§ Theorem [Friedan-Qiu-Shenker]

For $0<c<1$, $M_{\Delta,c}/N$ is unitary <u>only if</u>

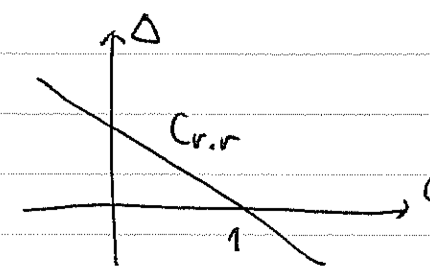
$$c = 1 - \frac{6}{m(m+1)} \qquad m=3,4,5,6,\cdots$$

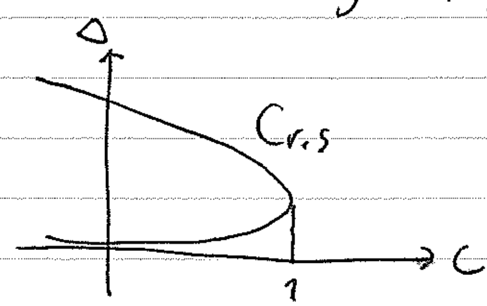
$$\Delta = \Delta_{r,s} = \frac{[r(m+1)-sm]^2-1}{4m(m+1)} \qquad 1\leq s\leq r\leq m-1$$

- In fact these representations <u>are</u> unitary.
  There is an explicit construction (e.g. "coset construction")
- $m=3 \Rightarrow c=\frac{1}{2}$, $\Delta_{1,1}=0$, $\Delta_{2,1}=\frac{1}{2}$, $\Delta_{2,2}=\frac{1}{16}$
  .. critical Ising !

We will follow the proof of [FQS].

We call the line $\Delta=\Delta_{r,r}$ and the curve $(\Delta-\Delta_{r,s})(\Delta-\Delta_{s,r})=0$ <u>vanishing curves</u> and denote them by $C_{r,r}$, $C_{r,s}$

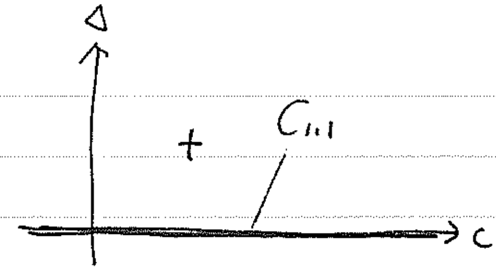




level 1

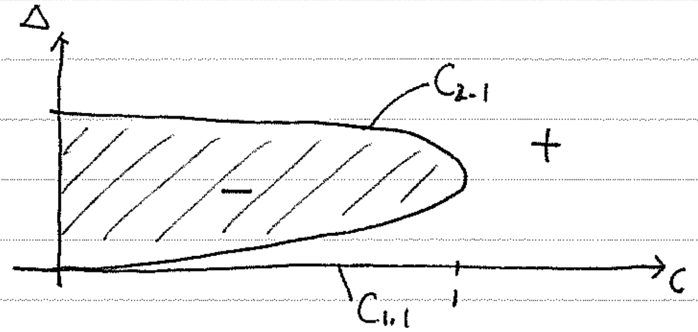
$\det M^{(1)} = \Delta = C_{1,1}$

We have a singular vector $v_{1,1}$ on $C_{1,1}$



level 2

$\det M^{(1)} = \Delta\,(\Delta-\Delta_{2,1}^{(c)})(\Delta-\Delta_{1,2}^{(c)}) = \Delta\cdot C_{2,1}(\Delta,c)$

