J_-^2 v_j},\ \underset{\cdots}{\cdots},\ \underset{j-n}{J_-^n v_j},\ \underset{\cdots}{\cdots} \} \quad \forall j \in \mathbb{C}$$

- <u>v_j is a highest weight vector $(J_3 = j)$</u>
- $J_-^n v_j \neq 0 \ \forall n$, and are linearly independent

 i.e. they are basis vectors of M_j $(\dim M_j = \infty)$.

Using the commutation relations of $J_3, J_\pm$, we find as before:

$$\boxed{J_+ (J_-^n v_j) = 2n\left(j - \frac{n-1}{2}\right) \cdot J_-^{n-1} v_j}$$

If $j \notin \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$, this is non-zero $\forall n \neq 0$.

- <u>M_j is irreducible</u> (non-unitary).

If $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$, this vanishes at $n = 2j+1$.

is non-zero at all other $n \neq 0$.

$\therefore$ <u>$J_-^{2j+1} v_j$ is another highest weight vector $(J_3 = -j-1)$</u>.

Such a vector is called a <u>singular vector</u>.

(Continued on $J_-^{2j+1} v_j$ for $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$)

- It generates an invariant subspace:

$$\mathrm{Span}\{ \underset{-j-1}{J_-^{2j+1} v_j},\ \underset{-j-2}{J_-^{2j+2} v_j},\ \underset{\cdots}{\cdots},\ \underset{-j-1-n}{J_-^{2j+1+n} v_j},\ \underset{\cdots}{\cdots} \} \subset M_j$$

Note: this is isomorphic to the Verma module at $-j-1$.

i.e. There is an embedding of representations

$$\boxed{M_{-j-1} \subset M_j}$$

- Therefore, $\underline{M_j \text{ is not irreducible}}$.

- Irreducible representation is obtained by factoring out (taking the quotient) by M_{-j-1}

Then, it is isomorphic to the spin j representation:

$$\boxed{V_j \cong M_j / M_{-j-1}.}$$

Character of representations

For any representation (ρ, V) of a group G, its character is defined by $\chi_{\rho,V}(g) = \mathrm{Tr}_V\, \rho(g)$.

It is a function on G which is conjugation invariant

$\chi_{\rho,V}(g_1 g_2 g_1^{-1}) = \chi_{\rho,V}(g_2)$.

If G is a Lie group, it is therefore determined by its values on maximal torus $T \subset G$.

Let V be a representation of $\mathfrak{su}(2) = \mathrm{Lie}(SU(2))$.

Its character is

$$\chi_V(z) = \mathrm{Tr}_V\left(z^{J_3}\right)$$

Spin j representation: $\chi_j(z) = z^j + z^{j-1} + \cdots + z^{-j+1} + z^{-j}$.

Verma module: $\chi_{M_j}(z) = z^j + z^{j-1} + \cdots = z^j(1 + z + z^2 + \cdots)$

$$= \frac{z^j}{1-z}.$$

relation $V_j \cong M_j / M_{-j-1} \Leftrightarrow \chi_{V_j}(z) = \chi_{M_j}(z) - \chi_{M_{-j-1}}(z)$.

Representations of Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} \delta_{n+m,0} (n^3 - n).$$

We try to repeat the story of $SU(2)$, with analogy:

$$J_+ \leftrightarrow L_1, L_2, L_3, \cdots$$

$$J_3 \leftrightarrow L_0, c \qquad \left(\begin{array}{l} c \text{ is here regarded as a} \\ \text{generator with } [L_n, c] = 0 \end{array}\right)$$

$$J_- \leftrightarrow L_{-1}, L_{-2}, L_{-3}, \cdots$$

- We are interested in <u>highest (or lowest) weight representations</u> for physical reason:

  highest weight vector $\leftrightarrow$ primary state/operator $|\Delta\rangle$

  other vectors $\leftrightarrow$ descendants. $L_{-h_1} \cdots L_{-h_s} |\Delta\rangle$

- We are particularly interested in <u>unitary representations</u> (though others may also be relevant).

  $\left(\text{recall } L_n^\dagger = L_{-n}.\right)$.

Verma module $M_{\Delta,c}$ (or just M_Δ)

$M_\Delta \ni |\Delta\rangle$ highest weight vector, $L_0|\Delta\rangle = \Delta|\Delta\rangle$.

