$$-\bar{\psi}^T A \psi + \bar{\eta}^T \psi + \bar{\psi}^T \eta = -(\bar{\psi}^T - \bar{\eta}^T A^{-1}) A (\psi - A^{-1}\eta) + \bar{\eta}^T A^{-1} \eta$$

$$\therefore Z(A,\eta,\bar{\eta}) = \int d^{2n}\psi \; e^{-(\bar{\psi} - A^{-T}\bar{\eta})^T A (\psi - A^{-1}\eta) + \bar{\eta}^T A^{-1}\eta}$$

$$= \left(\int d^{2n}\psi \; e^{-\bar{\psi}^T A \psi} \right) \cdot e^{\bar{\eta}^T A^{-1} \eta} = Z(A) \cdot e^{\bar{\eta}^T A^{-1}\eta}$$

$$\langle \psi_i \bar{\psi}_j \rangle = \frac{1}{Z(A)} \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \bar{\eta}_i} Z(A,\eta,\bar{\eta}) \Big|_{\eta=\bar{\eta}=0} = A^{-1}_{ij} = \psi_i \bar{\psi}_j$$

$$\langle \psi_i \psi_j \rangle = \frac{1}{Z(A)} \frac{\partial}{\partial \bar{\eta}_i} \frac{\partial}{\partial \bar{\eta}_j} Z(A,\eta,\bar{\eta}) \Big|_{\eta=\bar{\eta}=0} = 0$$

$$\langle \bar{\psi}_i \bar{\psi}_j \rangle = \frac{1}{Z(A)} \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} Z(A,\eta,\bar{\eta}) \Big|_{\eta=\bar{\eta}=0} = 0$$

$$\vdots$$

$$\langle \psi_i \psi_j \bar{\psi}_k \bar{\psi}_l \rangle = \psi_i \psi_j \bar{\psi}_k \bar{\psi}_l + \psi_i \psi_j \bar{\psi}_k \bar{\psi}_l$$

$$= -\psi_i \bar{\psi}_k \, \psi_j \bar{\psi}_l + \psi_i \bar{\psi}_l \, \psi_j \bar{\psi}_k$$

$$= -A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} . \quad \text{etc.}$$

define $:\mathcal{O}(\psi_1, \cdots, \psi_n, \bar{\psi}_1, \cdots \bar{\psi}_n):$ as before

$$\psi_i \psi_j = :\psi_i \psi_j: + \psi_i \psi_j = 0 \qquad \therefore \quad :\psi_i \psi_j: = \psi_i \psi_j$$

$$\psi_i \bar\psi_j = :\psi_i \bar\psi_j: + \psi_i \bar\psi_j \qquad \therefore \quad :\psi_i \bar\psi_j: = \psi_i \bar\psi_j - A^{-1}_{ij}$$

we can ignore $\psi_i \psi_j$, $\bar\psi_i \bar\psi_j$ all the time.

$$\psi_i \bar\psi_j \psi_k \bar\psi_l = :\psi_i \bar\psi_j \psi_k \bar\psi_l: + :\psi_i \bar\psi_j \psi_k \bar\psi_l: + :\psi_i \bar\psi_j \psi_k \bar\psi_l:$$

$$+ :\psi_i \bar\psi_j \psi_k \bar\psi_l: + :\psi_i \bar\psi_j \psi_k \bar\psi_l:$$

$$+ \psi_i \bar\psi_j \psi_k \bar\psi_l + \psi_i \bar\psi_j \psi_k \bar\psi_l$$

$$\langle :\psi_{i_1} \psi_{i_2} \cdots \psi_{i_s} \bar\psi_{j_1} \cdots \bar\psi_{j_r}: \rangle = 0 \qquad s \text{ or } r \geq 1$$