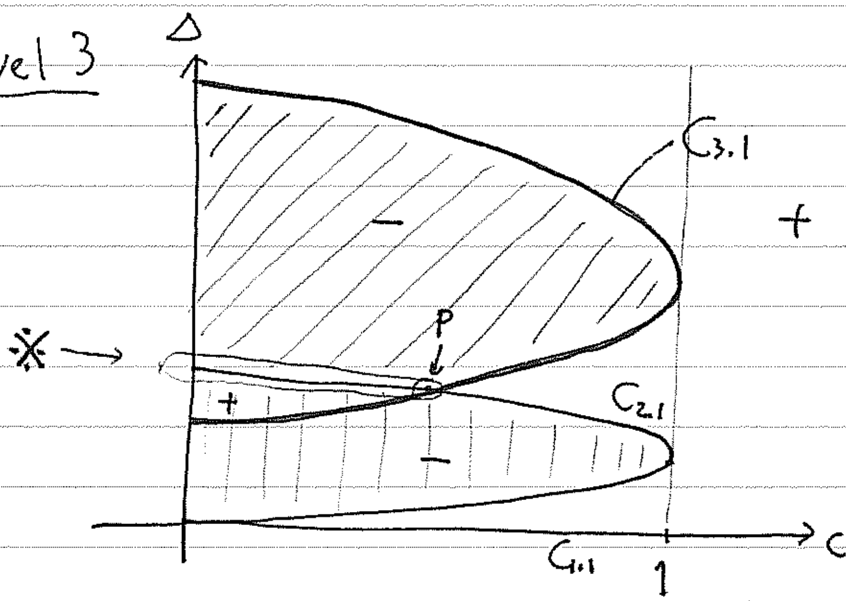


Interior is eliminated but NOT ("yet") the points <u>on $C_{2,1}$</u>

We have a singular vector $v_{2,1}$ on $C_{2,1}$

and a null vector $L_{-1}v_{1,1}$ on $C_{1,1}$

level 3



$\det M^{(2)} = \Delta^2\cdot C_{2,1}\cdot C_{3,1}$

/// region is newly eliminated (|||| already eliminated).

We have a singular vector $v_{3,1}$ on $C_{3,1}$, a null $L_{-1}v_{2,1}$ on $C_{2,1}$

and two nulls $L_{-2}v_{1,1}$, $L_{-1}^2 v_{1,1}$ on $C_{1,1}$.

What about the segment $\cdot\!\times\!\cdot$ of $C_{2,1}$? $\left(\Delta=\frac{1}{2},\ c=\frac{1}{2}\right)$

Let us look at the Verma module M_p at the intersection point p.

Claim The singular vector $V_{3,1}$ and the null vector $L_{-1}V_{2,1}$ are independent.

☺ Let us consider the Kac determinant for the Verma module $M(V_{2,1})$.

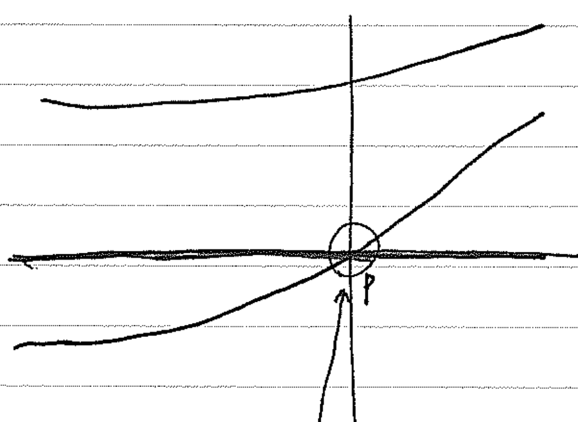
Note $\Delta(V_{2,1}) = \Delta_p + 2$. ($V_{2,1}$ appears as a singular vector at level 2 in M_{Δ_p})

level 1 in $M(V_{2,1})$

$$M^{(1)}_{\Delta_p+2,\,c_p} = 2(\Delta_p+2) \neq 0 \quad \left(\because \Delta_p > 0 \ \left(\text{it is } \tfrac{1}{2}\right)\right)$$

Thus $M(V_{2,1})$ does not have a singular vector at level 1 //

$\therefore$ The eigenvector that goes negative in the region inside $C_{3,1}$ is distinct from the descendant of $V_{2,1}$.



