§ Inclusion Relation of Verma Modules

Suppose (Δ_0, c_0) is a unitary point $(0 < c_0 < 1)$

<u>Q</u> : What is the set of all singular vectors in M_{Δ_0, c_0} ?

By Kac's formula, $\exists$ one $v_{r,s}$ at level rs for each (r,s) s.t.

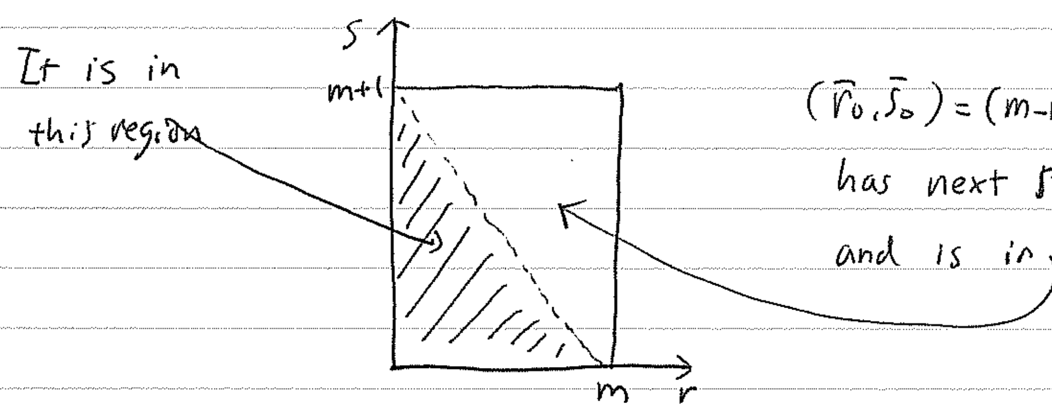
$$\Delta_0 = \Delta_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \quad \text{for } c_0 = 1 - \frac{6}{m(m+1)}.$$

$$\left[\text{It has } \Delta(v_{r,s}) = \Delta_{r,s} + rs = \frac{((m+1)r + ms)^2 - 1}{4m(m+1)}. \right]$$

<u>Note</u> : (r,s) is a solution $\Rightarrow$ $\pm(r,s) + n(m, m+1)$ are also solutions

Since m and $m+1$ are coprime these are the only solutions.

Let's denote by (r_0, s_0) the solution that minimizes rs.

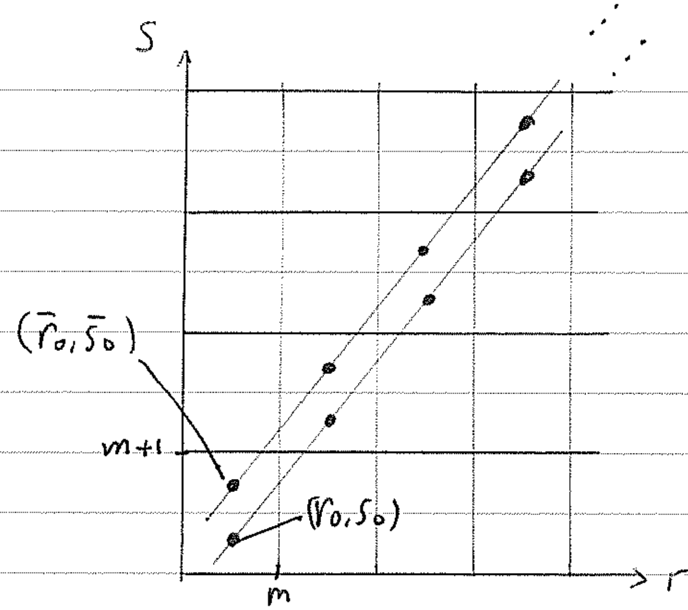


$(\bar{r}_0, \bar{s}_0) = (m - r_0, m+1-s_0)$ has next smallest rs and is in

A general soln is $(r,s) = (r_0, s_0) + n(m, m+1)$ or $(\bar{r}_0, \bar{s}_0) + n(m, m+1)$ $n \geq 0$

or

$$= \pm(r_0 + nm, s_0 + n(m+1)) \quad \begin{matrix} n \geq 0 \\ n \leq -1. \end{matrix}$$



Are there other singular vectors? — YES, $\exists$ another set, detected by Kac formula for $M(v_{r_0,s_0})$:

$\exists$ a singular vector $v'_{r,s} \in M(v_{r_0,s_0})$ at level rs $\left(\begin{array}{c}\text{level } r_0s_0+rs \\ \text{in } M_{\Delta_0,c_0}\end{array}\right)$ for each (r,s) s.t. $\Delta(v_{r_0,s_0}) = \Delta_{r,s}$.

i.e. $\left|(m+1)r_0 + ms_0\right| = \left|(m+1)r - ms\right|$.

