Unitarity Constraints

§ Elementary considerations.

- $$\| L_{-n}|\Delta\rangle \|^2 = \langle\Delta| L_n L_{-n} |\Delta\rangle = \langle\Delta|\left(2nL_0 + \frac{c}{12}(n^3-n)\right)|\Delta\rangle$$
$$= 2n\Delta + \frac{c}{12}(n^3-n) \geq 0 \quad \forall n$$

$n=1 \Rightarrow \boxed{\Delta \geq 0}$ $\qquad$ n large $\Rightarrow \boxed{c \geq 0}$

- If $c=0$, Only the trivial repr. is possible.

Proof $|\psi_1\rangle = L_{-n}^2|\Delta\rangle, \quad |\psi_2\rangle = L_{-2n}|\Delta\rangle$

$$\langle\psi_1|\psi_1\rangle = \langle\Delta|L_n^2 L_{-n}^2|\Delta\rangle = 2n(2\Delta+n)\langle\Delta|L_n L_{-n}|\Delta\rangle = 2n(2\Delta+n)\cdot 2n\Delta$$

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle = \langle\Delta|L_n^2 L_{-2n}|\Delta\rangle = 3n\cdot 2n\Delta$$

$$\langle\psi_2|\psi_2\rangle = \langle\Delta|L_{2n}L_{-2n}|\Delta\rangle = 4n\Delta$$

$$\det\left(\langle\psi_i|\psi_j\rangle_{1\le i,j\le 2}\right) = 16n^3\Delta^2(2\Delta+n) - 4\cdot 9n^2\Delta^2 = 4n^3\Delta^2(8\Delta-5n)$$

If $\Delta \neq 0$ this goes negative for large enough n. $\boxed{\therefore \Delta = 0}$

Also $\langle\Delta|L_{-\{k\}}^{\dagger} L_{-\{k'\}}|\Delta\rangle = 0 \quad \forall \{k\} \neq \{\phi\}$.

Thus the only non-zero state is $|0\rangle$. //

⊙ $M^{(n)}_{\Delta,c}$ is positive definite for large enough Δ's (c: fixed).

proof. • $\| L_{-\{k\}}|\Delta\rangle \|^2 = \langle\Delta| L_{k_s}\cdots L_{k_1} L_{-k_1}\cdots L_{-k_s}|\Delta\rangle$.

a power of Δ comes from commutator

$$[L_k, L_{-k}] = 2k L_0 + \frac{c}{12}(k^3-k).$$

$\therefore$ The largest power is $\prod_{i=1}^{s} 2k_i \Delta \sim \underset{\vee\atop 0}{C}\, \Delta^s$

• If $\{k\}\neq\{k'\}$, $\langle\Delta| L_{-\{k\}}^{\dagger} L_{-\{k'\}}|\Delta\rangle \sim C\Delta^p$

$$p < \max\{s, s'\} \qquad \left(\begin{matrix}\{k\}=\{k_1\cdots k_s\} \\ \{k'\}=\{k'_1\cdots k'_{s'}\}\end{matrix}\right).$$

• Thus $\| \sum_{\{k\}} C_{\{k\}} L_{-\{k\}}|\Delta\rangle \|^2$ has largest power in Δ

in the terms $\| C_{\{k\}} L_{-\{k\}}|\Delta\rangle \|^2 \sim \underset{\vee\atop 0}{C}\, \Delta^s$

for $\{k\}$'s with the largest "s". i.e.

$$\| \sum_{\{k\}} C_{\{k\}} L_{-\{k\}}|\Delta\rangle \|^2 = \underset{\vee\atop 0}{C}\, \Delta^{\max\{s\}} + \text{lower power in } \Delta.$$

This is positive for large enough Δ. //

⊙ In particular, $M^{(n)}_{\Delta,c}$ is positive definite in the region of (Δ, c) with $\det M^{(n)}_{\Delta,c} > 0$ which is connected to Δ large, c fixed.