← Plot of eigenvalues of $M^{(3)}_{\Delta,c}$ along $C_{2,1} \ni (\Delta, c)$

← zero from $L_{-1}V_{2,1}$

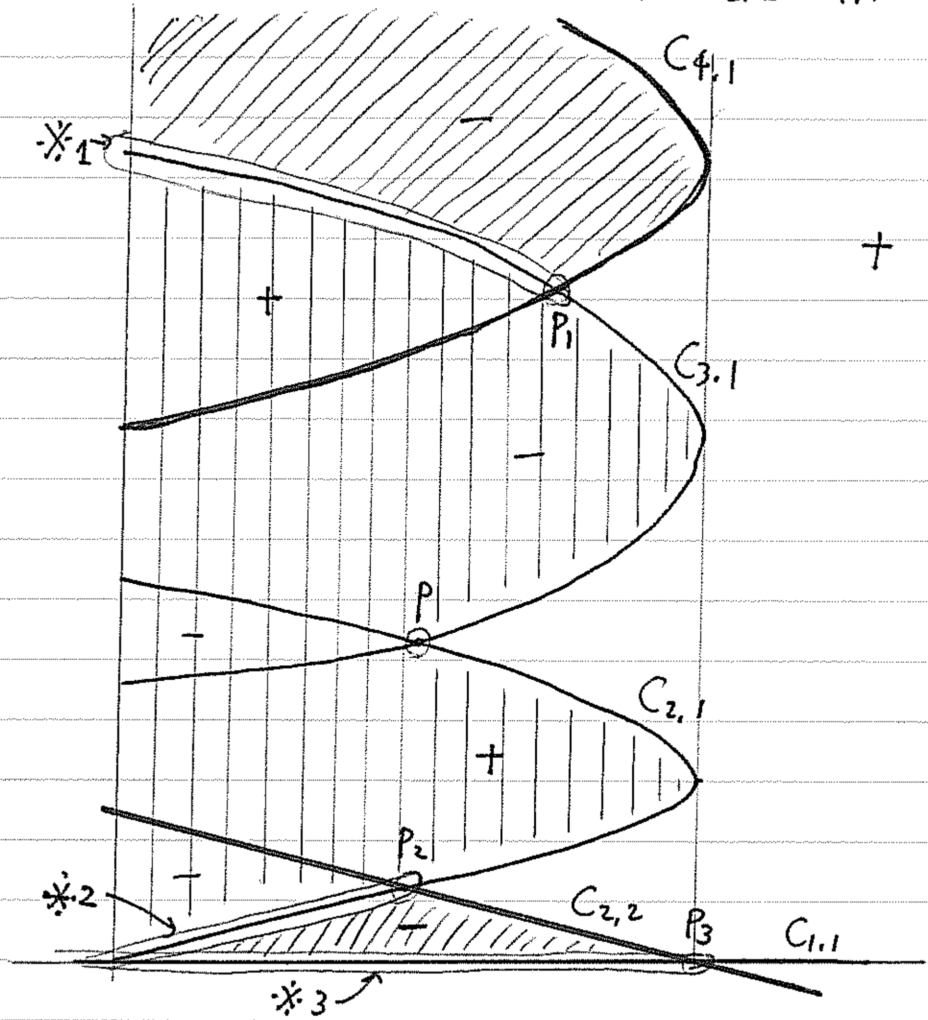
another zero from $V_{3,1}$

$\exists$ negative norm vector

$\boxed{\therefore \text{The open segment of } C_{2,1} \text{ to the left of } p \text{ is eliminated.}}$

(point p ("still") remains).

level 4 $\quad \det M^{(4)} = \Delta^3 C_{2,1}^2 C_{3,1} C_{2,2} C_{4,1}$



newly eliminated

already eliminated.

$P_1 = (\Delta = \frac{3}{2}, C = \frac{7}{10})$ [m=4]

$P_2 = (\Delta = \frac{1}{16}, C = \frac{1}{2})$ [m=3]

$P_3 = (\Delta = 0, C = 1)$

Open ※1 (left of P_1 on $C_{3,1}$) eliminated — P_1 remains

Open ※2 (left of P_2 on $C_{2,1}$) eliminated — P_2 remains

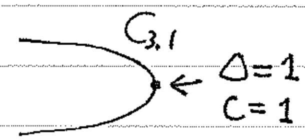
} for the same reason as ※ (left of p on $C_{2,1}$).

※3 **NOT** ("yet") eliminated!

$v_{2,2}$ can be a descendant of $v_{1,1}$. $\left(v_{2,2} \in M(v_{1,1})^{(3)} \right)$ — level 4−1=3

(actually is)

$$\begin{cases} \cdot\ \Delta(v_{1,1}) = 0+1 = 1 \\ \cdot\ \det M^{(3)}_{\underbrace{\Delta(v_{1,1})}_{=1},\, C=1} = 0 \end{cases}$$



Digression For $c=1$, $\Delta = \frac{\ell^2}{4}$ ($\ell \in \mathbb{Z}_{\geq 0}$), the set of (r,s) $r \geq s$

s.t. $\Delta = \Delta_{r,s}$ is $\{(\ell+1,1), (\ell+2,2), \dots\}$.

In fact, we have the inclusion relation

$$M_{\frac{\ell^2}{4},1} \supset M(v_{\ell+1,1}) \supset M(v_{\ell+2,2}) \supset M(v_{\ell+3,3}) \supset \cdots$$

☺ Note $\Delta(v_{\ell+r,r}) = \frac{\ell^2}{4} + (\ell+r)r = \frac{(\ell+2r)^2}{4}$

Relative level of $v_{\ell+r+1,r+1}$ & $v_{\ell+r,r}$ is

$$(\ell+r+1)(r+1) - (\ell+r)r = \ell+2r+1$$

$$\det M^{(\ell+2r+1)}_{\frac{(\ell+2r)^2}{4},1} = \prod_{1 \leq r's' \leq \ell+2r+1} \left(\frac{(\ell+2r)^2}{4} - \frac{(r'-s')^2}{4} \right)^{P(\ell+2r+1-r's')}$$