Solutions are $(r,s) = \pm(r_0+nm, -s_0+n(m+1))$ $\begin{array}{l} n \geq 1 \\ n \leq -1. \end{array}$

denote $(r_1, s_1) = (r_0+m, -s_0+m+1)$

$(\bar{r}_1, \bar{s}_1) = (-r_0+m, s_0+m+1)$.

<u>A</u>: $v_{r,s}$ and $v'_{r,s}$ are all of the singular vectors of M_{Δ_0,c_0}.

By studying Kac determinants, we find the following inclusion relation :

$$
\begin{tikzcd}
 & M_{\Delta_0, c_0} \arrow[dl] \arrow[dr] & \\
M(v_{r_0, s_0}) \arrow[d] \arrow[drr] & & M(v_{\bar{r}_0, \bar{s}_0}) \arrow[d] \arrow[dll] \\
M(v^1_{r_1, s_1}) \arrow[d] \arrow[drr] & & M(v^1_{\bar{r}_1, \bar{s}_1}) \arrow[d] \arrow[dll] \\
M(v_{r_0+m, s_0+(m+1)}) \arrow[d] \arrow[drr] & & M(v_{\bar{r}_0+m, \bar{s}_0+m+1}) \arrow[d] \arrow[dll] \\
M(v^1_{r_1+m, s_1+m+1}) \arrow[d] \arrow[drr] & & M(v^1_{\bar{r}_1+m, \bar{s}_1+m+1}) \arrow[d] \arrow[dll] \\
M(v_{r_0+2m, s_0+2(m+1)}) & & M(v_{\bar{r}_0+2m, \bar{s}_0+2(m+1)}) \\
\vdots & & \vdots
\end{tikzcd}
$$

where $M_1 \to M_2$ means $M_2 \subset M_1$.

In particular, the subspace of null vectors N
($\equiv$ the max inv. subspace)

$$= M(v_{r_0,s_0}) + M(v_{\bar{r}_0,\bar{s}_0}).$$

Also,
$$\begin{tikzcd} M_1 \arrow[d] \arrow[dr] & M_2 \arrow[d] \arrow[dl] \\ M_3 & M_4 \end{tikzcd}$$
means $M_1 \cap M_2 = M_3 + M_4$

☺ If M_1 & M_2 are invariant subspaces, then $M_1 \cap M_2$ is also an inv. subspace. By Feigin-Fuchs Thm, $M_1 \cap M_2$ must be a sum of Verma modules generated by singular vectors. Since $M_3 \subset M_1 \cap M_2$, $M_4 \subset M_1 \cap M_4$, and all other singular vectors in $M_1 \cap M_2$ are in M_3 or M_4 we have $M_1 \cap M_2 = M_3 + M_4$

In this notation, we also have the inclusion relation

for $\Delta = \frac{\ell^2}{4}, c=1$:
$$\begin{tikzcd} M_{\frac{\ell^2}{4},c} \arrow[d] \\ M(v_{\ell+1,1}) \arrow[d] \\ M(v_{\ell+2,1}) \arrow[d] \\ \vdots \end{tikzcd}$$

§ Character formula

Recall, for Verma module

$$\mathrm{ch}_{M_{\Delta,c}}(q) = \mathrm{Tr}_{M_{\Delta,c}}\, q^{L_0-\frac{c}{24}} = \frac{q^{\Delta-c/24}}{\prod_{n=1}^{\infty}(1-q^n)}$$

For $c=1$, $\Delta=\frac{\ell^2}{4}$, we have $N = M(v_{\ell+1,1})$ $\quad \Delta(v_{\ell+1,1}) = \frac{\ell^2}{4}+\ell+1$

$$\therefore\ \mathrm{ch}_{M_{\Delta,c}/N}(q) = \mathrm{ch}_{M_{\Delta,c}}(q) - \mathrm{ch}_N(q)$$

$$= q^{\frac{\ell^2}{4}-\frac{1}{24}}(1-q^{\ell+1})\Big/\prod_{n=1}^{\infty}(1-q^n) = \frac{q^{\frac{\ell^2}{4}-\frac{1}{24}}}{\prod_{\substack{n\neq\ell+1\\ n\geq 1}}(1-q^n)}$$