$:\psi_i \bar\psi_j:$ avoids the "singularity" as $i \to j$. (in higher dim)

$$:\psi_i \bar\psi_j: \cdot :\bar\psi_k \bar\psi_l: = \psi_i \bar\psi_j \psi_k \bar\psi_l + :\psi_i \bar\psi_j \psi_k \bar\psi_l:$$

$$+ :\psi_i \bar\psi_j \psi_k \bar\psi_l: + :\psi_i \bar\psi_j \psi_k \bar\psi_l:$$

avoids self contractions $\psi_i \bar\psi_j$, $\psi_k \bar\psi_l$

not pairs

Free field theory of just n anticommuting variables

$$S = \frac{1}{2} \sum_{i,j=1}^{n} \psi_i B_{ij} \psi_j$$

$\hookrightarrow$ we may assume $B_{ij} = -B_{ji}$

B: antisymmetric

$\Rightarrow \exists U \in SO(n, \mathbb{C})$ s.t.

$$B = U \begin{pmatrix} 0 & b_1 & & & & & \\ -b_1 & 0 & & & & & \\ & & 0 & b_2 & & & \\ & & -b_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & b_r \\ & & & & & -b_r & 0 \end{pmatrix} U^{-1} \quad \text{if } n = 2r$$

$$U \left(\begin{array}{cccccc|c} 0 & b_1 & & & & & \\ -b_1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & & 0 & b_r & \\ & & & & -b_r & 0 & \\ \hline & & & & & & 0 \end{array}\right) U^{-1} \quad \text{if } n = 2r+1$$

$\psi_i = \sum_j U_{ij} \psi_j'$: $\quad S = b_1 \psi_1' \psi_2' + b_2 \psi_3' \psi_4' + \cdots + b_r \psi_{2r-1}' \psi_{2r}'$

$$Z(B) = \int \underbrace{d\psi_1 \cdots d\psi_n}_{= \frac{1}{\det U} d\psi_1' \cdots d\psi_n'} \; e^{-\underbrace{\frac{1}{2} \psi^T B \psi}_{= b_1 \psi_1' \psi_2' + \cdots + b_r \psi_{2r-1}' \psi_{2r}'}}$$

$\frac{1}{\det U} = 1$

$$= \int d\psi_1' \cdots d\psi_n' \; e^{-b_1 \psi_1' \psi_2' - \cdots - b_r \psi_{2r-1}' \psi_{2r}'}$$

$$= \begin{cases} b_1 \cdots b_r & \text{if } n = 2r \text{ (even)} \\ 0 & \text{if } n = 2r+1 \text{ (odd)} \end{cases}$$

n even case

$$Z(B) = b_1 \cdots b_r \qquad \text{where } B = U \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & b_r \\ & & & -b_r & 0 \end{pmatrix} U^{-1}$$

$$U \in SO(n)$$

$$\|$$

Pfaffian of B, $Pf(B)$.

$$(Pf(B))^2 = b_1^2 \cdots b_r^2 = \det B.$$

$$\therefore \quad Pf(B) = \pm\sqrt{\det B}$$

but it ~~includes~~ knows which sgn to be taken.

As before, one can show $\langle \psi_i \psi_j \rangle = B^{-1}_{ij} = \overbrace{\psi_i \psi_j}$

$$:\psi_i \psi_j: \overset{\text{def}}{=} \psi_i \psi_j - \overbrace{\psi_i \psi_j}, \quad \text{etc.}$$

$$:\psi_i \psi_j::\psi_k \psi_l: = \text{no self contraction}$$

Free field theory of n commuting complex-variables $\mathbb{C}$

$$S = \sum_{i,j=1}^{n} \bar{z}_i A_{ij} z_j$$

assume A hermitian and positive eigenvalues.

$$Z(A) = \int_{\mathbb{C}^n} d^2 z_1 \cdots d^2 z_n \, e^{-z^\dagger A z}$$

$$= \frac{\pi^n}{\det A}$$

$$\langle z_i z_j \rangle = \langle \bar{z}_i \bar{z}_j \rangle = 0$$

$$\langle z_i \bar{z}_j \rangle = A^{-1}_{ij}$$

1-d QFT (Quantum Mechanics)

Variables are functions of t = time.

e.g. $X(t)$ real valued function

$$S(X) = \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \left\{ \frac{1}{2}\left(\frac{dX}{dt}\right)^2 - V(X) \right\} dt$$