$$= 0$$

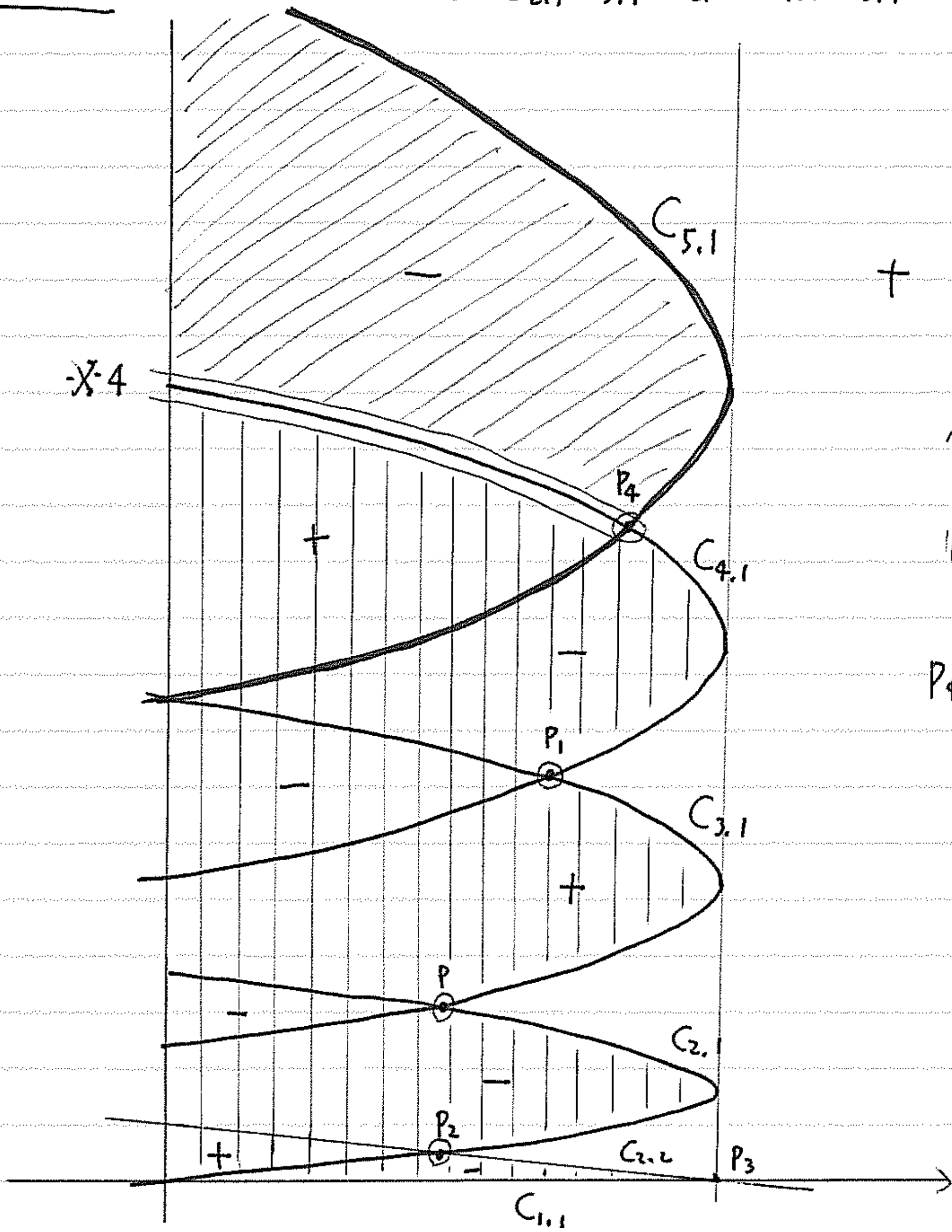
$$\uparrow$$

$r' = \ell+2r+1$

$s' = 1$

Thus indeed $M(v_{\ell+r,r})$ has a singular vector at the level of $v_{\ell+r+1,r+1}$. //

level 5 $\quad \det M^{(5)} = \Delta^5 C_{2,1}^3 C_{3,1}^2 C_{2,2} C_{4,1} C_{5,1}$