For a unitary (Δ_0, c_0), $0<c_0<1$:

$$N = M(v_{r_0,s_0}) + M(v_{\bar{r}_0,\bar{s}_0})$$

$$\mathrm{ch}_{M_{\Delta_0,c_0}/N} = \mathrm{ch}_{M_{\Delta_0,c_0}} - \underbrace{\mathrm{ch}_N}_{\mathrm{ch}_{M(v_{r_0,s_0})} + \mathrm{ch}_{M(v_{\bar{r}_0,\bar{s}_0})} - \mathrm{ch}_{M(v_{r_0,s_0})\cap M(v_{\bar{r}_0,\bar{s}_0})}}$$

$$= \mathrm{ch}_{M_{\Delta_0,c_0}} - \mathrm{ch}_{M(v_{r_0,s_0})} - \mathrm{ch}_{M(v_{\bar{r}_0,\bar{s}_0})} + \mathrm{ch}_{\underbrace{M(v_{r_0,s_0})\cap M(v_{\bar{r}_0,\bar{s}_0})}_{M(v'_{r_1,s_1}) + M(v'_{\bar{r}_1,\bar{s}_1})}}$$

$$= \mathrm{ch}_{M_{\Delta_0,c_0}} - \mathrm{ch}_{M(v_{r_0,s_0})} - \mathrm{ch}_{M(v_{\bar{r}_0,\bar{s}_0})} + \mathrm{ch}_{M(v'_{r_1,s_1})} + \mathrm{ch}_{M(v'_{\bar{r}_1,\bar{s}_1})} - \mathrm{ch}_{M(v'_{r_1,s_1})\cap M(v'_{\bar{r}_1,\bar{s}_1})}$$

$\vdots$

$$= \mathrm{ch}_{M_{\Delta_0,c_0}} - \sum_{n=0}^{\infty}\left\{\mathrm{ch}_{M(v_{r_0+nm,\, s_0+n(m+1)})} + \mathrm{ch}_{M(v_{\bar{r}_0+nm,\, \bar{s}_0+n(m+1)})}\right\}$$

$$+ \sum_{n=0}^{\infty}\left\{\mathrm{ch}_{M(v'_{r_1+nm,\, s_1+n(m+1)})} + \mathrm{ch}_{M(v'_{\bar{r}_1+nm,\, \bar{s}_1+n(m+1)})}\right\}$$

<u>Note</u> $\Delta(v_{r,s}) = \Delta_0 + rs$

$$\Delta(v'_{r,s}) = \Delta(v_{r_0,s_0}) + rs = \Delta_0 + r_0 s_0 + rs$$

$$\therefore \mathrm{ch}_{M_{\Delta_0,c_0}/N}(q) = \frac{q^{\Delta_0 - \frac{c_0}{24}}}{\prod_{n=1}^{\infty}(1-q^n)}\left\{1 - \sum_{n=0}^{\infty}\left(q^{(r_0+nm)(s_0+n(m+1))} + q^{(\bar{r}_0+nm)(\bar{s}_0+n(m+1))}\right)\right.$$

$$\left. + \sum_{n=0}^{\infty}\left(q^{(r_1+nm)(s_1+n(m+1))+r_0s_0} + q^{r_0s_0+(\bar{r}_1+nm)(\bar{s}_1+n(m+1))}\right)\right\}$$

<u>Note</u>
$$\begin{cases} \bar{r}_0 = m - r_0, & \bar{s}_0 = m+1-s_0 \\ r_1 = m + r_0, & s_1 = m+1-s_0 \\ \bar{r}_1 = m - r_0, & \bar{s}_1 = m+1+s_0 \end{cases}$$

$$= \frac{q^{\Delta_0 - \frac{c_0}{24}}}{\prod_{n=1}^{\infty}(1-q^n)}\left\{\sum_{n\in\mathbb{Z}} q^{r_0s_0 + (r_0+nm)(-s_0+n(m+1))} - \sum_{n\in\mathbb{Z}} q^{(r_0+nm)(s_0+n(m+1))}\right]$$