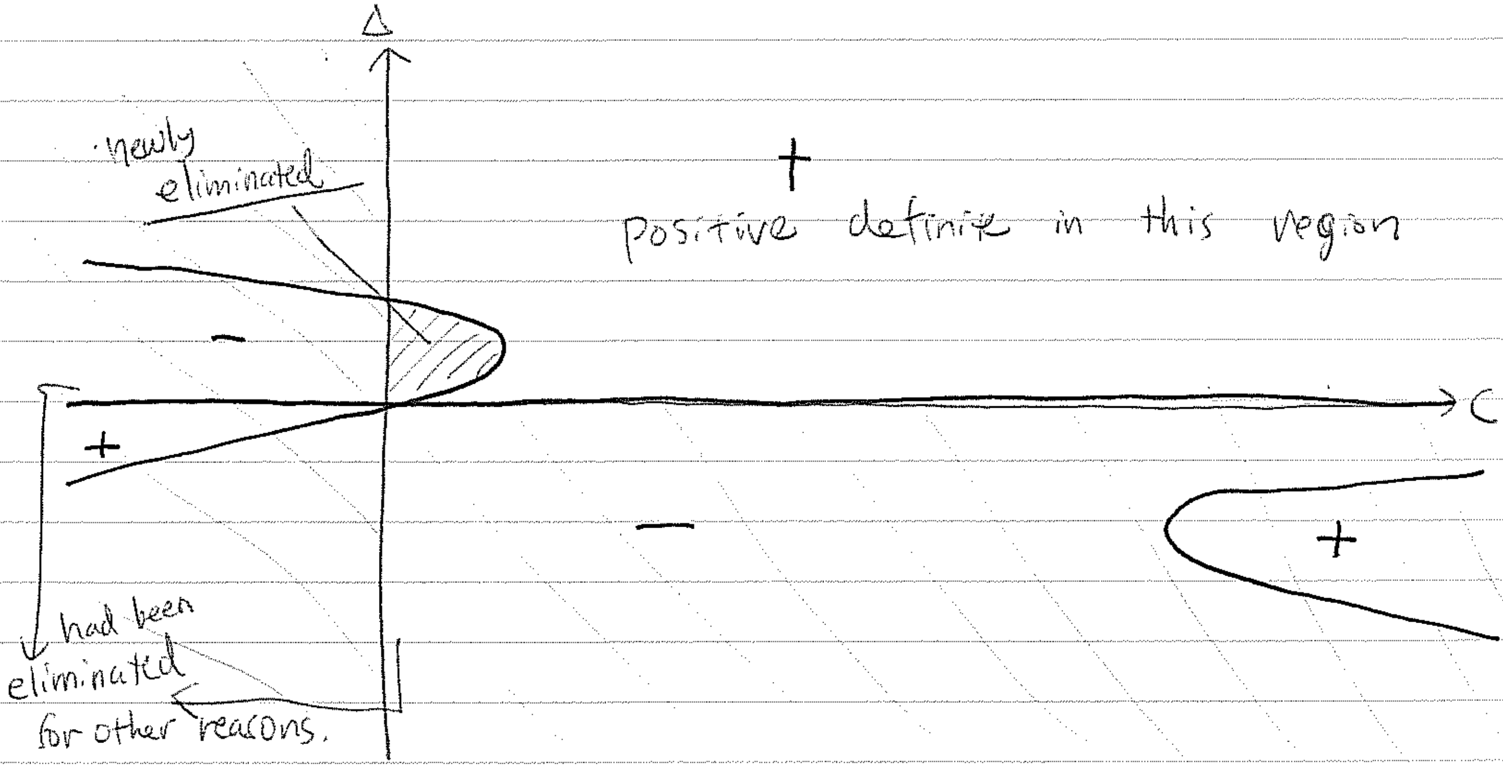
__Example__

$$M^{(2)}_{\Delta,c} = \begin{pmatrix} 4\Delta + \frac{c}{2} & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix}$$

$$\det M^{(2)}_{\Delta,c} = \left(4\Delta + \frac{c}{2}\right) 4\Delta(2\Delta+1) - 36\Delta^2$$

$$= 4\Delta\left(8\Delta^2 + (c-5)\Delta + \frac{c}{2}\right)$$

$$= \frac{1}{8}\Delta\left[\underbrace{(16\Delta + c-5)^2 - \left((c-13)^2 + 12^2\right)}_{\text{hyperbola w asymp. lines } \begin{cases} \Delta = -\frac{1}{2} \\ \Delta = -\frac{c}{8} + \frac{9}{8} \end{cases}}\right]$$



This indeed shows the power of $\det M^{(n)}_{\Delta,c}$!