- Other vectors are descendants of $|\Delta\rangle$
- We assume <u>no relations</u> other than those following from commutation relations

$$\left[\begin{aligned} \text{e.g. } L_{-2}L_{-1}|\Delta\rangle &= ([L_{-2}, L_{-1}] + L_{-1}L_{-2})|\Delta\rangle \\ &= -L_{-3}|\Delta\rangle + L_{-1}L_{-2}|\Delta\rangle \end{aligned}\right]$$

M_Δ has basis : $\{|\Delta\rangle\} \cup \{L_{-k_1}L_{-k_2}\cdots L_{-k_s}|\Delta\rangle \mid 1 \le k_1 \le \cdots \le k_s\}$

"level" is defined by $L_0 - \Delta$

$\downarrow$		#
0	$\|\Delta\rangle$	1
1	$L_{-1}\|\Delta\rangle$	1
2	$L_{-2}\|\Delta\rangle,\ L_{-1}L_{-1}\|\Delta\rangle$	2
3	$L_{-3}\|\Delta\rangle,\ L_{-1}L_{-2}\|\Delta\rangle,\ L_{-1}^3\|\Delta\rangle$	3
4	$L_{-4}\|\Delta\rangle,\ L_{-1}L_{-3}\|\Delta\rangle,\ L_{-2}L_{-2}\|\Delta\rangle,\ L_{-1}^2L_{-2}\|\Delta\rangle,\ L_{-1}^4\|\Delta\rangle$	5
5	$L_{-5}\|\Delta\rangle,\ L_{-1}L_{-4}\|\Delta\rangle,\ L_{-2}L_{-3}\|\Delta\rangle,\ L_{-1}^2L_{-3}\|\Delta\rangle,\ L_{-1}L_{-2}^2\|\Delta\rangle,\ L_{-1}^3L_{-2}\|\Delta\rangle,\ L_{-1}^5\|\Delta\rangle$	7
⋮		⋮
N		$P(N)$

partitions of N

Character $\chi_{M_\Delta}(q) = \mathrm{Tr}_{M_\Delta}\left(q^{L_0 - \frac{c}{24}}\right)$

$$= \sum_{N=0}^{\infty} q^{\Delta + N - \frac{c}{24}} \cdot P(N) = \frac{q^{\Delta - \frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q^n)}$$

Questions

- Is M_Δ irreducible?
- Is it unitary?
- If not irreducible, what kind of invariant subspaces are there?
- Can we find some invariant $N \subset M_\Delta$ such that M_Δ / N is unitary?

— The answers depend very much on the values of Δ, c.

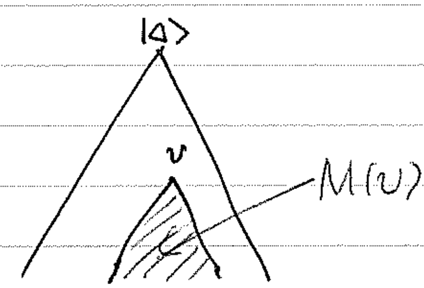
(Just as the properties of M_j depends on $j \in \mathbb{C}$)

$v \in M_\Delta$ is a <u>singular vector</u> if $L_n v = 0 \quad \forall n > 0$.

(i.e. a descendant of $|\Delta\rangle$ which <u>is also a primary</u>)

A singular vector generates an invariant subspace $M(v) \subset M_\Delta$ consisting of descendants <u>of v</u>. If $L_0 v = \Delta_v v$, this subrepresentation is isomorphic to the Verma module at Δ_v:

$$M(v) \cong M_{\Delta_v}.$$

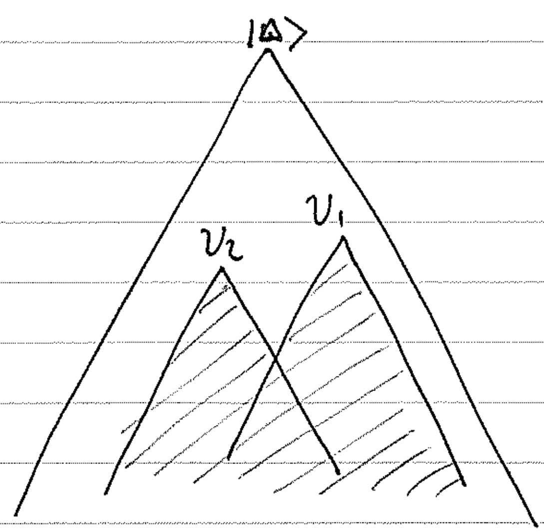


<u>Theorem</u> [Feigin-Fuchs]

Suppose $M \subset M_\Delta$ is an invariant subspace. Then $\exists v_1, v_2, \cdots, v_n \in M_\Delta$ singular vectors s.t.

$$M = M(v_1) + M(v_2) + \cdots + M(v_n)$$

not necessarily a direct sum. i.e. overlaps possible.