$c_0 = \frac{1}{2}$ $[m=3]$

$r_0=1,\ s_0=1 \Rightarrow (\bar{r}_0, \bar{s}_0) = (2,3),\ (r_1, s_1) = (4,3),\ (\bar{r}_1, \bar{s}_1) = (2,5)$

$$\left[1 - (q + q^6 + \cdots) + (q^{1+12} + q^{1+10} + \cdots)\right] \Big/ \prod_{n=1}^{\infty} (1-q^n)$$

$$\begin{aligned} = {} & 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \cdots \\ & - q - q^2 - 2q^3 - 3q^4 - 5q^5 - 7q^6 - 11q^7 - \cdots \\ & - q^6 - q^7 - \cdots \end{aligned}$$

$$= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + \cdots$$

Matches

$(r_0, s_0) = (2,1) \Rightarrow (\bar{r}_0, \bar{s}_0) = (1,3),\ (r_1, s_1) = (5,3),\ (\bar{r}_1, \bar{s}_1) = (1,5)$

$$\left[1 - (q^2 + q^3 + \cdots) + (q^{2+5} + \cdots)\right] \Big/ \prod_{n=1}^{\infty} (1-q^n)$$

$$\begin{aligned} = {} & 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \cdots \\ & - q^2 - q^3 - 2q^4 - 3q^5 - 5q^6 - 7q^7 - \cdots \\ & - q^3 - q^4 - 2q^5 - 3q^6 - 5q^7 - \cdots \\ & + q^7 + \cdots \end{aligned}$$

$$= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + \cdots$$

matches!

$(r_0, s_0) = (1,2) \Rightarrow (\bar{r}_0, \bar{s}_0) = (2,2),\ (r_1, s_1) = (4,2),\ (\bar{r}_1, \bar{s}_1) = (2,6)$

$$\left[1 - (q^2 + q^4 + \cdots) + (q^{2+8} + q^{2+12} + \cdots)\right] \Big/ \prod_{n=1}^{\infty} (1-q^n)$$

$$\begin{aligned} = {} & 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \cdots \\ & - q^2 - q^3 - 2q^4 - 3q^5 - 5q^6 - 7q^7 - \cdots \\ & - q^4 - q^5 - 2q^6 - 3q^7 - \cdots \end{aligned}$$

$$= 1 + q + q + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + \cdots$$

matches!

Alternatively $\Delta(v_{r,s}) = \Delta_{r,s} + rs = \dfrac{((m+1)r+ms)^2-1}{4m(m+1)}$

$\Delta(v'_{r,s}) = \Delta_{r,s} + rs =$ same expression.

(r,s) in the sum of $\mathrm{ch}_{M(v_{r,s})}$ are

$$(r,s) = \pm(r_0+nm, s_0+n(m+1)) \quad \begin{matrix} n\geq 0 \\ n\leq -1 \end{matrix}$$

(r,s) in the sum of $\mathrm{ch}_{M(v'_{r,s})}$ are

$$(r,s) = \pm(r_0+nm, -s_0+n(m+1)) \quad \begin{matrix} n\geq 1 \\ n\leq -1. \end{matrix}$$

$$\therefore \{\Delta(v_{r,s})\} = \left\{ \frac{((m+1)(r_0+nm)+m(s_0+n(m+1)))^2-1}{4m(m+1)} \right\}_{n\in\mathbb{Z}}$$

$$\{\Delta_0\} \cup \{\Delta(v'_{r,s})\} = \left\{ \frac{((m+1)(r_0+nm)+m(-s_0+n(m+1)))^2-1}{4m(m+1)} \right\}_{n\in\mathbb{Z}}$$

$n=0 \Leftrightarrow \Delta_0$

$$\therefore \mathrm{ch}_{M_{\Delta_0,c_0}/N}(q) = \sum_{n\in\mathbb{Z}} \frac{q^{\star - \frac{c_0}{24}} - q^{\star - \frac{c_0}{24}}}{\prod_{n=1}^{\infty}(1-q^n)}$$

$\dfrac{c_0}{24} = \dfrac{1}{24} - \dfrac{1}{4m(m+1)}$

$$= \sum_{n\in\mathbb{Z}} \frac{q^{\frac{(2nm(m+1)+(m+1)r_0-ms_0)^2}{4m(m+1)}} - q^{\frac{(2nm(m+1)+(m+1)r_0+ms_0)^2}{4m(m+1)}}}{q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)}$$

