Differential Equations for Correlation Functions

Suppose $\exists$ a null vector $|\chi\rangle$, $\langle \text{any}|\chi\rangle = 0$,

from descendants of a primary state $|\Delta\rangle$:

$$|\chi\rangle = \sum_{\{k\}} C_{\{k\}} L_{-\{k\}} |\Delta\rangle .$$

Then, we may regard the corresponding operator χ

to be zero. i.e. any correlation function involving χ

is zero:

$$\langle \chi(z) \phi_1(z_1) \cdots \phi_s(z_s) \rangle = 0.$$

i.e. $$\sum_{\{k\}} C_{\{k\}} \left\langle \left(L_{-\{k\}} \phi_\Delta \right)(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle = 0 \quad \text{---} (\star)$$

Using Ward identity

$$\langle (L_{-\{k\}} \phi_\Delta)(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle = \hat{\mathcal{L}}_{-\{k\}} \langle \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle,$$

$\uparrow$ differential operator wrt $z, z_1, \cdots, z_s$

$(\star)$ can be written as a differential equation

for $\langle \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \rangle$.

- The order of differential equation is generically the same as the level of the null vector.

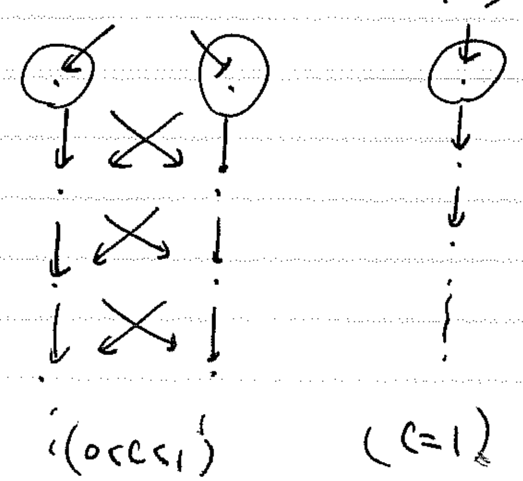
- If χ is a null vector, its descendants are all null vectors. But the differential equation for descendants can be derived from the differential eqn for χ, since

$$\langle L_{-n}\chi(z)\,\phi_1(z_1)\cdots\phi_s(z_s)\rangle \underset{\uparrow}{=} \hat{\mathcal{L}}_{-n}\langle \chi(z)\,\phi_1(z_1)\cdots\phi_s(z_s)\rangle .$$

$\phi_1 \cdots \phi_s$ are primaries

- Thus, independent differential equations are from the singular vectors that generate the space of all null vectors.

- In view of $|\Delta\rangle$ or $|\Delta\rangle$ there are <u>two</u> or <u>one</u> independent differential eqns for $0<c\leq 1$ unitary theories.



· By Kac formula, $\exists$ singular vector at level rs if

$$\Delta = \Delta_{r,s} = \frac{1}{4}(\alpha_+ r + \alpha_- s)^2 + \frac{c-1}{24}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

· For example, if $\Delta = \Delta_{1,2}$ or $\Delta_{2,1}$, there is a singular vector at level 2. Recall

$$M^{(2)} = \begin{pmatrix} 4\Delta + \frac{c}{2} & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix} \quad \text{wrt } \left(L_{-2}|\Delta\rangle,\ L_{-1}^2|\Delta\rangle\right).$$

Kernel of $M^{(2)}$ is $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ s.t. $\left(4\Delta + \frac{c}{2}\right)v_1 + 6\Delta\, v_2 = 0$,

$$6\Delta v_1 + 4\Delta(2\Delta+1)v_2 = 0.$$

(One follows from the other if $\Delta = \Delta_{1,2}$ or $\Delta_{1,2}$?)