[Thus, we would like to know what kind of singular vectors are there.]

To this end, and also to classify unitary representations, we introduce a <u>hermitian form</u> $(,)$ or $\langle | \rangle$ on M_Δ:

$$\langle \Delta | \Delta \rangle \stackrel{\text{def}}{=} 1$$

$$\left(L_{-\{k\}} |\Delta\rangle , L_{-\{k'\}} |\Delta\rangle \right) := \langle \Delta | L_{-\{k\}}^{+} L_{-\{k'\}} |\Delta\rangle$$

where RHS is computed using $L_{-n}^{+} = L_n$, Virasoro algebra and $\langle \Delta | \Delta \rangle = 1$

— This is nothing but the matrix $M_\Delta^{\{k\}\{k'\}}$ considered earlier!

• Subspaces of different levels are orthogonal to one another.

<u>Def</u> $v \in M_\Delta$ is called a <u>null vector</u>

if it is orthogonal to all vectors of M_Δ (including itself).

i.e. $\langle \varphi | v \rangle = 0 \quad \forall \varphi \in M_\Delta$.

• <u>A singular vector is null.</u>

<u>proof</u> Suppose v is singular. Since v is orthogonal to all vectors with different levels, enough to check $\langle \varphi | v \rangle = 0$ for φ at the same level. φ is a descendant

$$|\varphi\rangle = \underset{k_1 \geq 1}{L_{-k_1}} \cdots |\Delta\rangle . \quad \langle \varphi | v \rangle = \langle \Delta | \cdots L_{k_1} |\Delta\rangle = 0 \quad //$$

- The subspace N of M_Δ consisting of all null vectors is an invariant subspace.

  i.e. v null $\Rightarrow L_n v$ null $\quad \forall n \in \mathbb{Z}$.

  proof $\forall \varphi \in M_\Delta$, $\langle \varphi | (L_n | v \rangle) = (\langle \varphi | L_n) | v \rangle \underset{\substack{\uparrow \\ v \text{ null}}}{=} 0$ //.

- In particular (using FF Theorem), $\exists$ singular vectors $v_1, \cdots, v_n$

  s.t. $N = M(v_1) + \cdots + M(v_n)$.

- A null vector is not necessarily singular.

  e.g. v_i singular $(\Delta_{v_i} \neq 0) \Rightarrow L_{-1} v_i$ is a null vector.

  But $L_1 (L_{-1} v_i) \underset{\substack{\uparrow \\ L_1 v_i = 0}}{=} 2 L_0 v_i = 2\Delta_{v_i} v_i \neq 0$

  not a singular vector.

- M_Δ / N is an irreducible representation.

  proof suppose $W \subsetneq M_\Delta / N$ is an invariant subspace.

  Then (for $\pi : M_\Delta \to M_\Delta/N$ natural projection) $\pi^{-1}(W) \subsetneq M_\Delta$ is invariant. Thus $\exists$ singular vectors $w_1, w_2, \cdots, w_r$ s.t. $\pi^{-1}(W) = \sum_i M(w_i)$. But w_i singular $\Rightarrow$ null $\therefore \pi^{-1}(W) \subset N$. $\therefore W = \{0\}$ //

Question : Is M_Δ/N unitary ?

For this, the hermitian form $(,)$ induced on M_Δ/N must be positive definite.

Let us take a closer look at $M_\Delta^{\{k_i\},\{k_i'\}}$.

$M_{\Delta,c}^{(n)}$ = the level n block of $M_{\Delta,c}^{\{k_i\},\{k_i'\}}$

(a $P(n) \times P(n)$ matrix.)

its determinant can tell us some thing:

- $\det M_{\Delta,c}^{(n)} = 0 \Rightarrow \exists$ null vector at level n.
- $\det M_{\Delta,c}^{(n)} < 0 \Rightarrow \exists$ negative eigenvalue (and non-deg.)
  $\Rightarrow (,)$ cannot be positive definite
  $\boxed{\text{ELIMINATED!}}$
- $\det M_{\Delta,c}^{(n)} > 0 \Rightarrow$ Still O.K.