newly eliminated

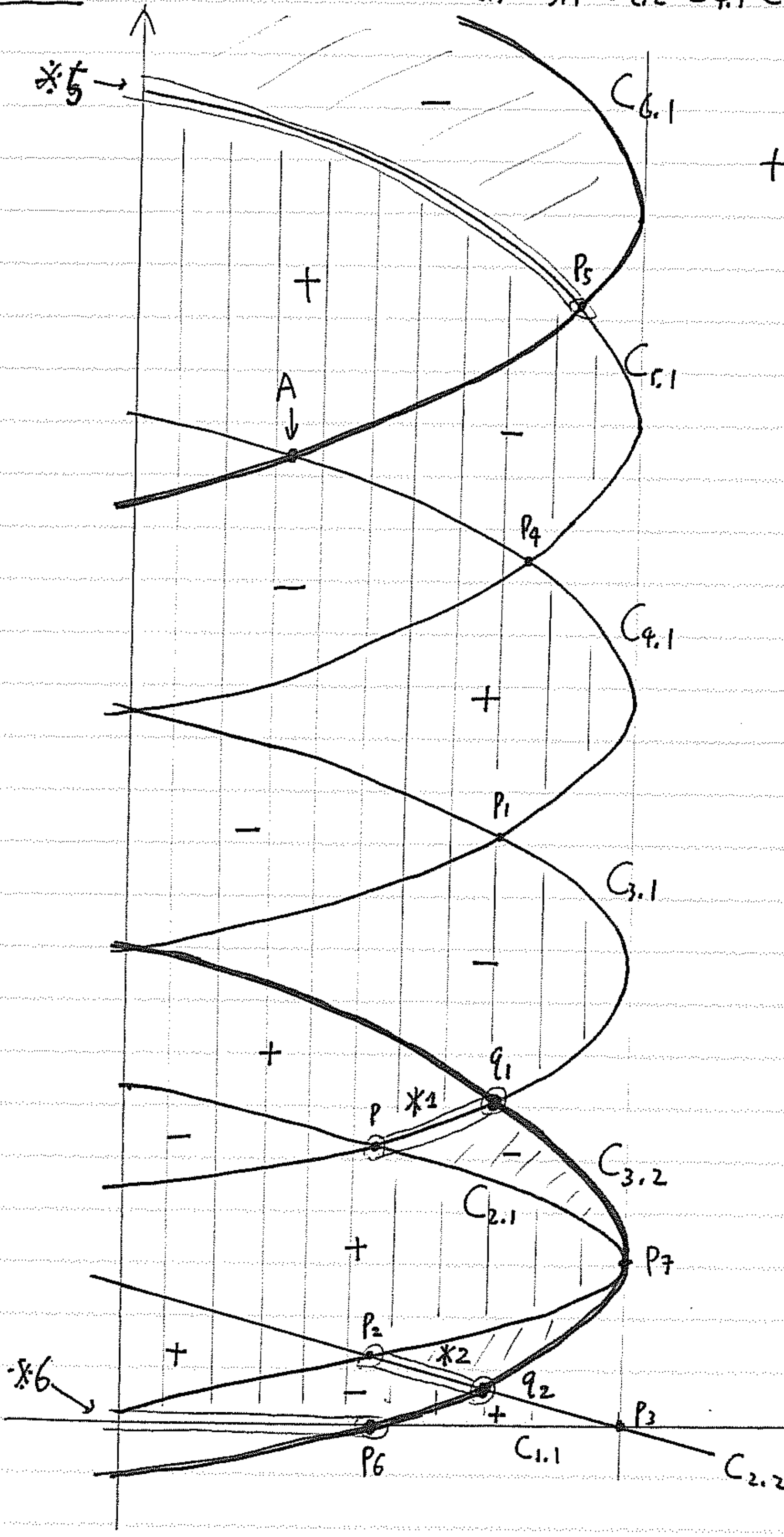
already eliminated

$P_4 : \Delta = 3,\ C = \frac{4}{5} \quad [m=5]$

Open ※4 (left of p_4 on $C_{4,1}$) is eliminated, p_4 ("still") remains.

level 6

$$\det M^{(6)} = \Delta^7 C_{2,1}^5 C_{3,1}^3 C_{2,2}^2 C_{4,1}^2 C_{5,1} C_{3,2} C_{6,1}$$



//// newly eliminated

|||| already eliminated

P_5 : $c=\frac{6}{7}$, $\Delta=5$, $(m=6)$

P_6 : $c=\frac{1}{2}$, $\Delta=0$, $(m=3)$

q_1 : $c=\frac{7}{10}$, $\Delta=\frac{3}{5}$ $(m=4)$

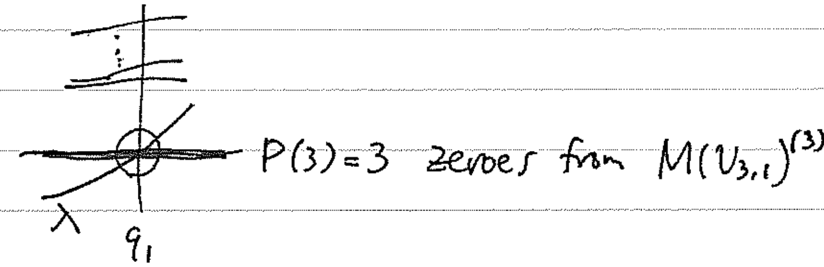
q_2 : $c=\frac{7}{10}$, $\Delta=\frac{3}{80}$ $(m=4)$

Open ※5 (left of p_5 on $C_{5,1}$) eliminated, p_5 remains.

Open ※6 (left of p_6 on $C_{1,1}$) eliminated, p_6 remains.