§ Kac determinant

$$\boxed{\det M_{\Delta,c}^{(n)} = c_n \cdot \prod_{\substack{r,s=1 \\ 1 \leq rs \leq n}}^{n} (\Delta - \Delta_{r,s})^{P(n-rs)}}$$

- $c_n > 0$ a positive number (depends only on n).
- $P(n-rs)$: # of partitions of $n-rs$
- $$\Delta_{r,s} = \frac{\left[(\sqrt{1-c}+\sqrt{25-c})r + (\sqrt{1-c}-\sqrt{25-c})s\right]^2 - 4(1-c)}{6 \cdot 16}$$

- If $\Delta = \Delta_{r,s}$ for some $r, s \geq 1$, a null vector $V_{r,s}$ shows up in $M_{\Delta,c}$ at level rs.
- $V_{r,s}$ is actually a singular vector.
- Descendants of $V_{r,s}$ are again null, and these produce the factor $(\Delta - \Delta_{r,s})^{P(n-rs)}$ at $n > rs$.

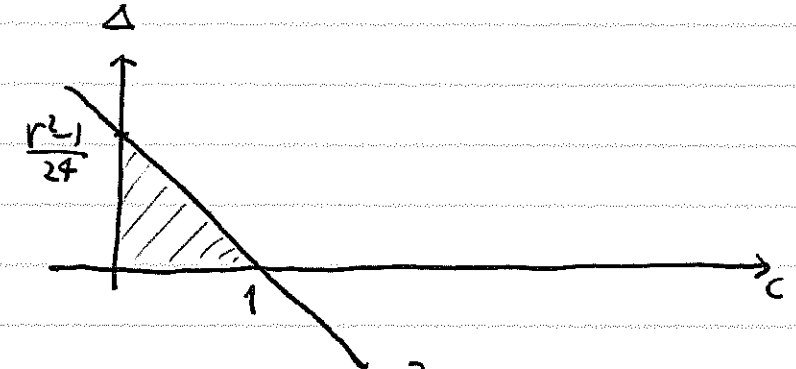
higher level.

We are interested in the sign of $\det M^{(n)}_{\Delta,c}$.