Lagrangian

— a particle in $\mathbb{R}$ under influence of potential $V(X)$

Equation of motion (E.O.M.): extremize $S(X)$

with fixed boundary value $X(t_1)$, $X(t_2)$.

$$\delta S = \int_{t_1}^{t_2} \left\{ \frac{dX}{dt}\frac{d\delta X}{dt} - V'(X)\delta X \right\} dt$$

$$= \int_{t_1}^{t_2} \delta X \left(-\frac{d}{dt}\frac{dX}{dt} - V'(X) \right) dt = 0$$

$$\Rightarrow \boxed{\frac{d^2X}{dt} = -V'(X)}$$

Now, consider the space of trajectories

$$\mathcal{F}(x_2, t_2 ; x_1, t_1) = \left\{ x : [t_1, t_2] \to \mathbb{R} \;\middle|\; \begin{array}{l} x(t_1) = x_1 \\ x(t_2) = x_2 \end{array} \right\}$$

& Compute the integral :

$$Z(x_2, t_2 ; x_1, t_1) = \int_{\mathcal{F}(x_2, t_2 ; x_1, t_1)} \mathcal{D}x \; e^{iS(x)}$$

$$|Z(x_2, t_2 ; x_1, t_1)|^2 = \text{probability of finding a particle}$$

at x_2 on $t = t_2$

if it was at x_1 on $t = t_1$.

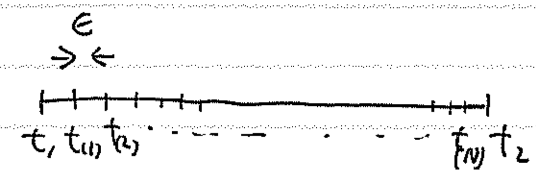
<u>But</u> What is this space $\mathcal{F}(x_2, t_2 ; x_1, t_1)$ precisely ?

& How do we define the measure $\mathcal{D}x$?

Two ways

① Discretization (Lattice theory)

Replace $[t_1, t_2]$ by a finite # of points



$$\mathcal{F} \underset{\rightsquigarrow}{\text{replace}} \mathbb{R}^N \ni (X_{(1)}, \cdots, X_{(N)}) \quad ; \quad X_{(i)} = X(t_{(i)})$$

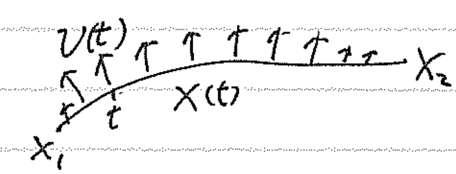
$$S(X) \rightsquigarrow \sum_{i=0}^{N} \epsilon \left\{ \frac{1}{2} \left(\frac{X_{(i+1)} - X_{(i)}}{\epsilon} \right)^2 - V(X_i) \right\} \qquad \begin{array}{l} X_{(0)} = X_1 \\ X_{(N+1)} = X_2 \end{array}$$

After the computation, take the limit $N \to \infty$.

② Continuum.

Introduce a metric on $\mathcal{F}(X_2, t_2 ; X_1, t_1)$

$$v \in T_{X(t)} \mathcal{F}(X_2, t_2 ; X_1, t_1)$$



$$(v, v) = \int_{t_1}^{t_2} dt \, v(t)^2$$

$\mathcal{D}X$ = Riemannian volume form of $\mathcal{F}(X_2, t_2 ; X_1, t_1)$

Euclidean Version:

Wick rotation $\boxed{t \to -i\tau}$

$$S(X) = \int dt \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 - V(X) \right)$$

$$\to \int -i\,d\tau \left(-\frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 - V(X) \right) =: i\,S_E(X)$$

$$S_E(X) = \int d\tau \left\{ \frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 + V(X) \right\}$$

$$\underbrace{Z_E(X_2,\tau_2;X_1,\tau_1)}_{} = \int \mathcal{D}X\, e^{-S_E(X)}$$

$\mathcal{F}(X_2,\tau_2;X_1,\tau_1)$

If $V(X)$ ~~positive~~ grows at $X=\pm\infty$, $S_E(X) \geq 0$
≥ 0 and

and $e^{-S_E(X)}$ works as a convergence factor.