<u>※1</u> a segment of $C_{3,1}$ between p and q_1

$v_{3,1} \notin M(v_{3,1})^{(3)}$. $\Rightarrow \exists$ a single negative eigenvalue λ to the immediate left of q_1.

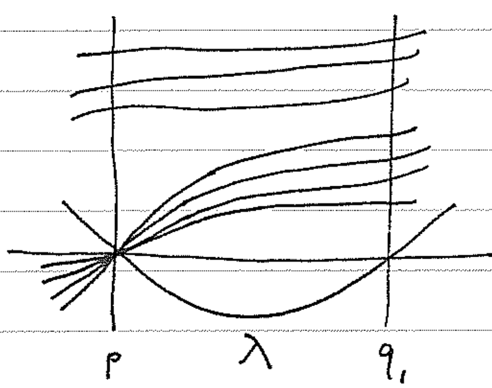


It continues to be negative at least until p, since there is no new zero there.

Thus the open segment between p and q_1 is eliminated.

At p there are $P(4)=5$ zeroes from $M(v_{2,1})^{(4)}$.

There is a possibility that λ is one of them:



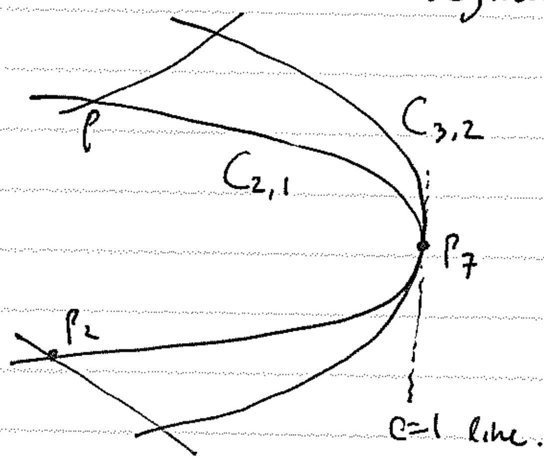
Because of this possibility, p is NOT ("yet") eliminated

$\therefore$ $\boxed{\text{$p$ and q_1 "still" remains, but the open segment between them is eliminated}}$

For the same reason

> p_2 and q_2 "still" remain, but
> the open segment between them is eliminated

What about the segments of $C_{2,1}$ between



p and p_7 & p_2 and p_7 ?

$p_7 \in C_{2,1} \cap C_{3,1}$

As we have seen,

$U_{3,2}$ is a descendant of $U_{2,1}$.

Thus, these segments survive.

General proof

The curve $C_{r,s}$ appears at level $n = rs$ and remains at all higher levels.

Def A first intersection on $C_{r,s}$ at level $n' \geq n$ is an intersection point with another curve at level n' which has the largest value of C.

Remark
- For "the largest C-value", we consider upper and lower parts of $C_{r,s}$ separately. So, $C_{r,s}$ may have two first intersections at each level.
- We only consider intersection points in $0 < C < 1$. Thus, we exclude $\Delta = \frac{\ell^2}{4}$, $C = 1$. (for $C_{\ell+r,r} \cap C_{\ell+s,s}$)

e.g At level 3, p is a first intersection for both $C_{3,1}$ & $C_{2,1}$

At level 4, p, p_1, p_2 are first int. for respective curves. (p_3 is excluded as a rule).

At level 6, p_5, p_4, p_1, q_2, p_6 are first intersections. A, p, p_2 are non-first intersections. (p_3, p_7 are not counted as intersections)

- When the curve $C_{r,s}$ appears at level $n=rs$, it has at most two first intersections.

  The segment to the left of a first intersection is in the region already considered at lower level.

  [It is eliminated except possibly the points of intersection with other curves.]

- When a new curve $C_{r',s'}$ appears at level $n'=r's'>n$ and makes a new first intersection on $C_{r,s}$ by crossing it, <u>we claim</u> that the open segment on $C_{r,s}$ between the new and the previous first intersections is eliminated.

  [or the whole segment to the left of the new first int. if there were no previous intersections]

- To prove this, it is enough to show that, at the new first intersection point, the singular vector $v_{r',s'}$ is <u>not</u> a descendant of $v_{r,s}$, i.e. $v_{r's'} \notin M(v_{r,s})^{(n'-n)}$.

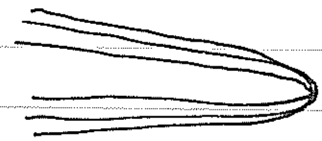
  [This will be explicitly checked after we <u>classify first intersections</u>.]

- Thus, only the first intersections remain.

  i.e. <u>Only the first intersections may possibly have unitary $M_{\Delta,c}/N$.</u>

Classification of first intersections

Note that the curves $C_{r,s}$ are classified by the value $|r-s|$.