Using $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ Dedekind's eta

$$\Theta_N[\lambda](\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{(Nn+\lambda)^2}{2N}}$$ Θ - function, $(\lambda \in \mathbb{Z}/N\mathbb{Z})$

$$\boxed{\mathrm{ch}_{M_{\Delta_0,c_0}/N}(q) = \frac{\Theta_{2m(m+1)}[(m+1)r_0 - ms_0](\tau)}{\eta(\tau)} - \frac{\Theta_{2m(m+1)}[(m+1)r_0 + ms_0](\tau)}{\eta(\tau)}}$$

Modular transformation property

- $\tau \to \tau+1$: $\quad \mathrm{ch}(q) \to \mathrm{ch}(q)\, e^{2\pi i(\Delta_0 - \frac{c_0}{24})}$

- $\tau \to -1/\tau$: $\quad \eta(\tau) \to \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$

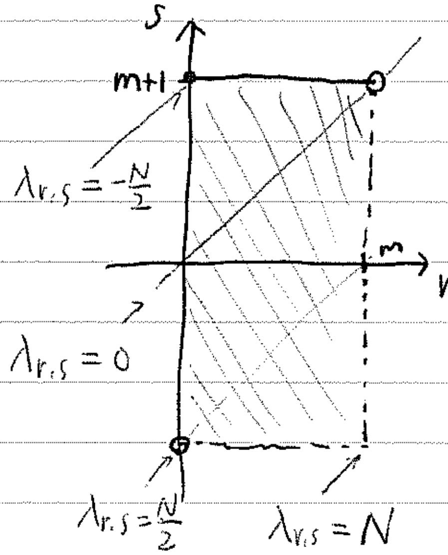
By Poisson resummation, we also find

$$\Theta_N[\lambda](-1/\tau) = \sqrt{-i\tau} \sum_{\mu=1}^{N} \frac{1}{\sqrt{N}} e^{2\pi i \lambda\mu/N}\, \Theta_N[\mu](\tau).$$

but $\mathrm{ch} = \dfrac{\Theta_N[\lambda_{r_0,s_0}](\tau)}{\eta(\tau)} - \dfrac{\Theta_N[\lambda_{r_0,-s_0}](\tau)}{\eta(\tau)} \longrightarrow\ ?$

$N = 2m(m+1)$, $\lambda_{r,s} = (m+1)r - ms$

Some preparation.



$$\lambda_{r,s} = (m+1)r - ms$$

~~The~~ A fundamental domain for $\mu \in \mathbb{Z}_N$ is given by the rectangle (left).

i.e. $\forall \mu$, $\exists 1.$ (r,s) in the rectangle s.t. $\mu = \lambda_{r,s} \mod N$.

$\exists 1.$ (r_*, s_*) s.t. $(m+1)r_* - ms_* \equiv 1 \mod N$.

denote $\omega_* = (m+1)r_* + ms_* \equiv 1 + 2ms_* \equiv -1 + 2(m+1)r_*$.

☆ $\omega_*^2 = ((m+1)r_* - ms_*)^2 + 4m(m+1)r_*s_* \equiv 1 \mod 2N$.

☆ $\omega_* \lambda_{r,s} = (1 + 2ms_*)(\lambda_{r,-s} - 2ms)$

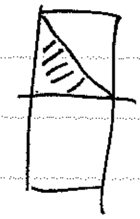
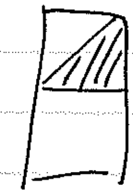
$$= \lambda_{r,-s} + 2ms_* \underbrace{\lambda_{r,-s}}_{(m+1)r + \cancel{ms}} - 2ms\,\overbrace{((m+1)r_* + \cancel{ms_*})}^{\omega_*}$$

$$\equiv \lambda_{r,-s} \mod N.$$

☆ ω_* and $N = 2m(m+1)$ are coprime.