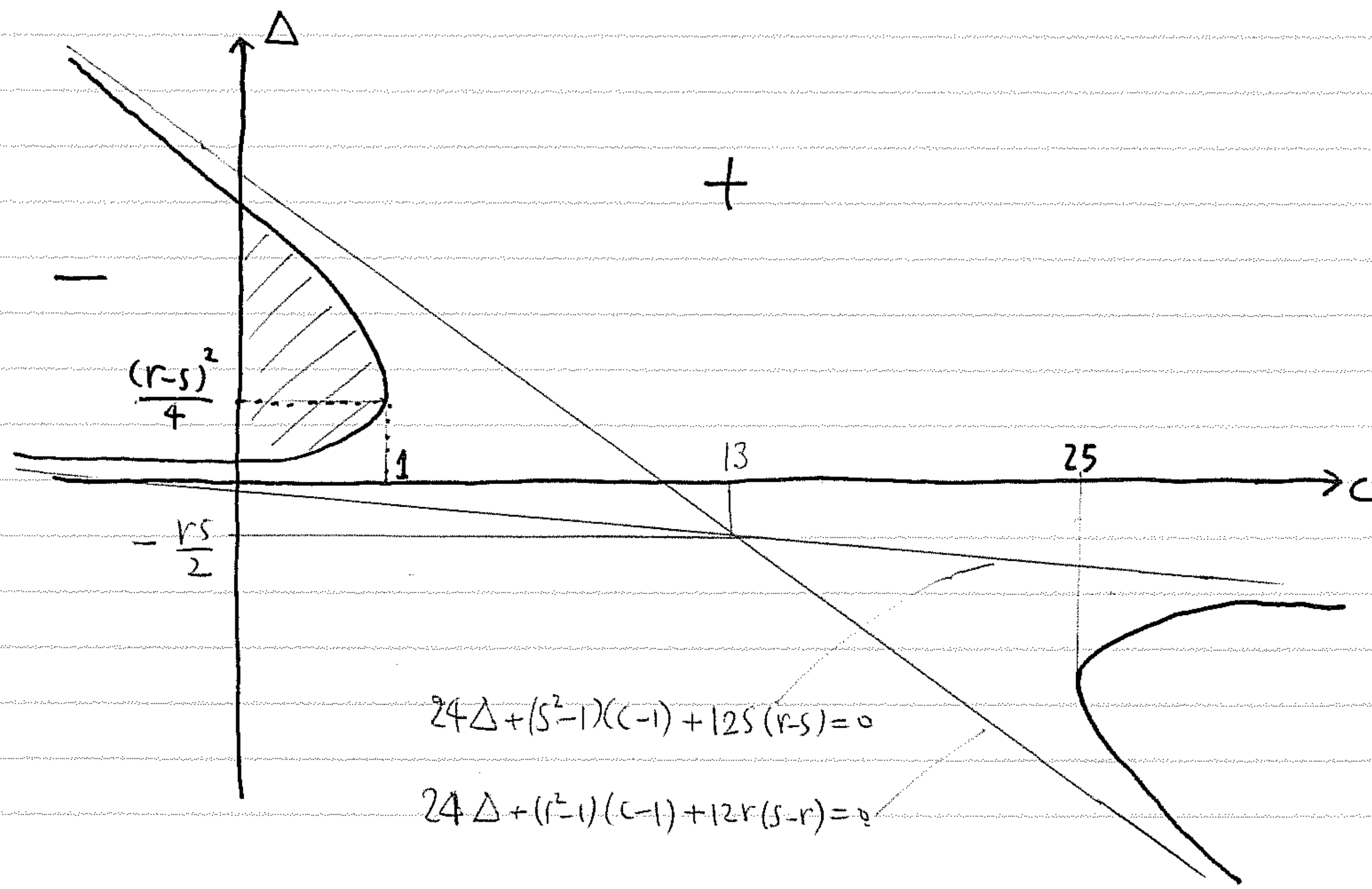
Kac's formula can be written as

$$\det M^{(n)}_{\Delta,c} = c_n \cdot \prod_{r=1}^{n} (\Delta - \Delta_{r,r})^{P(n-r^2)} \cdot \prod_{1 \le s < r \le n} \left[(\Delta - \Delta_{r,s})(\Delta - \Delta_{s,r}) \right]^{P(n-rs)}$$

$$\underline{\Delta - \Delta_{r,r} = \Delta + \frac{r^2-1}{24}(c-1)}$$



$$(\Delta - \Delta_{r,s})(\Delta - \Delta_{s,r}) = \frac{\left[3\cdot 16\,\Delta - (1-c)(r^2+s^2-2) - 12(r-s)^2\right]^2 - (c-13)^2(r^2-s^2)^2 + 12^2(r^2-s^2)^2}{(3\cdot 16)^2}$$



- $c>1,\ \Delta>0$. $\det M^{(n)}_{\Delta,c} > 0 \quad \forall n$

  . connected to $\Delta \to \infty$, c fixed.

  $\therefore$ $M^{(n)}_{\Delta,c}$ is positive definite

  The Verma module $M_{\Delta,c}$ is irreducible and unitary.

- $c=1$ $\quad \det M^{(n)}_{\Delta,c} \geq 0 \quad \forall n$

  $= 0$ for $\Delta = \frac{(r-s)^2}{4}$

  $\Delta \neq \frac{\ell^2}{4}$ $(\forall \ell \in \mathbb{Z}) \Rightarrow$ $M_{\Delta,c}$ is irreducible and unitary.

  $\Delta = \frac{\ell^2}{4}$ $(\ell \in \mathbb{Z}) \Rightarrow$ $M_{\Delta,c}$ is not irreducible ($\exists$ null vector)

  The irreducible repr $M_{\Delta,c}/N$ is

  possibly unitary

  it is actually unitary

☆ Suppose $\exists$ a primary $\mathcal{O}$ with conformal weight Δ in a unitary CFT with central charge c.

Then $M(\mathcal{O}) := \left\{ L_{-\{k\}}\mathcal{O} \;\middle|\; \begin{smallmatrix}\{k\}=\phi \\ \text{or } 1\le k_1\le\cdots\le k_s\end{smallmatrix} \right\}$

$\cong M_{\Delta,c}/N$ as a representation of Virasoro.

<u>proof</u> $M(\mathcal{O})$ must be of the form $M_{\Delta,c}/W$ for some invariant subspace $W \subset M_{\Delta c}$.

We have seen N is maximum $\Rightarrow$ $W \subset N$.

But, in order not to have null states, need $N \subset W$.

$\therefore W = N$ //.

- We know a unitary CFT with $c=1$: Sigma model on S^1_R.