The singular vector is s.t. $v_2 = -\frac{3}{2(2\Delta+1)}v_1$ i.e.

$$|\chi\rangle = L_{-2}|\Delta\rangle - \frac{3}{2(2\Delta+1)}L_{-1}^2|\Delta\rangle$$

This leads to the eqn

$$0 = \left\langle \left(L_{-2}\, \phi_\Delta(z) - \frac{3}{2(2\Delta+1)} L_{-1}^2\, \phi_\Delta(z) \right) \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle$$

If $\phi_1, \cdots, \phi_s$ are primaries, Ward identity is

$$\left\langle L_{-2} \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle = - \sum_{i=1}^{s} \left((z_i - z)^{-1} \partial_{z_i} + \frac{\partial}{\partial z_i} (z_i - z)^{-1} \Delta_i \right) \times \left\langle \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle$$

$$= \sum_{i=1}^{s} \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \left\langle \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle .$$

Thus the equation is

$$\boxed{\begin{aligned} 0 = & \sum_{i=1}^{s} \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \left\langle \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle \\ & - \frac{3}{2(2\Delta+1)} \frac{\partial^2}{\partial z^2} \left\langle \phi_\Delta(z)\, \phi_1(z_1) \cdots \phi_s(z_s) \right\rangle . \end{aligned}} \tag{*}$$

This is a strong constraint on the correlation functions.

In some cases, the set of such equations <u>determines</u> the correlation functions.

non zero only if $\Delta = \Delta_1$, $\tilde{\Delta} = \tilde{\Delta}_1$

· For 2-point function $\langle \phi_\Delta(z)\, \phi_1(z_1) \rangle = \dfrac{C_{\Delta 1}}{(z-z_1)^{2\Delta}(\bar{z}-\bar{z}_1)^{2\tilde{\Delta}}}$:

(*) has no new information.

· For 3-point function $\langle \phi_\Delta(z)\, \phi_1(z_1)\, \phi_2(z_2) \rangle$

$$= \frac{C_{\Delta 12}}{(z-z_1)^{\Delta+\Delta_1-\Delta_2}(z-z_2)^{\Delta+\Delta_2-\Delta_1}(z_1-z_2)^{\Delta_1+\Delta_2-\Delta} \times \text{c.c.}\ (\Delta_i \to \tilde{\Delta}_i)}$$:

(*) gives a constraint on $\Delta, \Delta_1, \Delta_2$ for $C_{\Delta 12} \neq 0$

[Constraint on OPE → see below]

· For 4-point function $\langle \phi_\Delta(z)\, \phi_1(z_1)\phi_2(z_2)\, \phi_3(z_3) \rangle$

$$= \begin{bmatrix} \text{Known prefactor} \\ \text{including } z_i, \bar{z}_i, z, \bar{z} \end{bmatrix} \times F(x, \bar{x}) \qquad x = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \text{cross ratio}$$

(*) becomes a second order Ordinary differential eqn for $\underbrace{F(x, \bar{x})}_{x \text{ dependence of}}$

With the corresponding eqn for $\bar{x}$

and with appropriate boundary conditions,

$F(x, \bar{x})$ can be determined exactly!

[We will see this in an example later]

Constraints on OPE

$$\varphi_{\Delta_0}(z)\,\varphi_\Delta(z_1) = \sum_{\Delta'} (z-z_1)^{\Delta'-\Delta_0-\Delta}\left\{ \varphi_{\Delta'}(z_1) + O(z-z_1) \right\}$$

Assume $\Delta_0 = \Delta_{2,1}$ or $\Delta_{1,2}$ and apply the eqn (*) to

$$\langle \varphi_{\Delta_0}(z)\,\varphi_\Delta(z_1)\cdots\rangle = \sum_{\Delta'} (z-z_1)^{\underbrace{\Delta'-\Delta_0-\Delta}_{\displaystyle \overset{!!}{K}}} \langle \varphi_{\Delta'}(z_1)\cdots\rangle + \cdots .$$

Look at the most singular terms as $z\to z_1$:

$$\frac{\Delta}{(z-z_1)^2} + \frac{-K}{(z-z_1)^2} - \frac{3}{2(2\Delta_0+1)}\frac{K(K-1)}{(z-z_1)^2} + \cdots = 0$$

less singular

$$\Rightarrow \quad \Delta - K - \frac{3}{2(2\Delta_0+1)}K(K-1) = 0$$

i.e. $\quad K(K-1) + \frac{2}{3}(2\Delta_0+1)K - \frac{2}{3}(2\Delta_0+1)\Delta = 0$

i.e. $\quad \left(K - \frac{1}{2} + \frac{1}{3}(2\Delta_0+1)\right)^2 = \left(-\frac{1}{2}+\frac{1}{3}(2\Delta_0+1)\right)^2 + \frac{2}{3}(2\Delta_0+1)\Delta$