$\therefore \quad \omega_* \times : \mathbb{Z}_N \xrightarrow{\cong} \mathbb{Z}_N.$

$$\therefore \; ch = \frac{\Theta_N[\lambda](\tau)}{\eta(\tau)} - \frac{\Theta_N[\omega_*\lambda](\tau)}{\eta(\tau)}$$

where λ is from  or equally 

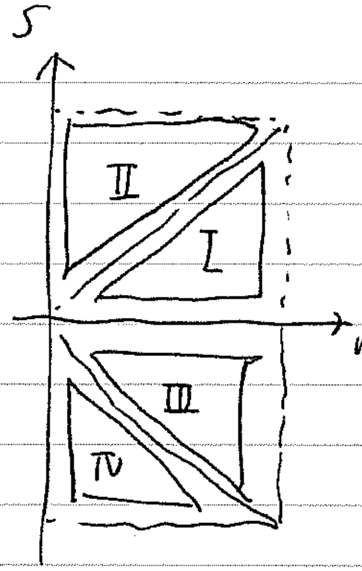
$$\left[\begin{array}{l} \text{c.f.}\ \Theta_N[\lambda] = \Theta_N[-\lambda] \\[1em] \therefore\ ch = 0 \quad \text{if} \quad \omega_*\lambda = \lambda \quad \text{or} \quad -\lambda \\ \qquad\qquad\qquad\qquad\quad \Updownarrow \qquad\qquad\quad \Updownarrow \\ \qquad\qquad\qquad\qquad \lambda \in (m+1)\mathbb{Z} \qquad \lambda \in m\mathbb{Z} \end{array}\right]$$

$$ch(-1/\tau) = \frac{\Theta_N[\lambda](-1/\tau)}{\eta(-1/\tau)} - \frac{\Theta_N[\omega_*\lambda](-1/\tau)}{\eta(-1/\tau)}$$

$$= \sum_{\mu=1}^{N} \frac{1}{\sqrt{N}} \left(e^{2\pi i \lambda\mu/N} \frac{\Theta_N[\mu](\tau)}{\eta(\tau)} - e^{2\pi i\, \omega_*\lambda\mu/N} \frac{\Theta_N[\mu](\tau)}{\eta(\tau)} \right)$$

$$= \sum_{\mu=1}^{N} \frac{1}{\sqrt{N}} \left(e^{2\pi i \lambda\mu/N} \frac{\Theta_N[\mu](\tau)}{\eta(\tau)} - e^{2\pi i\, \overbrace{\omega_*\lambda\, \omega_*}^{1 \bmod 2N}\mu/N} \frac{\Theta_N[\omega_*\mu](\tau)}{\eta(\tau)} \right)$$

$$= \sum_{\mu=1}^{N} \frac{1}{\sqrt{N}} e^{2\pi i \lambda\mu/N} \left(\frac{\Theta_N[\mu](\tau)}{\eta(\tau)} - \frac{\Theta_N[\omega_*\mu](\tau)}{\eta(t)} \right)$$



$$\mathrm{I} \leftrightarrow \mathrm{II} \quad : \quad \mu \leftrightarrow -\mu$$

$$\mathrm{I} \leftrightarrow \mathrm{III} \quad : \quad \mu \leftrightarrow \omega_* \mu$$
$$(\& \, \mathrm{II} \leftrightarrow \mathrm{IV})$$

$$\therefore \sum_{\mu} = \sum_{\mu \in \mathrm{I}} + \underbrace{\sum_{\mu \in \mathrm{II}}}_{\substack{\Updownarrow \\ \mu = -\mu' \\ \mu' \in \mathrm{I}}} + \underbrace{\sum_{\mu \in \mathrm{III}}}_{\substack{\Updownarrow \\ \mu = \omega_* \mu' \\ \mu' \in \mathrm{I}}} + \underbrace{\sum_{\mu \in \mathrm{IV}}}_{\substack{\Updownarrow \\ \mu = \omega_* \mu' \\ \mu' \in \mathrm{I}}}$$

$$\therefore \mathrm{ch}_\lambda(-1/\tau) = \sum_{\mu \in \mathrm{I}} \frac{1}{\sqrt{N}} \left(e^{2\pi i \lambda \mu/N} + e^{-2\pi i \lambda \mu/N} - e^{2\pi i \lambda \omega_* \mu/N} - e^{-2\pi i \lambda \omega_* \mu/N} \right) \times \mathrm{ch}_\mu(\tau)$$

$$\Updownarrow$$

$$\mathrm{ch}_{r_0, s_0}(-1/\tau) = \sum_{r_0', s_0'} 2\sqrt{\frac{2}{m(m+1)}} (-1)^{1 + r_0 s_0' + s_0 r_0'} \sin\left(\pi \frac{m+1}{m} r_0 r_0'\right) \sin\left(\pi \frac{m}{m+1} s_0 s_0'\right) \times \mathrm{ch}_{r_0' s_0'}(\tau) .$$