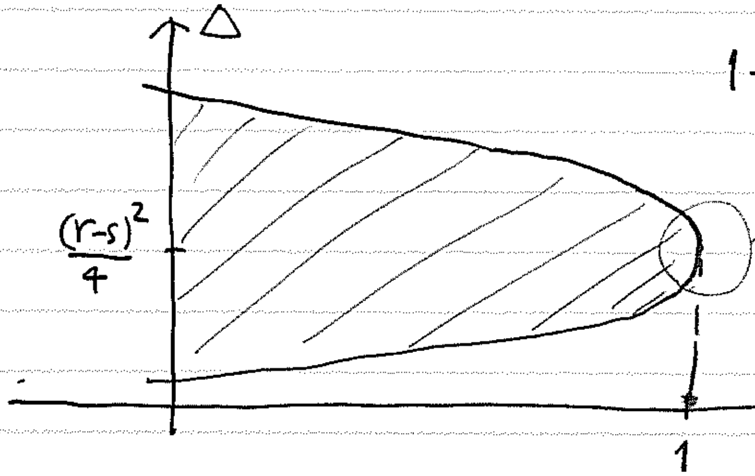
It has a primary $\mathcal{O}_{l,m} = e^{i\frac{l}{R}X} e^{iRm\hat{X}}$ with $\Delta = \frac{1}{4}\left(\frac{l}{R} - Rm\right)^2$

For $R=1$, $\Delta = \frac{1}{4}(l-m)^2$.

$M_{\frac{l^2}{4},1}/N \cong M(\mathcal{O}_{l,0})$ $\therefore$ <u>unitary</u>

Thus we are left with $0 < c < 1$.

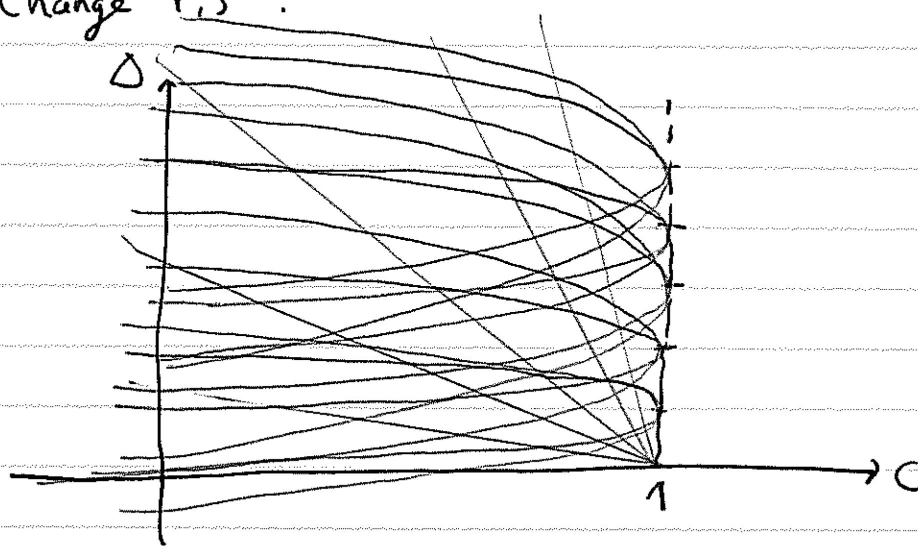
excluded region at $n = rs$:



$$1 - c = \frac{\left(3\cdot 16\Delta - 12(r-s)^2 - (1-c)(r^2+s^2-2)\right)^2}{(25-c)(r^2-s^2)^2}$$

$$\sim \left(\frac{3\cdot 16}{5(r^2-s^2)}\right)^2 \left(\Delta - \frac{(r-s)^2}{4}\right)^2$$

Change r, s :



It looks like all regions $c < 1$ are excluded!

But wait! We know a unitary CFT

with $c = \frac{1}{2}$: Critical Ising model

(Majorana fermion).

§ Theorem [Friedan-Qiu-Shenker]

For $0<c<1$, $M_{\Delta,c}/N$ is unitary only if

$$c = 1 - \frac{6}{m(m+1)} \qquad m=3,4,5,6,\cdots$$

$$\Delta = \Delta_{r,s} = \frac{[r(m+1)-sm]^2-1}{4m(m+1)} \qquad 1\leq s\leq r\leq m-1$$

- In fact these representations are unitary.
  There is an explicit construction (e.g. "Coset construction")
- $m=3 \Rightarrow c=\frac{1}{2}$, $\Delta_{1,1}=0$, $\Delta_{2,1}=\frac{1}{2}$, $\Delta_{2,2}=\frac{1}{16}$
  -- critical Ising!

We will follow the proof of [FQS].

We call the line $\Delta=\Delta_{r,r}$ and the curve $(\Delta-\Delta_{r,s})(\Delta-\Delta_{s,r})=0$ vanishing curves and denote them by $C_{r,r}$, $C_{r,s}$