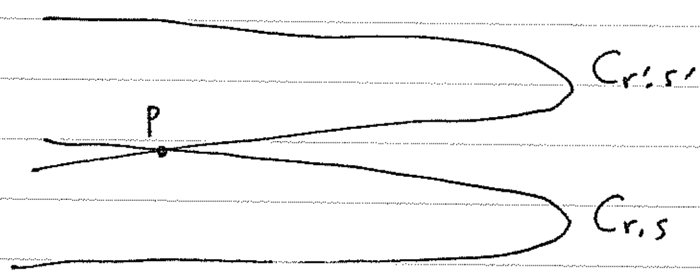


$\leftarrow$ Curves of the same $|r-s|$.

— They form a family.

Intuitively, first intersections are between curves from ~~the~~ adjacent families. This is in fact true, and can be proved as follows.

Suppose p is an intersection of curves $C_{r,s}$, $C_{r',s'}$ which are not from adjacent families.



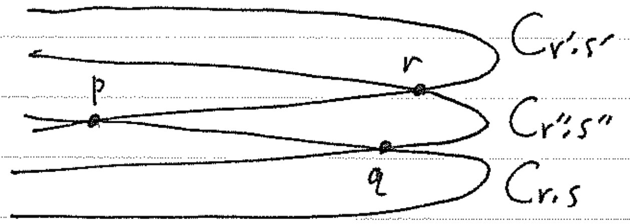
$\leftarrow$ assume this situation with $r \geq s$, $r' \geq s'$

so $r'-s' > r-s+1$

p is determined by $-\{(m_p+1)s' - m_p r'\} = (m_p+1)r - m_p s$

i.e. $m_p(r'-s'-(r-s)) = r+s'$.

Then we can find another curve $C'' = C_{r'',s''}$ at level lower than either $n = rs$ or $n' = r's'$, so that it has intersection with $C_{r,s}$ or $C_{r',s'}$ with larger value of c.



For q, take $r'' = r'-1$, $s'' = s'$. Then $r''s'' < r's'$, and

$$m_q(\underbrace{r''-s''-(r-s)}_{r'-s'-(r-s)-1}) = \underbrace{r+s''}_{r+s'} \quad \therefore \boxed{m_q > m_p} \quad \boxed{\therefore C_q > C_p}$$

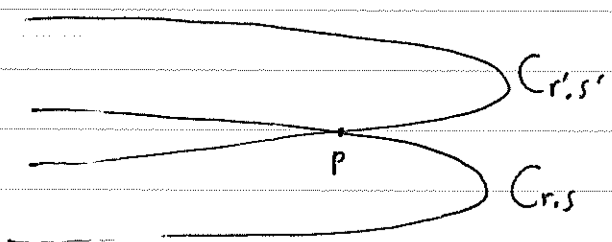
For r, take $r'' = r$, $s'' = s-1$. Then $r''s'' < rs$, and

$$m_r(\underbrace{r'-s'-(r''-s'')}_{r'-s'-(r-s)-1}) = \underbrace{r''+s'}_{r+s'} \quad \therefore m_r > m_p \quad \therefore \boxed{C_r > C_p}$$

This means that p cannot be a first intersection.

//.

Suppose $C_{r,s}$ and $C_{r',s'}$ are from adjacent families.



$$r'-s' = r-s+1$$

(assume $r' \geq s'$, $r \geq s$)

Then m_p is determined by $m_p(\underbrace{r'-s'-(r-s)}_{= 1}) = r+s'$

i.e. $\boxed{m_p = r+s'}$ It is an integer ≥ 2 !

Also, $s' \geq 1 \Rightarrow \boxed{1 \leq s \leq r \leq m_p - 1}$

Remark The equation for $m = m_p, r, s, r', s'$ can also be written as

$$-\{(m+1)s' - mr'\} = (m+1)r - ms.$$

i.e. $(m+1)(r+s') = m(s+r')$

For $m \in \mathbb{Z}_{\geq 2}$, m and $m+1$ are coprime integers.

$\Rightarrow$ $r+s' = nm$ & $s+r' = n(m+1)$ for some $n \in \mathbb{Z}$.