Convenient parametrization for $\Delta_i = \frac{\alpha_i^2}{4} + \frac{c-1}{24} =: \Delta(\alpha_i)$

$$\left(K = \frac{\alpha'^2-\alpha^2}{4} - \Delta_0 \right)$$

$$\therefore \left(\frac{\alpha'^2-\alpha^2}{4} - \frac{1}{6}(2\Delta_0+1)\right)^2 = \underbrace{\left(-\frac{1}{2}+\frac{1}{3}(2\Delta_0+1)\right)^2 + \frac{2}{3}(2\Delta_0+1)\frac{c-1}{24}}_{\displaystyle \overset{\Vert}{0} \leftarrow \det M^{(2)}_{\Delta_0,c} = 0} + \frac{2}{3}(2\Delta_0+1)\frac{\alpha^2}{4}$$

$$\Delta_0 = \Delta_{2,1} = \Delta(2\alpha_+ + \alpha_-) \quad \Rightarrow \quad \frac{2}{3}(2\Delta_0 + 1) = \alpha_+^2$$

$$\Delta_0 = \Delta_{1,2} = \Delta(\alpha_+ + 2\alpha_-) \quad \Rightarrow \quad \frac{2}{3}(2\Delta_0 + 1) = \alpha_-^2$$

$$\therefore \quad \left(\frac{\alpha'^2 - \alpha^2}{4} - \frac{\alpha_\pm^2}{4}\right)^2 = \frac{\alpha_\pm^2 \alpha^2}{4} \qquad \text{for } \Delta_0 = \begin{cases} \Delta_{2,1} \\ \Delta_{1,2} \end{cases}$$

$$\Updownarrow$$

$$\alpha'^2 - \alpha^2 - \alpha_\pm^2 = 2\alpha_\pm \alpha \ \text{ or } -2\alpha_\pm \alpha$$

$$\Updownarrow$$

$$\alpha'^2 = (\alpha + \alpha_\pm)^2 \ \text{ or } \ (\alpha - \alpha_\pm)^2 .$$

Thus, we found the following rule :

$$\boxed{\begin{array}{l} \text{The OPE of } \phi_{\Delta_0} \text{ for } \Delta_0 = \begin{cases} \Delta_{2,1} \\ \Delta_{1,2} \end{cases} \text{ and} \\ \phi_\Delta \text{ with } \Delta = \Delta(\alpha) \text{ can include primaries } \phi_{\Delta'} \\ \text{only with } \Delta' = \Delta(\alpha'), \ \alpha' = \alpha + \alpha_\pm \text{ or } \alpha - \alpha_\pm \end{array}}$$

Schematically, we have the "Fusion rule" :

$$\text{“}\left[\phi_{\Delta_{2,1}}\right] \times \left[\phi_{\Delta(\alpha)}\right] \subseteq \left[\phi_{\Delta(\alpha+\alpha_+)}\right] + \left[\phi_{\Delta(\alpha-\alpha_+)}\right]\text{”}$$

$$\text{“}\left[\phi_{\Delta_{1,2}}\right] \times \left[\phi_{\Delta(\alpha)}\right] \subseteq \left[\phi_{\Delta(\alpha+\alpha_-)}\right] + \left[\phi_{\Delta(\alpha-\alpha_-)}\right]\text{”}.$$

["$\subseteq$" because $\left[\phi_{\Delta_{2,1 \ (1,2)}}\right] \times \left[\phi_{\Delta(\alpha)}\right]$ may not contain both $\left[\phi_{\Delta(\alpha+\alpha_\pm)}\right]$ and $\left[\phi_{\Delta(\alpha-\alpha_\pm)}\right]$]

· It is straightforward to check that the eqn (*) for three point function $\langle \phi_\Delta(z)\, \phi_1(z_1)\, \phi_2(z_2) \rangle_{*_0}$ $\left(\Delta = \begin{cases} \Delta_{2,1} \\ \Delta_{1,2} \end{cases}\right)$ is satisfied only if $\Delta, \Delta_1, \Delta_2$ obey this constraint.

i.e. $\Delta_1 = \frac{\alpha_1^2}{4} + \frac{c-1}{24}$, $\Delta_2 = \frac{\alpha_2^2}{4} + \frac{c-1}{24}$

with $|\alpha_1 - \alpha_2| = |\alpha_\pm|$.