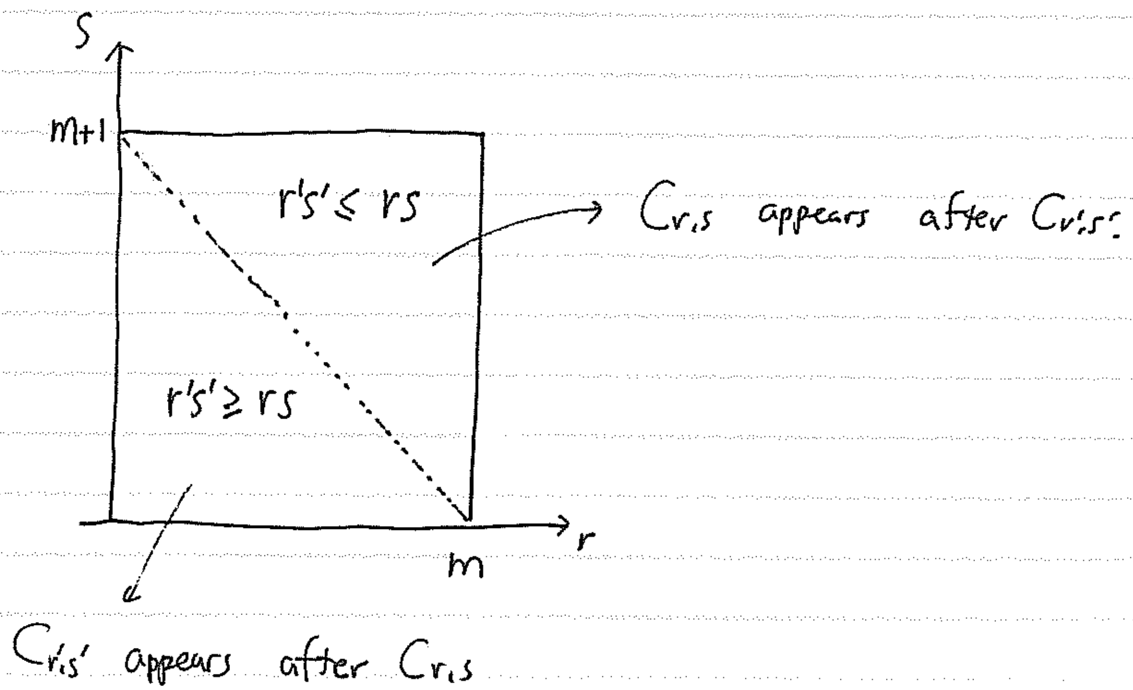
Since $r'-s' = r-s+1$, we must have $n=1$.

Thus, we find

$$\boxed{(r', s') = (m+1-s,\ m-r)}$$

The Relative level $r's' - rs = m(m+1) - r(m+1) - sm$.

This can be positive or negative.



It remains to show that, when a new curve $C_{r',s'}$ appears at level $n'=r's'>n=rs$ and makes a new first intersection p by crossing $C_{r,s}$, the singular vector $v_{r',s'}$ is not a descendant of $v_{r,s}$ (@ p).

i.e. $v_{r',s'} \notin M(v_{r,s})^{(n'-n)}$.

[We know that either $|r'-s'|-|r-s|=+1$ or -1.
Assume the former (the latter case can be treated similarly).
Then, we know $(r',s')=(m+1-s, m-r)$.]

Use Kac's formula:

$$\det M^{(n'-n)}_{\underbrace{\Delta(v_{r,s})}_{\Delta_{r,s}+rs}, C_p} = \prod_{1\le r''s''\le n'-n} \left(\Delta_{r,s}+rs-\Delta_{r'',s''}\right)^{P(n'-n-r''s'')}$$

We want to show $\Delta_{r,s}+rs \neq \Delta_{r'',s''}$ if $1\le r''s''\le n'-n$

$$\frac{((m+1)r+ms)^2-1}{4m(m+1)} \qquad \frac{((m+1)r''-ms'')^2-1}{4m(m+1)}$$

i.e. $\left|(m+1)r+ms\right| \overset{(*)}{=} \left|(m+1)r''-ms''\right|$ if $r''s''\le r's'-rs$.

Since m and $(m+1)$ are coprime, a general solution for $(*)$ is

$(r'',s'') = \pm(r,-s)\pm n(m,m+1)$. $\pm n \ge 1$ is needed for $r'',s''\ge 1$.

$$r''s'' \leq \underbrace{r's' - rs}_{} \quad ?$$

$$\| \qquad = (m+1-s)\overset{-rs}{(m-r)} = m(m+1) - sm - r(m+1)$$

$$(r+nm)(-s+n(m+1)) = -rs - nms + n(m+1)r + n^2 m(m+1)$$

$$\therefore \ (n^2-1)m(m+1) + (n+1)r(m+1) - s(n-1)m \overset{?}{\leq} rs.$$

$\boxed{n=1}$: $\quad 2r(m+1) \overset{?}{\leq} rs \qquad$ No!

$\boxed{n=-1}$: $\quad 2sm \overset{?}{\leq} rs \qquad$ No!

$\boxed{|n| \geq 2}$: $\quad$ impossible.

Thus there is no solution.

i.e. $\det M^{(n'-n)}_{\Delta(v_{r,s}),\,c_p} \neq 0.$

i.e. $M(v_{r,s})$ @ p has no singular vector at level $n'-n$.

$\therefore$ $v_{r',s'}$ cannot be a descendant of $v_{r,s}$.

Q.E.D.

This complete the proof of [FQS] Theorem.